Extremes Day

In honour of Laurens de Haan

Extremes, Risk, Safety and the Environment

CEAUL / FCUL

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ISBN: 972-8859-48-1
Depósito Legal: 239151/06
Number of copies: 60
February 2006

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[1]Edition partially supported by FCT / POCTI and POCI / FEDER.

Preface

The Extremes Day will be centered around the "*Gulbenkian Lecture*" of Laurens de Haan. The Calouste Gulbenkian Foundation conferred a Gulbenkian Professorship to Laurens de Haan, Professor at the Faculty of Economics, Erasmus University Rotterdam, Honorary Professor at Peking University, China, and Honorary Doctor at Lisbon University, Portugal.

Laurens de Haan has one of the most prominent careers in the XX^{th} century Statistics. He may indeed be considered as one of the world exponents in the area of *Statistics of Extremes* and his Ph.D. thesis, written at 1970, and entitled *On Regular Variation and its Application to the Weak Convergence of Sample Extremes*, is still an almost compulsory reference in the field. Laurens de Haan has contributed to the development of well-built theories in areas like *Extended Regular Variation*, *Multivariate Extremes*, *Semi-Parametric Estimation* and *Extremes for Dependent Sequences*. Recently, he has been paying special attention to the field of *Extremes in Infinite-Dimensional Spaces*. Beyond the building of a unified and rigorous *Extreme Value Theory*, Laurens de Haan has also had a pioneering work in the solution of important environmental problems, related to the modelling of rivers, sea and dams, and the specification of new standards for the Dutch sea defences.

Since 1997, Laurens de Haan has regularly visited Lisbon, and this has led to the development of joint research work with several members of CEAUL (Centro de Estatística e Aplicações da Universidade de Lisboa), as well as inspired the scientific cooperation with other members of the Portuguese statistical community. On the grounds of the strong cooperation developed between Laurens de Haan and members of DEIO/FCUL (Departamento de Estatística e Investigação Operacional/Faculdade de Ciências da Universidade de Lisboa) and CEAUL, M. Ivette Gomes (DEIO, FCUL) put forward a proposal for the award of a Gulbenkian Professorship, which naturally has been granted.

Under this Gulbenkian Professorship, Laurens de Haan has been a visiting professor at FCUL (DEIO), from the 1st of January until the 31st of December 2005. On the occasion of his Gulbenkian Lecture, entitled *ON EXTREME VALUE THEORY. OR: HOW TO LEARN FROM ALMOST DISASTROUS EVENTS*, CEAUL and the ERAS project, POCI/MAT 58876/2004, are organizing an "Extremes Day" in honour of Laurens de Haan. In this "Extremes Day" we are interested in detecting the new advances in the field of *Statistics of Extremes*, and their applications to *Risk, Safety and the Environment*, the main topics of ERAS project.

We thank Calouste Gulbenkian Foundation and the Faculty of Science, University of Lisbon, for sponsoring this event. And we indeed thank Professor Laurens de Haan for his generous sharing of ideas with the members of CEAUL and DEIO.

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Contents

ON EXTREME VALUE THEORY OR: HOW TO LEARN FROM ALMOST DISASTROUS EVENTS[2]

•• Gulbenkian Lecture ••

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Everybody has a personal bank account. Every month the salary is added to the balance. Now suppose that you are a big spender and you spend most of the money every month. It would be unpleasant if the balance would go down to zero during the month but this has never happened. You could be afraid of this event to happen and you would like to know the probability that in fact the balance would be depleted during a month. This is a difficult statistical problem since you want to estimate the probability of an event that has never happened and this seems impossible.

Other similar examples are:

- Banks and insurance companies want to (have to) assess the probability that they go bankrupt in some given period of time. The regulator forces them to do so.
- A communication tower is much affected by wind storms, but the tower has never collapsed. Since much depends on this tower, one needs to know the probability of collapse.

Problems of this kind can be attacked using a special branch of the area of mathematical statistics called extreme value theory. The theory has been developed over the last 70 years by scientists mainly from Europe and notably by Professor Tiago de Oliveira from this university. We are now able to provide a reliable answer to the mentioned questions.

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[2]Partially supported by FCT/POCTI and POCI/FEDER.

I want to explain some ideas and results of extreme value theory, in particular what is needed to answer the stated problems.

I shall do so by discussing a specific problem in which I have been involved. It has to do with the coastal protection of the Netherlands against floods.

In some places the low lying parts of the country are protected by natural sand dunes and in other parts by man-made dykes. The two situations call for different approaches. In the case of protection by sand dunes the height of the dunes is usually sufficient but the problem is that during a heavy storm big chunks of the sand dunes are washed away. Hence the safety is expressed not in terms of meters (i.e. the height) but in terms of square meters (i.e. the sand content).

In case the strip of coast is protected by a dyke only the height is important since usually a storm does not inflict damage to the well fortified outer part of the dyke. But then the question is: how high should we build the dyke in order to achieve a sufficiently low probability of overflowing? This is a kind of non standard problem in statistics since there may never have been a flood at that point of the coast. I shall try to explain the main idea behind the approach to solve this problem. I shall not use any mathematics or mathematical formulas, I shall try to explain everything just by showing some graphs.

MODELING EXTREMAL DEPENDENCE

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Abstract:
The classical multivariate extreme value theory cannot distinguish asymptotically between exact independence of the components of a random vector and a moderate dependence which vanishes as the observations become more and more extreme. For instance, the limit distribution of the standardized maxima of the components of a bivariate normal random vector is the same for all correlations less than 1; hence it does not reflect the strength of the dependence between the components. By contrast, the model proposed by Ledford and Tawn (1997) captures the dependence structure between the components of the exceedances over a high (yet finite) threshold much more accurately. It has thus proved to be very useful for modeling the extremal dependence, if one can take neither asymptotic dependence nor exact independence of the components for granted. For example, large claims in different lines of business of a non-life insurer often exhibit a clear positive dependence which vanishes asymptotically when one considers the exceedances over increasing thresholds.

We discuss some estimators of the main parameters of the model by Ledford and Tawn. Moreover, a graphical tool to asses the goodness-of-fit is introduced. In analogy to the well-known Hill pp-plot, to this end one checks whether differences of the logarithm of certain empirical probabilities lie approximately on a certain plane. In addition to this purely data-analytic tool, we derive asymptotic confidence intervals which enables us to check whether the observed deviations from the ideal plane can be explained by random effects or whether they indicate that the model assumptions are violated. These asymptotic results are based on approximations to certain empirical processes established by Draisma *et al.* (2004). The practical usefulness of these tools is demonstrated by examples from the insurance business and finance.

TOPICS ON MULTIVARIATE AND INFINITE-DIMENSIONAL EXTREMES[3]

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Abstract: We shall discuss a few main results and concepts in multivariate and infinite-dimensional extreme value theory, like limit distribution of normalized maxima and exponent measure. They will be introduced in the multivariate context, and hopefully the infinite-dimensional case will follow smoothly. The usefulness of the theory is exemplified with the problem of failure set probability estimation in both situations.

Key words and phrases: multivariate and infinite-dimensional extreme value theory, exponent measure, failure set estimation.

1. Introduction

One basic topic in multivariate extreme value theory is the limit distribution of normalized maxima. Let $(X, Y, \ldots, Z)$ be a random vector with distribution function (d.f.) F. A common approach is to take componentwise maxima, and examine the convergence in distribution of the random vector,

$$\left(\frac{\max(X_1,\ldots,X_n)-b_n}{a_n}, \frac{\max(Y_1,\ldots,Y_n)-d_n}{c_n}, \cdots, \frac{\max(Z_1,\ldots,Z_n)-f_n}{e_n}\right)$$

where $a_n, c_n, \ldots, e_n > 0$ and $b_n, d_n, \ldots, f_n$ real, are normalizing constants and $\{(X_i, Y_i, \ldots, Z_i)\}_{i=1}^n$ is an independent and identically distributed (i.i.d.) sample from F (see e.g. de Haan and Resnick (1977) [36]; Resnick (1987) [54]; de Haan and Ferreira (2006) [33]).

[3]Partially supported by FCT/POCTI and POCI/FEDER.

Then the exponent measure can be obtained, a noteworthy feature in extreme value theory. For instance we shall see its usefulness in estimating the probability of failure set.

Similarly, extreme value theory in function space often starts with limit theory for normalized maximum of i.i.d. random functions. Let $C[0,1]$ be the space of continuous functions f on $[0,1]$ equipped with the supremum norm, $\sup_{s\in[0,1]}|f(s)|$, and X a random function with sample space in $C[0,1]$. Then we mean to examine the convergence in distribution in $C[0,1]$ of the random process

$$\left\{\max_{1\le i\le n}\frac{X_i(s)-b_s(n)}{a_s(n)}\right\}_{s\in[0,1]} \tag{1}$$

where the maximum is taken pointwise for each $s\in[0,1]$ and, $a_s(n)>0$ and $b_s(n)$ real, are continuous functions for $s\in[0,1]$ (see e.g. Giné, Hahn, and Vatan (1990) [21]; de Haan and Lin (2001) [34]; de Haan and Ferreira (2006) [33]). One can then obtain the exponent measure and we shall use it in the estimation of the probability of a failure set in function space.

2. Multivariate Extremes

In the multivariate setting we discuss the bivariate case. The extension to higher dimensions should de straightforward.

Let $\{(X_i,Y_i)\}_{i=1}^n$, be i.i.d. random vectors from $F(x,y)=P(X\le x, Y\le y)$. If there exist $a_n, c_n>0$, and b_n, d_n real, and G is a d.f. with non-degenerate marginals such that,

$$\begin{aligned} &P\left(\frac{\max(X_1,\dots,X_n)-b_n}{a_n}\le x, \frac{\max(Y_1,\dots,Y_n)-d_n}{c_n}\le y\right)\\ &= F^n(a_nx+b_n, c_ny+d_n)\to^d G(x,y), \qquad n\to\infty, \end{aligned} \tag{2}$$

then we have convergence in distribution of the joint normalized maxima and G is a multivariate extreme value distribution.

Consider the standardization to Pareto marginals, i.e. the random vector $(1-F_1(X))^{-1},(1-F_2(Y))^{-1})$ where F_1, F_2 are the marginal d.f.'s of F supposed continuous. The exponent measure can be obtained as the result of the following limit, defined for all Borel sets $A\subset[0,\infty)^2$ with $\inf_{x,y\in A}\max(x,y)>0$ and $\nu(\partial A)=0$,

$$\nu(A) := \lim_{n\to\infty} nP\left\{\left(\frac{1}{n(1-F_1(X_1))}, \frac{1}{n(1-F_2(Y))}\right)\in A\right\}. \tag{3}$$

It turns out that the exponent measure ν is finite on $[0,\infty)^2\setminus[0,a]$, $\forall a>0$ and, in particular, it possesses the following homogeneity property: for all sets A as above,

$$\nu(cA)=c^{-1}\nu(A), \quad \forall c>0, \tag{4}$$

where the set cA is obtained by multiplying all elements of A by the constant c.

From the exponent measure or related functions, and their properties, one finds ways to characterize multivariate extreme value distributions. For instance via the function L and the level sets Q_c (Huang (1992) [42]; de Haan and Ferreira (2006) [33]) or the dependence functions from Sibuya (1960) [56] and Pickands (1981) [53].

Let us now see another application of the exponent measure. Suppose C_n is a given failure set, in particular containing large values of X or Y, and we want to estimate $p_n = P((X,Y) \in C_n)$ on the basis of an i.i.d. sample from F. Since we want to apply the previous asymptotic results, the failure set must depend on the sample size n in order to preserve the extreme nature of the problem. Then p_n can be rewritten as

$$p_n = P\left\{\left(\frac{1}{n(1-F_1(X_1))}, \frac{1}{n(1-F_2(Y))}\right) \in Q_n\right\}$$

where

$$Q_n := \left\{\left(\frac{1}{n(1-F_1(x))}, \frac{1}{n(1-F_2(y))}\right) : (x,y) \in C_n\right\},$$

and if $Q_n = c_n S$, for some $c_n \to \infty$ and S a fixed (open) Borel set in $[0,\infty)^2$ with $\inf_{x,y\in S} \max(x,y) > 0$ and $\nu(\partial S) = 0$, from (3) and (4),

$$p_n \approx \frac{\nu(c_n S)}{n} = \frac{\nu(S)}{c_n n}, \qquad \text{as } n \to \infty.$$

This motivates $\hat{p}_n = \hat{\nu}(\hat{S})/(nc_n)$. The proposal is then to use the empirical measure to estimate ν. While the set Q_n might not contain any observation, likewise the set C_n, the set S must contain enough observations which is possible via the sequence c_n. For the estimation of S one must deal with the estimation of the marginal d.f.'s.

The consistency of such an estimator, under appropriate additional conditions, was obtained in Ferreira, de Haan and Lin (2005) [14], and an application can be found in Ferreira and de Haan (2005) [13].

3. Extremes in $C[0,1]$

Let X be a random function in $C[0,1]$ with continuous marginal d.f.'s $F_s(x) := P\{X(s) \le x\}$, for each $s \in [0,1]$. Suppose the process (1) converges in distribution to some limit process on $C[0,1]$, with non-degenerate marginals. Then the exponent measure can be obtained from the limit

$$\nu(A) := \lim_{n\to\infty} nP\left(\left\{\frac{1}{n\{1-F_s(X_i(s))\}}\right\}_{s\in[0,1]} \in A\right)$$

for every Borel set $A \subset \{f \in C[0,1] : f \ge 0\}$ such that $\inf\{\sup_{s\in[0,1]} |f(s)| : f \in A\} > 0$ and $\nu(\partial A) = 0$. Note the similarity with the multivariate situation and, in particular, the same homogeneity property holds: $\nu(cA) = c^{-1}\nu(A)$, $\forall c > 0$, where the set cA is obtained by multiplying all elements of A by c.

Let us now apply the results to estimate $p_n := P\{X(s) > f_n(s)$ for some $s \in [0,1]\}$, on the basis of an i.i.d. sample and where f_n is a deterministic function on $C[0,1]$ possibly well far away from the observations. Similarly as before rewrite p_n as

$$P\left(\frac{1}{n\left\{1-F_s(X(s))\right\}} > \frac{1}{n\left\{1-F_s(f_n(s))\right\}} \text{ for some } s \in [0,1]\right)$$

and if $n^{-1}\left\{1-F_s(f_n(s))\right\}^{-1} = c_n h(s)$, for some $c_n \to \infty$,

$$p_n \approx \frac{\nu\{g : g(s) > c_n h(s) \text{ for some } s \in [0,1]\}}{n} = \frac{\nu\{g : g(s) > h(s) \text{ for some } s \in [0,1]\}}{n c_n}.$$

Consequently the construction of the estimator is quite similar to the previous case.

The consistency of such an estimator, under appropriate additional conditions, was obtained in Ferreira, de Haan and Lin (2005) [14].

EXTREMES OF INTEGER-VALUED MOVING AVERAGE PROCESSES

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This paper aims to analyze the extremal properties of integer-valued moving average sequences obtained as discrete analogues of conventional moving averages replacing scalar multiplication by binomial thinning. In particular, we consider the case in which the scalar coefficients are replaced by random coefficients, since in real applications the thinning probabilities may depend on several factors changing in time. Furthermore, the extremal behavior of periodic integer-valued moving average sequences is also considered. For these models, we find an unexpected phenomenon: when assessing their extremal properties, the extremal index seems not to be the object to look at.

ISSUES REGARDING EXTREME VALUE THEORY IN ENVIRONMENTAL AND NAVAL APPLICATION

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Introduction and generalities

When faced with modeling of high values it is natural that one thinks first of Extreme Value Theory (EVT). Here we focus on the possible need for caution in its application, and the suggestion of modified methods where needed. Specifically our discussion will hinge on what might be loosely described as the "coordination of period and level" which is implicit in the theory. We offer three examples from personal involvement — two from naval architecture, where there are mismatches between period and level and EVT does not provide the most appropriate means for describing the large values of concern. The third case concerns environmental regulation, where extremal theory fits but the resulting data does not satisfactorily support compliance decisions. In all cases we find at least partial remedies at hand via "Broad sense EVT", defined to include related areas of Central Limit Theory, level crossing problems, Palm distributions for Gaussian models, and exact modeling of probabilities of surrogate rare events.

The "Usual, Customary and Reasonable"[4] use of EVT concerns the distribution of the maximum of n iid random variables with df F, emanating from the almost trivial result:

"Theorimo" 1. *For a sequence of levels* $\{u_n\}$*,* $P\{M_n \le u_n\}$ *converges to a limit* $\rho = e^{-\tau}$ *iff* $n(1 - F(u_n)) \to \tau$*.*

The particular case $u_n = a_n^{-1}x + b_n$ for suitable sequences $a_n > 0$, $b_n \in \mathbb{R}$, leads to the customary distributional limit

$$P\{a_n(M_n - b_n \le x)\} \longrightarrow G(x), \tag{1}$$

[4] As with physicians' charges

for a df G of extreme value type.

A common situation occurs when one wishes to estimate $P\{M_n \le c\}$ for a fixed (high or critical) "level" c determined by an application at hand for large values of a "period" n. For example c may be dike height in classic Dutch dike design, n the time period of concern (perhaps several hundred years), and it may be quite reasonable to match c with $u_n = a_n^{-1}x + b_n$ via (1). In the environmental case c may be a critical safe level for human health of a hazardous pollutant, and critical levels of stress or roll in the naval applications considered.

Two of the three cases to be considered have various "mismatch features". For example the levels involved may be moderately high (certainly so by physical standards) but may not be "extreme" in the sense needed for direct extremal theory application. Or equivalently the levels may be "extreme" by physical standards, but the time period too short to apply (1) or the *Theorimo*. For the third (environmental) case EVT fits well but its limited data use causes problems of statistical inference. We discuss these in the light of the alternative approaches suggested above.

Case 1: Capsize prediction of vessels in high seas.

Capsize potential has always been a factor in the compromises involved in the design of ships, particularly in terms of other factors such as speed and maneuverability of naval vessels. Many different capsize modes are possible and may be individually or collectively present at any one time. Perhaps the most immediately obvious is that of heavy rolling, the modeling of which has been the focus of numerous joint US-Canada naval studies, emphasizing the fitting of Type 1 extreme value distributions to maximum roll in tank or simulated runs under specified conditions for the ocean and vessel.

It is our conclusion that these studies employ time periods which may be too short for convincing application of EVT, and the use of the specific double exponential form has no obvious intuitive advantage over other parametric fits. One may also debate the logic of the use of Type 1 fits for situations with bounded variables (roll angles). This does not suggest that it is fruitless to apply EVT to describe excessive roll, but at least substantially more extensive studies appear to be desirable.

On the other hand this may also provide a useful venue for consideration of lower levels, and surrogates for maximum capsize angle, presumably amenable to analysis using CLT rather than EVT methods, and having friendlier statistical properties. Yet another general approach based more on physical considerations, is to attempt to identify characteristics of wave trains which lead to capsize and evaluate their probabilities theoretically for specific ocean spectra. Such an approach suggested in simple form by deKat has been developed and used especially for capsize threats in following seas the calculation of the relevant wave statistics being now significantly enhanced by the development and use by I. Rychlik of the "WAFO" program library of Lund University.

Case 2: Structural safety of vessels in stormy (hurricane) conditions.

The structural integrity of a vessel can be threatened in a multitude of ways by adverse sea conditions and especially storms, as popularly evidenced in "The Perfect Storm". Two special types of threat have received special attention — the first being stress caused by the bending of a vessel as it is subjected to buoyancy changes along its length from wave motion ("wave induced" (WI) stress). This is a low frequency effect substantially mimicking the wave motion. The other threat is a so-called "whipping" stress which causes high frequency vibrations in the hull when the bow of the ship emerges from a rough ocean, and crashes ("slams") back down impacting the ocean surface.

Using a Gaussian field model for the ocean surface and a model of Ochi for the stress at a slam as proportional to the square of (vertical) reentry speed, one may obtain the relevant ("Palm") distributions for the two stresses and their sum, the "total stress" at a slam instant. It is a standard exercise to show that the resulting extreme value distribution for the maximum stress in a long period is exponential — totally corresponding to the whipping component alone. While this is a very pleasant and simple finding, it illustrates the inadequacy of blind reliance on EVT methods, though correctly computed. In practice in this case the time required for the "acquisition" of the exponential limit is so long that destruction of the hull would occur much earlier from the WI stresses.

In this case the problem may be solved by returning to a "pre-EVT limit" framework. Since the distribution of total stress at slams is known, and it is a reasonable assumption that their occurrence is Poisson, the probability that the maximum stress at slams should not exceed a level c is readily calculated. Since it is observed that invariably the maximum stress in a period is at (the largest) slam, the distribution of maximum stress is thus obtained. Such calculations have been done for typical ship parameters, operating under hurricane conditions, giving excellent agreement with actual tank model test data.

Case 3: EVT vs CLT in Environmental Assessment.

A variety of lessons may be learned from the earlier US regulation of ground-level ("tropospheric") ozone which was solidly based on EVT considerations. In conformity with this the level exceedances were reasonably assumed to be Poisson, and a standard of no more than 1 per year on average established as a criterion for compliance. This "ExEx" criterion appeared to be well suited for its purpose of evaluating occurrence of high levels of ozone. In particular there was no mismatch of the extreme level and the period considered. However a problem of testing conformity became very much in evidence. An area was declared to be "in compliance" with the criterion if there were no more than 3 exceedances in a 3 year period. But the small exceedance numbers near compliance resulted in oscillations of areas in and out of compliance.

This behavior reflects the poor "power" properties of the compliance determination procedure, regarded as a statistical test, not because of inappropriate formulation but from woefully inadequate data — consisting essentially of just the few highest values. A sensible remedy is clear — to rephrase the regulation to count exceedances of a lower level, and of course allow more of them for compliance. This means in particular that the test statistic (number of exceedances) may not be Poisson any more but for a judiciously chosen lower level can be normal, leading to significant advantage in its application. That is the use of CLT in lieu of our perhaps more favourite EVT seemed to be the logical procedure. In fact the regulating agency ($USEPA$) did appear to appreciate the technical issues favoring such a CLT procedure, but adopted a quite different complex regulation which is more "ad hoc" in its approach.

BIAS-CORRECTED HILL ESTIMATOR UNDER A THIRD ORDER FRAMEWORK[5]

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Abstract: In this paper we are interested in an adequate estimation of the dominant component of the bias of Hill's estimator of a positive tail index γ, in order to remove it from the classical Hill estimator in different asymptotically equivalent ways. The asymptotic distributional properties of the proposed estimators of γ are derived and the estimators are compared not only asymptotically, but also for finite samples through Monte Carlo techniques.

1. Introduction

In the field of Extremes, we usually say that a model F is heavy-tailed whenever the tail function is regularly varying, with a negative index of regular variation equal to $\{-1/\gamma\}$, $\gamma > 0$, or equivalently, the quantile function $U(t) = F^{\leftarrow}(1-1/t), t > 1$, with $F^{\leftarrow}(x) = \inf\{y : F(y) \geq x\}$, is of regular variation with index γ. This means that,

$$1 - F(x) \in RV_{-1/\gamma} \qquad \Longleftrightarrow \qquad U(t) \in RV_{\gamma}, \tag{1}$$

with the usual notation RV_α for the class of regularly functions with index of regular variation α. The second order parameter ρ (≤ 0), rules the rate of convergence in the first order condition (1), and is the non-positive parameter appearing in the limiting relation

$$\lim_{t \longrightarrow \infty} \frac{\ln U(tx) - \ln U(t) - \gamma \ln x}{A(t)} = \frac{x^\rho - 1}{\rho} \tag{2}$$

which we assume to hold for all $x > 0$ and where $|A(t)| \in RV_\rho$ (Geluk and de Haan, 1987). We shall assume everywhere that $\rho < 0$.

To obtain information on the distributional behaviour of second order parameters' estimators, we shall further assume that the rate of convergence in (2) is ruled by a

[5]Partially supported by FCT/POCTI and POCI/FEDER.

function $B(t)$ such that $B(t)$ is also of regular variation with the same index ρ, i.e., we assume that for all $x > 0$,

$$\lim_{t\longrightarrow\infty} \frac{\frac{\ln U(tx)-\ln U(t)-\gamma\ln x}{A(t)} - \frac{x^\rho-1}{\rho}}{B(t)} = \frac{x^{2\rho}-1}{2\rho}. \tag{3}$$

We shall indeed assume that we are in a large sub-classe of Hall's class of models (Hall, 1982; Hall and Welsh, 1985), with tail function

$$1-F(x) = Cx^{-1/\gamma}\left(1 + D_1 x^{\rho/\gamma} + D_2 x^{2\rho/\gamma} + o(x^{2\rho/\gamma})\right), \quad \text{as } x \longrightarrow \infty. \tag{4}$$

Consequently, we may choose $A(t) = \gamma\beta t^\rho$, $B(t) = \beta' t^\rho$, $\beta \neq \beta'$, $\rho < 0$.

For k intermediate, i.e., a sequence of integer values such that

$$k = k_n \longrightarrow \infty, \qquad k_n = o(n), \quad \text{as } n \longrightarrow \infty, \tag{5}$$

and under the third order framework in (3), Hill's estimator (Hill, 1975), $H(k)$, has the following asymptotic distributional representation,

$$H(k) \stackrel{d}{=} \gamma + \frac{\gamma}{\sqrt{k}} Z_k^{(1)} + \frac{A(n/k)}{1-\rho} + \frac{A(n/k)B(n/k)}{1-2\rho}(1 + o_p(1)), \tag{6}$$

where $Z_k^{(1)}$ is an asymptotically standard normal r.v.

The dominant component of the bias of Hill's estimator, given by $A(n/k)/(1-\rho) = \gamma\,\beta(n/k)^\rho$, is estimated through $H(k)\widehat{\beta}(n/k)^{\widehat{\rho}}/(1-\widehat{\rho})$ and directly removed from $H(k)$, through two asymptotically equivalent expressions,

$$\overline{H}(k) \equiv \overline{H}_{\widehat{\beta},\widehat{\rho}}(k) := H(k)\left(1 - \tfrac{\widehat{\beta}}{1-\widehat{\rho}}\left(\tfrac{n}{k}\right)^{\widehat{\rho}}\right), \tag{7}$$

and

$$\overline{\overline{H}}(k) \equiv \overline{\overline{H}}_{\widehat{\beta},\widehat{\rho}}(k) := H(k)\exp\left(-\tfrac{\widehat{\beta}}{1-\widehat{\rho}}\left(\tfrac{n}{k}\right)^{\widehat{\rho}}\right), \tag{8}$$

where $\widehat{\beta}$ and $\widehat{\rho}$ are adequate estimators of the second order parameter β and ρ, respectively, that will be estimated externally. Such a decision is related to the discussion in Gomes and Martins (2002) on the advantages of an external estimation of the second order parameter ρ, versus an internal estimation at the same level k.

We shall consider here particular members of the class of estimators of the second order parameter ρ proposed in Fraga Alves *et al.* (2003). We have considered the estimator of β obtained in Gomes and Martins (2002). Under adequate general conditions, any of these estimators is consistent and asymptotically normal.

Remark 1. *To estimate the second order parameters, we shall work with the level*

$$k_1 = \left[n^{0.995}\right]. \tag{9}$$

If $\rho > -49.75$ *then* $\widehat{\rho}(k_1) - \rho = O_p\left(n^{0.005\rho}\right)$*, and consequently, for any intermediate level* k*,* $(\widehat{\rho}(k_1) - \rho)\ln(n/k) = o_p(1)$*, and* $\sqrt{k}\ A(n/k)(\widehat{\rho}(k_1) - \rho)\ln(n/k) = o_p(1)$ *whenever* $\sqrt{k}\ A(n/k) \longrightarrow \lambda$*, finite. We also know that* $\widehat{\beta}(k) - \beta \sim -\beta\ln(n/k)(\widehat{\rho} - \rho)$*.*

2. Asymptotic behaviour of the estimators

Since both estimators have similar asymptotically properties we will only show results for $\overline{H}(k)$.

Theorem 1. *Under the third order framework in* (3)*, further assuming that* $A(t) = \gamma\ \beta\ t^{\rho}$*, and for levels* k *intermediate, we get,*

$$\overline{H}_{\beta,\rho}(k) \stackrel{d}{=} \gamma + \frac{\gamma}{\sqrt{k}} Z_k^{(1)} + A(n/k)\left(\frac{B(n/k)}{1-2\rho} - \frac{A(n/k)}{\gamma(1-\rho)^2}\right)(1+o_p(1)), \tag{10}$$

where $Z_k^{(1)}$ *is an asymptotically standard normal r.v. Consequently,* $\sqrt{k}\left(\overline{H}_{\beta,\rho}(k) - \gamma\right)$ *is asymptotically normal with variance equal to* γ^2*, and with a null mean value not only when* $\sqrt{k}A(n/k) \longrightarrow 0$*, but also when* $\sqrt{k}\ A(n/k) \longrightarrow \lambda$ *and* $\sqrt{k}\ A^2(n/k) \longrightarrow 0$*. If* $\sqrt{k}\ A^2(n/k) \longrightarrow \lambda_A$ *and* $\sqrt{k}\ A(n/k)B(n/k) \longrightarrow \lambda_B$*, then we have a asymptotic mean value equal to* $\lambda_B/(1-2\rho) - \lambda_A/(\gamma(1-\rho)^2)$*.*

Theorem 2. *Under the conditions of the previous Theorem, let us consider that* $\widehat{\rho} - \rho = o_p(1/\ln n)$*. Then* $\overline{H}(k)$ *is consistent for* γ*, but we can only assure that* $\sqrt{k}\ \left(\overline{H}(k) - \gamma\right)$ *is asymptotically normal with variance equal to* γ^2 *and null mean value if* $\sqrt{k}\ A(n/k) \longrightarrow \lambda$*, finite. We may still go to levels* k*:* $\sqrt{k}\ A(n/k) \to \infty$*, provided we compute* $\hat{\rho}$ *at* k_{01}*, optimal for the* ρ*-estimation, and* $k = o(k_{01})$*, as* $n \to \infty$*, or if* $\widehat{\rho}$ *is a consistent estimator of* ρ*, such that* $(\widehat{\rho} - \rho)\ln n = o_p(1/\sqrt{k}\ A(n/k))$*.*

3. Finite sample behavior of the estimators

We have here implemented multi-sample simulation experiments of size 10,000 = 1,000(runs) × 10(replicates), in order to obtain, for the Fréchet and Burr models, the distributional behaviour of the the Hill estimator and the new estimators $\overline{H}(k)$ and $\overline{\overline{H}}(k)$ in (7) and (8), respectively. We have simulated the mean value, E, the mean squared error, MSE, the optimal simulated level, $k_0^{\bullet} := \arg\min_k MSE[\bullet]$ and the relative efficiency, defined as $Reff_{\widehat{\gamma}} = \sqrt{MSE[\widehat{\gamma}(k_0^{\widehat{\gamma}})]/MSE[H(k_0^H)]}$.

Remark 2. *Note that an indicator higher than one means a better performance than the Hill estimator. Consequently, the higher these indicators are, the better the new estimators perform, comparatively to the Hill estimator.*

In Table 1 we present the $Reff$ indicator with 95% confidence interval, based on the t_9 quantile, for the Fréchet and Burr models. In Figure 1 we picture the mean values (left) and the mean squared error (right) of the Hill, $\overline{H}_{\beta,\rho}$ and $\overline{H}$ estimators for a Burr model with $(\gamma,\ \rho) = (1, -1)$.

Table 1: Reff indicator

n	$\overline{H}_{\beta,\rho}(k)$	$\overline{H}(k)$	$\overline{\overline{H}}_{\beta,\rho}(k)$	$\overline{\overline{H}}(k)$
Fréchet parent: $\gamma = 1,\ \rho = -1$				
100	1.545 ± 0.029	1.176 ± 0.011	1.478 ± 0.027	1.324 ± 0.017
500	1.709 ± 0.025	1.170 ± 0.019	1.641 ± 0.020	1.295 ± 0.023
1000	1.784 ± 0.017	1.229 ± 0.011	1.700 ± 0.018	1.346 ± 0.014
5000	1.987 ± 0.026	1.453 ± 0.014	1.883 ± 0.021	1.530 ± 0.015
10000	2.072 ± 0.032	1.593 ± 0.018	1.953 ± 0.036	1.640 ± 0.018
50000	2.328 ± 0.029	2.018 ± 0.020	2.189 ± 0.031	1.850 ± 0.011
Burr parent: $\gamma = 1,\ \rho = -0.5$				
100	2.881 ± 0.047	1.409 ± 0.016	2.075 ± 0.025	1.345 ± 0.013
500	3.382 ± 0.049	1.352 ± 0.012	2.316 ± 0.019	1.322 ± 0.011
1000	3.643 ± 0.031	1.312 ± 0.011	2.438 ± 0.023	1.290 ± 0.012
5000	4.190 ± 0.060	1.234 ± 0.006	2.745 ± 0.039	1.225 ± 0.006
10000	4.422 ± 0.076	1.215 ± 0.011	2.853 ± 0.041	1.211 ± 0.011
50000	5.109 ± 0.091	1.180 ± 0.009	3.251 ± 0.048	1.178 ± 0.009
Burr parent: $\gamma = 1,\ \rho = -1$				
100	1.902 ± 0.031	1.885 ± 0.033	1.684 ± 0.021	1.724 ± 0.028
500	2.127 ± 0.025	2.113 ± 0.025	1.850 ± 0.022	1.928 ± 0.022
1000	2.235 ± 0.026	2.275 ± 0.024	1.936 ± 0.022	2.083 ± 0.024
5000	2.525 ± 0.046	2.734 ± 0.061	2.143 ± 0.035	2.544 ± 0.055
10000	2.633 ± 0.040	3.011 ± 0.049	2.229 ± 0.038	2.784 ± 0.048
50000	2.919 ± 0.028	3.813 ± 0.061	2.450 ± 0.039	3.500 ± 0.052
Burr parent: $\gamma = 1,\ \rho = -2$				
100	1.425 ± 0.019	1.181 ± 0.014	1.371 ± 0.016	1.173 ± 0.014
500	1.543 ± 0.025	1.149 ± 0.007	1.471 ± 0.019	1.144 ± 0.008
1000	1.579 ± 0.018	1.137 ± 0.012	1.504 ± 0.015	1.134 ± 0.012
5000	1.703 ± 0.028	1.142 ± 0.012	1.619 ± 0.026	1.140 ± 0.012
10000	1.761 ± 0.034	1.141 ± 0.010	1.672 ± 0.026	1.139 ± 0.010
50000	1.890 ± 0.027	1.136 ± 0.012	1.794 ± 0.024	1.136 ± 0.011

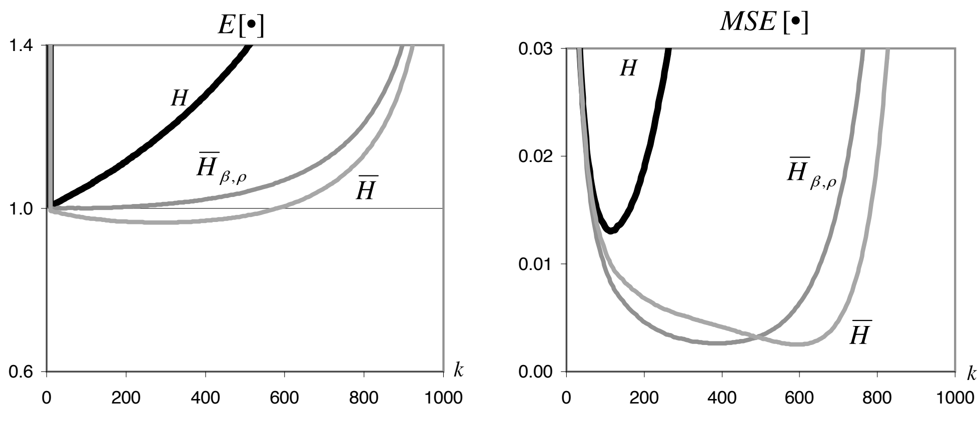

Figure 1: Underlying Burr parent with $\gamma = 1$ and $\rho = -1$.

MIXED MOMENT ESTIMATOR[6]

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Abstract: A new estimator of the extreme value index is developed. Its very simple form is born out of a limiting relation introduced by Laurens de Haan (1970). A striking feature of what we call the Mixed Moment estimator is that its variance coincides with the variance of the maximum likelihood estimator when the index is non-negative and falls off rapidly when γ becomes negative.

Key words and phrases: estimation, max-domain of attraction, regular variation theory.

1. Introduction

Let $X_1, X_2, \ldots$ be independent random variables with common unknown distribution function F. If for some constants $a_n > 0$ and $b_n \in \mathbb{R}$, we have

$$\lim_{n\to\infty} P\big\{a_n^{-1}\big(\max(X_1,\ldots,X_n) - b_n\big) \le x\big\} = G(x), \quad x \in \mathbb{R} \tag{1}$$

for some non-degenerate function G, then F is in the domain of attraction of G and (notation: $F \in \mathcal{D}(G)$) and G must be the eneralized Extreme Value distribution

$$G(x) = G_\gamma(x) := \begin{cases} \exp(-(1+\gamma x)^{-1/\gamma}), & 1+\gamma x > 0 \quad \text{if} \quad \gamma \neq 0 \\ \exp(-\exp(-x)), & x \in \mathbb{R} \quad \text{if} \quad \gamma = 0. \end{cases}$$

This distribution has three possible forms: Fréchet ($\gamma > 0$), Gumbel ($\gamma = 0$) and Weibull ($\gamma < 0$). Hence, estimation of the extreme value index γ is an impending

[6]Partially supported by FCT/POCTI and POCI/FEDER.

problem.

A necessary and sufficient condition for $F \in \mathcal{D}(G_\gamma)$ is the *first order extended regular variation property* of U:

$$\lim_{t\to\infty} \frac{U(tx)-U(t)}{a(t)} = \begin{cases} \frac{x^\gamma - 1}{\gamma}, & \gamma \neq 0 \\ \log x, & \gamma = 0, \end{cases} \tag{2}$$

for every $x > 0$ and some positive measurable function a, with U as the tail quantile function defined by the generalized inverse

$$U(t) := \Big(\frac{1}{1-F}\Big)^{\leftarrow}(t) = \inf\big\{x : F(x) \geq 1 - \frac{1}{t}\big\}.$$

According to Theorems 2.6.1 and 2.6.2 of (de Haan, 1970), a distribution function F with right endpoint $x_F := \sup\{x : F(x) < 1\}$ in $(0, \infty]$ belongs to $\mathcal{D}(G_\gamma)$ if and only if

$$\lim_{t\uparrow x_F} \frac{\big(1-F(t)\big)\int_t^{x_F}\int_y^{x_F}\big(1-F(x)\big)\frac{dx}{x^2}\,dy}{t^2\Big(\int_t^{x_F}\big(1-F(x)\big)\frac{dx}{x^2}\Big)^2} = \varphi(\gamma) := \begin{cases} 1+\gamma, & \gamma > 0 \\ \frac{1-\gamma}{1-2\gamma}, & \gamma \leq 0. \end{cases} \tag{3}$$

The left hand side of (3) can be written alternatively as

$$\frac{\big(1-F(t)\big)^{-1}\Big\{\int_t^\infty \log\frac{x}{t}\,dF(x) - \int_t^\infty\big(1-\frac{t}{x}\big)\,dF(x)\Big\}}{\Big\{\big(1-F(t)\big)^{-1}\int_t^\infty\big(1-\frac{t}{x}\big)\,dF(x)\Big\}^2} \tag{4}$$

Let $X_1, X_2, \ldots$ be i.i.d. random variables with distribution function F and $\big\{X_{i,n}\big\}_{i=1}^n$ their ascending n-th order statistics. We can build a statistic starting from (4) by replacing F with its empirical counterpart F_n and t by the order statistic $X_{n-k,n}$ with $k < n$. The result is

$$\hat\varphi_n(k) := \frac{M_{n,0}(k) - M_{n,1}(k)}{\big(M_{n,1}(k)\big)^2}, \tag{5}$$

where

$$M_{n,0}(k) := \frac{1}{k}\sum_{i=0}^{k-1}\log\Big(\frac{X_{n-i,n}}{X_{n-k,n}}\Big) \qquad \text{and} \qquad M_{n,1}(k) := 1 - \frac{1}{k}\sum_{i=0}^{k-1}\frac{X_{n-k,n}}{X_{n-i,n}}.$$

It is precisely the statistic (5) that is at the origin of what we call the **mixed moment estimator** for the extreme value index $\gamma \in \mathbb{R}$:

$$\hat\gamma_n(k) := \frac{\hat\varphi_n(k) - 1}{1 + 2\,\min\big(\hat\varphi_n(k) - 1, 0\big)}. \tag{6}$$

2. Main results

Theorem 3. *Suppose F has right endpoint $x_F = U(\infty) := \lim_{t\to\infty} U(t) > 0$ and satisfies* (2) *for some $\gamma \in \mathbb{R}$. Let $k = k_n$ be an intermediate sequence, i.e., a sequence of positive integers k_n such that $k_n \to \infty$ and $k_n = o(n)$ as $n \to \infty$. Then, the* ***mixed moment estimator*** *$\hat{\gamma}_n(k)$ introduced in* (6) *is a consistent estimator for the tail index $\gamma \in \mathbb{R}$, i.e., the following limit in probability holds:*

$$\hat{\gamma}_n(k) \underset{n\to\infty}{\overset{P}{\longrightarrow}} \gamma.$$

Apart from (2), we shall need a second order condition, specifying the inherent rate of the convergence (deHaanStadtmuller, 1996; Drees, 1998). As such, assume the existence of a function A not changing sign eventually and tending to zero as $t \to \infty$ guaranteeing that

$$\lim_{t\to\infty} \frac{\frac{U(tx)-U(t)}{a(t)} - \frac{x^\gamma - 1}{\gamma}}{A(t)} = H_{\gamma,\rho}(x) =: \frac{1}{\rho}\Big(\frac{x^{\gamma+\rho}-1}{\gamma+\rho} - \frac{x^\gamma - 1}{\gamma}\Big), \tag{7}$$

for all $x > 0$, where $\rho \leq 0$ is the second order parameter governing the rate of convergence. Under these circumstances, we say that the function U is of *second order extended regular variation* (notation: $U \in 2ERV(\gamma,\rho)$). We remark that $\lim_{t\to\infty} A(tx)/A(t) = x^\rho$, for every $x > 0$.

Theorem 4. *Given that U satisfies* (7) *with the restriction $\gamma \neq \rho$, assume $U(\infty) := \lim_{t\to\infty} U(t) > 0$. Let $k = k_n$ be an intermediate sequence. Then*

$$\sqrt{k}\,\big(\hat{\gamma}_n(k) - \gamma\big) \underset{n\to\infty}{\overset{d}{\longrightarrow}} N(\lambda\, bias,\, \sigma^2), \tag{8}$$

1. *if* (7) *holds with $(0 < \gamma < -\rho$ and $l \neq 0)$ or $\gamma = 0$ or $\gamma = -\rho$;*
2. *if* (7) *holds with $(|\gamma| > -\rho$ and $\rho \leq 0)$ or $(0 < \gamma < -\rho$ and $l = 0)$ and $k = k_n$ is such that $\sqrt{k}\,A\big(n/k\big) \underset{n\to\infty}{\longrightarrow} \lambda \in \mathbb{R}$;*
3. *if* (7) *holds with $\rho < \gamma < 0$ and $k = k_n$ is such that $\sqrt{k}\,a\big(n/k\big) \underset{n\to\infty}{\longrightarrow} \lambda$, λ finite;*

where $l := \lim_{t\to\infty}(U(t) - a(t)/\gamma)$, $l \in \mathbb{R}$ for $0 < \gamma < -\rho$, and with

$$\sigma^2 = \sigma^2(\gamma) := \begin{cases} (1+\gamma)^2, & \gamma \geq 0 \\ \left(\frac{1-\gamma}{1-2\gamma}\right)^2 \frac{6\gamma^2 - \gamma + 1}{(1-2\gamma)^5(1-3\gamma)(1-4\gamma)}, & \gamma < 0. \end{cases}$$

THE EXTREMAL INDEX OF PERIODIC RANDOM FIELDS

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Abstract: We present a family of local dependence conditions for **T**-periodic random fields under which we can calculate the extremal index from the joint distribution of a finite number $s_1 s_2$ of variables.

1. Introduction

The results about limiting crossing probabilities of nonstationary random fields (Pereira and Ferreira, 2005) can be applied to simple forms of nonstationarity like periodic random fields. However, specific local conditions for these random fields, as those for the stationary random fields, may give a better insight into the limiting clustering of high values.

In this paper we consider that $\mathbf{X} = \{X_{\mathbf{n}} : \mathbf{n} \in \mathbb{N}^m\}$ is a **T**-periodic random field on $\mathbb{N}^m$, where $\mathbb{N}$ is the set of all positive integers and $m \geq 2$.

Definition 1. *The random field* $\mathbf{X}$ *on* $\mathbb{N}^m$ *is periodic if there exist integers* $T_j \geq 1,\ j = 1, 2, \ldots, m,$ *such that, for all* $\mathbf{i} \in I_1 \times \cdots \times I_m = \left\{i_1^{(1)}, \ldots, i_{n_1}^{(1)}\right\} \times \cdots \times \left\{i_1^{(m)}, \ldots, i_{n_m}^{(m)}\right\}$ *with* $1 \leq i_1^{(j)} < \cdots < i_{n_j}^{(j)},\ j = 1, \ldots, m,$ *we have* $(X_{\mathbf{i}}, \mathbf{i} \in I_1 \times \cdots \times I_m) \stackrel{d}{=} \left(X_{\mathbf{i}+(T_1,0,\ldots,0)}, \mathbf{i} \in I_1 \times \cdots \times I_m\right), \cdots, (X_{\mathbf{i}}, \mathbf{i} \in I_1 \times \cdots \times I_m) \stackrel{d}{=} \left(X_{\mathbf{i}+(0,0,\ldots,T_m)}, \mathbf{i} \in I_1 \times \cdots \times I_m\right)$, *where* $\stackrel{d}{=}$ *denotes the equality in distribution.*

If $T_j,\ j = 1, 2, \ldots, m$, are the smallest integers satisfying the above definition we say that $\mathbf{X}$ is a $\mathbf{T} = (T_1, \ldots, T_m)$-periodic random field. When $T_j = 1, j = 1, 2, \ldots, m$, the random field $\mathbf{X}$ is stationary.

We shall consider the conditions and results for $m = 2$, since it is notationally simplest.

For a family of real levels $\{u_{\mathbf{n}}\}_{\mathbf{n}\geq\mathbf{1}}$ and a subset $\mathbf{I}$ of the rectangle of points $\mathbf{R_n} = \{1,\ldots,n_1\}\times\{1,\ldots,n_2\}$, we will denote the event $\{X_{\mathbf{i}} \leq u_{\mathbf{n}}, \mathbf{i}\in\mathbf{I}\}$ by $\{M_{\mathbf{n}}(\mathbf{I}) \leq u_{\mathbf{n}}\}$ or simply by $\{M_{\mathbf{n}} \leq u\}$ when $\mathbf{I} = \mathbf{R_n}$. For each $i = 1, 2$, we say the pair $\mathbf{I}$ and $\mathbf{J}$ is in $S_i(l)$ if the distance between $\Pi_i(\mathbf{I})$ and $\Pi_i(\mathbf{J})$ is greater or equal to l, where $\Pi_i, i = 1, 2$, denote the cartesian projections. The distance between two points $\mathbf{i} = (i_1,\ldots,i_m)$ and $\mathbf{j} = (j_1,\ldots,j_m)$ in $\mathbb{N}^m$ is defined by $|\mathbf{i}-\mathbf{j}| = \underset{1\leq l\leq m}{max} |i_l - j_l|$ and the distance between two subsets $\mathbf{I}$ and $\mathbf{J}$ of $\mathbb{N}^m$ is defined by $d(\mathbf{I},\mathbf{J}) = inf\,\{|\mathbf{i}-\mathbf{j}| : \mathbf{i}\in\mathbf{I}, \mathbf{j}\in\mathbf{J}\}$.

Restrictions on clustering of high values for stationary and $\mathbf{T}$-periodic time series have been considered in the form of $D^{(k)}(u_n), k \geq 1$, conditions introduced in Chernick *et al.* (1991) (see also Ferreira and Martins, 2003; Ferreira, 1994). In Ferreira and Pereira (2005) is introduced the condition $D^{(\mathbf{s})}(u_{\mathbf{n}})$, $\mathbf{s} = (s_1, s_2)\in\mathbb{N}^2$, which is the $D^{(k)}(u_n)$ tailored for homogeneous random fields. That condition and the coordinatewise-mixing $\Delta(u_{\mathbf{n}})-$condition introduced in Leadbetter and Rootzén (1988) enable us to compute the extremal index from the joint distribution of a finite number $s_1 s_2$ variables.

In this paper we generalize the definition of extremal index of a homogeneous random field (Choi, 2002) to a $\mathbf{T}$-periodic random field and we compute it under a local mixing condition analogous to that considered in Ferreira and Pereira (2005).

2. Computing the extremal index

Definition 2. *The $\mathbf{T}-$periodic random field $\mathbf{X}$ has extremal index $\theta_{\mathbf{X}}$ when, for each $\tau > 0$, there exists a sequence of real numbers $\left\{u_{\mathbf{n}}^{(\tau)}\right\}_{\mathbf{n}\geq\mathbf{1}}$ such that $n_1 n_2 \frac{1}{T_1 T_2}\sum_{\mathbf{i}\leq(T_1,T_2)} P\left(X_{\mathbf{i}} > u_{\mathbf{n}}^{(\tau)}\right) \underset{\mathbf{n}\to\infty}{\longrightarrow} \tau$ and $P(M_{\mathbf{n}} \leq u_{\mathbf{n}}^{(\tau)}) \underset{\mathbf{n}\to\infty}{\longrightarrow} e^{-\theta_{\mathbf{X}}\tau}$ with $\theta_{\mathbf{X}}$ independent of τ.*

In the following we describe the asymptotic behavior of the partial maximum $M_{\mathbf{n}}$, $\mathbf{n} \geq \mathbf{1}$, under the coordinatewise-mixing condition $\Delta(u_{\mathbf{n}})$ (Leadbetter and Rootzén, 1988) and a local dependence condition that generalizes the $D^{(\mathbf{s})}(u_n)$ condition (Ferreira and Pereira, 2005). The coordinatewise-mixing $\Delta(u_{\mathbf{n}})-$condition exploits the past and future separation one coordinate at a time and is defined as follows.

Definition 3. *The random field $\mathbf{X}$ satisfies the coordinatewise-mixing condition $\Delta(u_{\mathbf{n}})$ if there exist sequences of integer valued constants $\{k_{n_i}\}_{n_i\geq 1}$, $\{l_{n_i}\}_{n_i\geq 1}$, $i = 1, 2$, such that, as $\mathbf{n} = (n_1, n_2) \longrightarrow \infty$, we have*

$$(k_{n_1}, k_{n_2}) \longrightarrow \infty,\ \left(\frac{k_{n_1} l_{n_1}}{n_1}, \frac{k_{n_2} l_{n_2}}{n_2}\right) \longrightarrow \mathbf{0} \tag{2.1}$$

and $(k_{n_1}\Delta_1, k_{n_1}k_{n_2}\Delta_2) \longrightarrow \mathbf{0}$, where Δ_i are the components of the mixing coefficient *defined as follows:*

$$\Delta_1 = sup\left|P\left(M_{\mathbf{n}}(\mathbf{I}_1) \le u_{\mathbf{n}}^{(1)}, M_{\mathbf{n}}(\mathbf{I}_2) \le u_{\mathbf{n}}^{(2)}\right) - P\left(M_{\mathbf{n}}(\mathbf{I}_1) \le u_{\mathbf{n}}^{(1)}\right) P\left(M_{\mathbf{n}}(\mathbf{I}_2) \le u_{\mathbf{n}}^{(2)}\right)\right|,$$

where the supremum is taken over pairs $\mathbf{I}_1$ *and* $\mathbf{I}_2$ *in* $S_1(l_{n_1})$ *such that* $|\Pi_1(\mathbf{I}_2)| \le \frac{n_1}{k_{n_1}}$,

$$\Delta_2 = sup\left|P\left(M_{\mathbf{n}}(\mathbf{I}_1) \le u_{\mathbf{n}}^{(1)}, M_{\mathbf{n}}(\mathbf{I}_2) \le u_{\mathbf{n}}^{(2)}\right) - P\left(M_{\mathbf{n}}(\mathbf{I}_1) \le u_{\mathbf{n}}^{(1)}\right) P\left(M_{\mathbf{n}}(\mathbf{I}_2) \le u_{\mathbf{n}}^{(2)}\right)\right|,$$

where the supremum is taken over pairs of $\mathbf{I}_1$ *and* $\mathbf{I}_2$ *in* $S_2(l_{n_2})$ *such that* $\Pi_1(\mathbf{I}_1) = \Pi_1(\mathbf{I}_2)$ *and* $|\Pi_2(\mathbf{I}_2)| \le \frac{n_2}{k_{n_2}}$.

Under the $\Delta(u_{\mathbf{n}})$ $-$ condition we have the asymptotic independence for maxima over disjoint rectangles $\left\{(i-1)\left[\frac{n_1}{k_{n_1}T_1}\right]T_1+1,\ldots,i\left[\frac{n_1}{k_{n_1}T_1}\right]T_1\right\} \times \left\{(j-1)\left[\frac{n_2}{k_{n_2}T_2}\right]T_2+1,\ldots,j\left[\frac{n_2}{k_{n_2}T_2}\right]T_2\right\}$, $i = 1,\ldots,k_{n_1}, j = 1,\ldots,k_{n_2}$. For sake of simplicity we write $[\frac{\mathbf{n}}{\mathbf{kT}}]\,\mathbf{T}$ for $\left(\left[\frac{n_1}{k_{n_1}T_1}\right]T_1, \left[\frac{n_2}{k_{n_2}T_2}\right]T_2\right)$.

Proposition 1. *Suppose that the* $\mathbf{T}-$*periodic random field* $\mathbf{X}$ *verifies the coordinatewise-mixing condition* $\Delta(u_{\mathbf{n}})$*. Then*

$$P\left(M_{\mathbf{n}} \le u_{\mathbf{n}}\right) - P^{k_{n_1}k_{n_2}}\left(M_{[\frac{\mathbf{n}}{\mathbf{kT}}]\mathbf{T}} \le u_{\mathbf{n}}\right) \xrightarrow[\mathbf{n}\to\infty]{} 0.$$

In the following consider $R^*_{\mathbf{i},\mathbf{j}} = \{i_1, i_1+1,\ldots,j_1\} \times \{i_2, i_2+1,\ldots,j_2\} - \{\mathbf{i}\}$. In particular, for $\mathbf{i} = \mathbf{1}$ we write simply $R^*_{\mathbf{j}}$.

Definition 4. *The* $\mathbf{T}-$*periodic random field* $\mathbf{X}$ *satisfies the condition* $D^{(\mathbf{s})}_{\mathbf{T}}(u_{\mathbf{n}})$*, for some* $\mathbf{s} \in \mathbb{N}^2$*, if there exist sequences of integer valued constants* $\{k_{n_i}\}_{n_i\ge1}$, $\{l_{n_i}\}_{n_i\ge1}$, $i = 1, 2$, *verifying (2.1) and*

$$n_1n_2\frac{1}{T_1T_2}\sum_{\mathbf{i}\le(T_1,T_2)}\sum_{\mathbf{j}\le[\frac{\mathbf{n}}{\mathbf{kT}}]\mathbf{T}} P\left(X_{\mathbf{i}} > u_{\mathbf{n}}, M_{\mathbf{n}}\left(\mathbf{R}^*_{\mathbf{i},\mathbf{i}+\mathbf{s}-\mathbf{1}}\right) \le u_{\mathbf{n}}, X_{\mathbf{j}} > u_{\mathbf{n}}\right) \xrightarrow[\mathbf{n}\to\infty]{} 0.$$

By considering $\mathbf{s} = \mathbf{1} = (1,1)$ and $\mathbf{s} = \mathbf{2} = (2,2)$ we obtain the generalization of the local conditions considered in (Pereira and Ferreira, 2005) to $\mathbf{T}-$periodic random fields.

When $\mathbf{T} = \mathbf{1} = (1,1)$ we obtain the condition $D^{(\mathbf{s})}(u_{\mathbf{n}})$ considered in Ferreira and Pereira (2005), for homogeneous random fields. Under $D^{(\mathbf{s})}(u_{\mathbf{n}})$ they compute the extremal index of $\mathbf{X}$ from the distribution of the first s_1s_2 variables of $\mathbf{X}$. We will extend their result for periodic random fields.

Proposition 2. *Let* $\mathbf{X}$ *be a* $\mathbf{T}-$*periodic random field satisfying the conditions* $\Delta(u_{\mathbf{n}})$ *and* $D^{(\mathbf{s})}_{\mathbf{T}}(u_{\mathbf{n}})$ *for some* $\mathbf{s} \in \mathbb{N}^2$*. Then*

$$n_1n_2\frac{1}{T_1T_2}\sum_{\mathbf{i}\le(T_1,T_2)} P\left(X_{\mathbf{i}} > u_{\mathbf{n}}, M_{\mathbf{n}}\left(\mathbf{R}^*_{\mathbf{i},\mathbf{i}+\mathbf{s}-\mathbf{1}}\right) \le u_{\mathbf{n}}\right) \xrightarrow[\mathbf{n}\to\infty]{} \nu > 0$$

if and only if $P\left(M_{\mathbf{n}} \le u_{\mathbf{n}}\right) \xrightarrow[\mathbf{n}\to\infty]{} exp(-\nu)$.

Proof: Under the condition $\Delta(u_{\mathbf{n}})$ we have $P\left(M_{\mathbf{n}} \leq u_{\mathbf{n}}\right) - P^{k_{n_1}k_{n_2}}\left(M_{\left[\frac{\mathbf{n}}{\mathbf{kT}}\right]\mathbf{T}} \leq u_{\mathbf{n}}\right) = o(1)$ and so it is enough to prove that $k_{n_1}k_{n_2}P\left(M_{\left[\frac{\mathbf{n}}{\mathbf{kT}}\right]\mathbf{T}} > u_{\mathbf{n}}\right) = \nu + o(1)$. Since

$$\begin{aligned}
& k_{n_1}k_{n_2}P\left(M_{\left[\frac{\mathbf{n}}{\mathbf{kT}}\right]\mathbf{T}} > u_{\mathbf{n}}\right) \\
= \; & k_{n_1}k_{n_2}\left[\frac{n_1}{k_{n_1}T_1}\right]\left[\frac{n_2}{k_{n_2}T_2}\right]\sum_{\mathbf{i}\leq(T_1,T_2)} P\left(X_{\mathbf{i}} > u_{\mathbf{n}}, M_{\mathbf{n}}\left(\mathbf{R}^*_{\mathbf{i},\mathbf{i}+\mathbf{s}-\mathbf{1}}\right) \leq u_{\mathbf{n}}\right) \\
& -k_{n_1}k_{n_2}\left[\frac{n_1}{k_{n_1}T_1}\right]\left[\frac{n_2}{k_{n_2}T_2}\right]\sum_{\mathbf{i}\leq(T_1,T_2)} P\left(X_{\mathbf{i}} > u_{\mathbf{n}}, M_{\mathbf{n}}\left(\mathbf{R}^*_{\mathbf{i},\mathbf{i}+\mathbf{s}-\mathbf{1}}\right) \leq u_{\mathbf{n}},\right. \\
& \qquad\qquad\qquad\qquad \left. M_{\mathbf{n}}\left(\mathbf{R}^*_{\mathbf{i}+\mathbf{s}-\mathbf{1},\left[\frac{\mathbf{n}}{\mathbf{kT}}\right]\mathbf{T}}\right) > u_{\mathbf{n}}\right),
\end{aligned}$$

the result follows by applying the $D^{(\mathbf{s})}_{\mathbf{T}}(u_{\mathbf{n}})-$ condition, since

$$\begin{aligned}
& k_{n_1}k_{n_2}\left[\frac{n_1}{k_{n_1}T_1}\right]\left[\frac{n_2}{k_{n_2}T_2}\right]\sum_{\mathbf{i}\leq(T_1,T_2)} P\left(X_{\mathbf{i}} > u_{\mathbf{n}}, M_{\mathbf{n}}\left(\mathbf{R}^*_{\mathbf{i},\mathbf{i}+\mathbf{s}-\mathbf{1}}\right) \leq u_{\mathbf{n}}\right., \\
& \qquad\qquad\qquad\qquad \left. M_{\mathbf{n}}\left(\mathbf{R}^*_{\mathbf{i}+\mathbf{s}-\mathbf{1},\left[\frac{\mathbf{n}}{\mathbf{kT}}\right]\mathbf{T}}\right) > u_{\mathbf{n}}\right) \\
\leq \; & k_{n_1}k_{n_2}\left[\frac{n_1}{k_{n_1}T_1}\right]\left[\frac{n_2}{k_{n_2}T_2}\right]\sum_{\mathbf{i}\leq(T_1,T_2)} \sum_{\substack{\mathbf{j}\leq\left[\frac{\mathbf{n}}{\mathbf{kT}}\right]\mathbf{T} \\ |\mathbf{i}-\mathbf{j}|\geq max\{s_1,s_2\}}} P\left(X_{\mathbf{i}} > u_{\mathbf{n}}, M_{\mathbf{n}}\left(\mathbf{R}^*_{\mathbf{i},\mathbf{i}+\mathbf{s}-\mathbf{1}}\right) \leq u_{\mathbf{n}},\right. \\
& \qquad\qquad\qquad\qquad \left. X_{\mathbf{j}} > u_{\mathbf{n}}\right) = o(1).
\end{aligned}$$

□

As a consequence of this result we compute the extremal index as follows.

Corollary 1. *If the* $\mathbf{T}-$*periodic random field* $\mathbf{X}$ *satisfies the conditions* $\Delta(u^{(\tau)}_{\mathbf{n}})$ *and* $D^{(\mathbf{s})}_{\mathbf{T}}(u^{(\tau)}_{\mathbf{n}})$ *for each* τ *then the extremal index of* $\mathbf{X}$, θ, *exists if and only if there exists*

$$\nu = \lim_{\mathbf{n}\to\infty} n_1 n_2 \frac{1}{T_1T_2}\sum_{\mathbf{i}\leq(T_1,T_2)} P\left(X_{\mathbf{i}} > u^{(\tau)}_{\mathbf{n}}, M_{\mathbf{n}}\left(\mathbf{R}^*_{\mathbf{i},\mathbf{i}+\mathbf{s}-\mathbf{1}}\right) \leq u^{(\tau)}_{\mathbf{n}}\right)$$

and in this case it holds $\theta = \frac{\nu}{\tau}$.

Example 1. *Let* $\mathbf{Y} = \{Y_{\mathbf{n}}\}_{\mathbf{n}\geq\mathbf{1}}$ *be an i.i.d. random field and* $\{u_{\mathbf{n}}\}_{\mathbf{n}\geq\mathbf{1}}$ *a sequence of real numbers such that* $n_1n_2P\left(Y_{\mathbf{1}} > u_{\mathbf{n}}\right) \xrightarrow[\mathbf{n}\to\infty]{} \tau$.
From $\mathbf{Y}$ *we shall define a* $\mathbf{T} = (2,3)$ *periodic random field that satisfies* $D^{(2,2)}_{\mathbf{T}}(u_{\mathbf{n}})$ *condition.*
Let $\mathbf{X} = \{X_{\mathbf{n}}\}_{\mathbf{n}\geq\mathbf{1}}$ *be such that* $X_{(i,2s+1)} = Y_{(i,2s+1)}$ *for* $s \geq 0$ *and* $i \geq 1$, $X_{(i,2s)} = Y_{(i,2s)}$ *if* $\left[\frac{i}{3}\right] = 1$ *and* $X_{(i,2s)} = max\left\{Y_{(i,2s)}, Y_{(i,2s-1)}\right\}$ *if* $\left[\frac{i}{3}\right] \in \{2,3\}$.
Since $X_{\mathbf{i}}$ *and* $X_{\mathbf{j}}$ *are independent if* $|\mathbf{i}-\mathbf{j}| > 1$ *the condition* $D^{(2,2)}_{\mathbf{T}}(u_{\mathbf{n}})$ *can be easily verified and the extremal index is* $\theta = \frac{7}{9}$.

Acknowledgements. I am grateful to Prof. Helena Ferreira for her valuable suggestions.

REVISITING THE EXTREMAL INDEX: CONDITIONS AND ESTIMATION[7]

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Abstract: The main objective of statistics of extremes is the estimation of parameters of rare events. Most of statistical procedures deal with the i.i.d. setup, but in practical situations the assumption of independence is not realistic. Under adequate weak dependence conditions, the classical limiting results hold true and we get the same max-stable limit for the suitable linear normalization. It appears now a parameter, the so-called *extremal index* that is a quantity which, in an intuitive way, allows one to characterize the relationship between the dependence structure of the data and the behaviour of the exceedances over a high threshold u_n. This is a relevant parameter for any inferential procedure. Let us consider a stationary sequence $\{X_n\}_{n\geq 1}$, with marginal distribution function F. Under general local and asymptotic dependence conditions, there exists the extremal index θ, $0 \leq \theta \leq 1$. For a max-autoregressive process of order one and for a random repetition model, the conditions for the existence of extremal index are revisited and some classical estimators are compared, via simulation, to another estimator obtained averaging the estimators calculated over different high levels.

Key words and phrases: Estimators, extremal index, simulation, dependence conditions

1. Introduction

Let $\{X_n\}_{n\geq 1}$ be a stationary sequence with marginal distribution function F and $M_n = \max(X_i : i = 1, \cdots, n)$. Let $\{a_n > 0\}$ and $\{b_n\}$ be sequences of constants such that as $n \to \infty$, $P\{(M_n - b_n)/a_n \leq x\}$ converges in distribution to a non-degenerate distribution function G. Suppose that a *distributional mixing condition* $D(u_n)$ is satisfied for $u_n = a_n x + b_n$, for each real x. Then G is a member of the generalized extreme value family of distributions (Leadbetter, 1974, and Leadbetter *et al.*, 1983).

[7]Research partially supported by FCT/POCI/MAT/58876 and CMA/FCT/FEDER.

The condition $D(u_n)$ alone is sufficient to guarantee that the central classical result concerning the possible extremal types, holds also to stationary sequences.

Choosing thresholds $u_n = u_n(\tau)$ such that $n\,\overline{F}(u_n) \to \tau$ for some $\tau > 0$, Chernick (1981) shows that, under $D(u_n)$ condition, any limit (function) for $P[M_n \leq u_n(\tau)]$ must be of the form

$$P[M_n \leq u_n(\tau)] \to \exp\{-\theta\tau\} \tag{1}$$

for some θ with $0 \leq \theta \leq 1$, called the *extremal index* of the sequence $\{X_n\}$.

Leadbetter (1983) shows that if $\{X_n\}$ has a non-zero extremal index θ, then any limiting distribution for the maximum must be of the same type as if the terms were i.i.d. with the same normalizing constants if $\theta = 1$, and simply modified constants for $0 < \theta < 1$.

Let $M_n^* = \max(X_i^* : i = 1, \cdots, n)$, where $\{X_i^*\}_{n\geq 1}$ are the associated independent sequence with the same d.f. F as each of the stationary sequence. If there exists sequences of constants $\{a_n > 0\}$ and $\{b_n\}$ and a non-degenerate distribution function G_1 such that $P\left[(M_n^* - b_n)/a_n \leq x\right] \to G_1(x)$, then if $D(u_n)$ holds with $u_n = a_n x + b_n$ for each x such that $G_1(x) > 0$ and if $P\left[(M_n - b_n)/a_n \leq x\right]$ converges for some x, then Leadbetter (1983) shows that

$$P\left[(M_n - b_n)/a_n \leq x\right] \to G_2(x) := G_1(x)^{\theta}, \quad \text{as} \quad n \to \infty, \tag{2}$$

for $\theta \in [0, 1]$.

2. Dependence conditions and extremal index estimation

Extensions of classical extreme value theory to stationary sequences generally make use of two types of dependence restriction: a *weak mixing condition* restricting long range dependence, the *distributional mixing condition* $D(u_n)$ (Leadbetter, 1974) and a *local condition* restricting the clustering of high level exceedances, the *condition* $D'(u_n)$ (Leadbetter, 1974). The first type, weaker than the usual forms of dependence restrictions such as *strong mixing* (Rosenblat, 1956) is the basic assumption in the development of the theory.

Strong mixing implies $D(u_n)$ for any sequence $\{u_n\}$. In contrast to the $D(u_n)$ condition, the $D'(u_n)$ condition limits the amount of short-range dependence in the process at extreme values, see Leadbetter (1974).

The conditions $D(u_n)$ and $D'(u_n)$ ensure that the extremes of the stationary sequences $\{X_n\}$ have the same qualitative behaviour as the extremes of an associated i.i.d. sequence. The main problem is to verify conditions $D(u_n)$ and $D'(u_n)$. We will study the behaviour of two different stationary processes. These processes are substantially concerned with cases where the index θ is less than one. The case $\theta = 0$ is pathological, although not impossible, see Denzel and O'Brien (1975) or Leadbtter *et.al.* (1983). If $\theta = 1$ the exceedances of an increasing threshold occur singly in the limit, if $\theta < 1$ the exceedances tend to cluster in the limit.

Let us then consider the models:

I - Random repetition model

O' Brien (1974) considers the following model: Let $\{Y_i\}$ and $\{J_i\}$ be independent sequences of i.i.d. r.v.'s, with

$$P(Y_i \le x) = F(x) \quad \text{and} \quad P(J_i = 1) = \theta = 1 - P(J_i = 0)$$

where F is the standard exponential distribution and $\theta \in (0, 1]$. The stationary sequence $\{X_n\}$ is defined as follows:

$$X_1 = Y_1 \quad \text{and} \quad X_i = J_i Y_i + (1 - J_i) X_{i-1}, \quad \text{for} \quad i \ge 2. \tag{3}$$

It is easy to check that the marginal d.f. of $\{X_n\}$ is F.

O'Brien (1974) shows that the sequence $\{X_n\}$ is strong mixing which implies that $D(u_n)$ holds. For this process we have

$$P[M_n - \ln n \le x] \to \exp\{-\theta \exp\{-x\}\} =: G_2(x) = G_1(x)^\theta \quad \text{as} \quad n \to \infty,$$

and $G_1(x) = \exp\{-\exp\{-x\}\}$; so the extremal index is θ. When $\theta = 1$, the random repetition model is independent and the $D'(u_n)$ condition holds. On the other hand, after some calculations and for $\theta < 1$, we have

$$\left[n \sum_{j=2}^{n} P\{X_1 > u_n, X_j > u_n\} \right] = n(n-1)\,(1 - F(u_u))^2\,(1 - (1-\theta)^n),$$

which does not tend to zero as $n \to \infty$ when $u_n = x + \ln n$, i.e., the condition $D'(u_n)$ is not satisfied.

II - Max-autoregressive process (ARMAX)

de Haan and Ferreira (2006) give the following process: Let $\{Z_i\}_{i \ge 1}$ be independent standard Fréchet random variables and $\beta \in [0, 1)$. The process $\{X_i\}$ is defined by,

$$X_1 = Z_1 \quad \text{and} \quad X_i = \max\{\beta X_{i-1}, (1-\beta) Z_i\}, \quad \text{for} \quad i \ge 2. \tag{4}$$

Then $\{X_n\}$ is a stationary sequence with marginal standard Fréchet distribution. A stationary solution of the recursion is

$$X_i = \max_{j \ge 0} \{\beta^j (1-\beta) Z_{i-j}\}, \quad \text{for} \quad i \ge 1.$$

showing that the ARMAX process is a special case of the moving-maximum process. The moving maximum process satisfies the standard distributional mixing condition $D(u_n)$, see Hall *et al.* (2002).

For $0 < x < \infty$,

$$P[M_n / n \le x] \to \exp\{-(1-\beta)/x\} =: G_2(x) = G_1(x)^{1-\beta} \quad \text{as} \quad n \to \infty,$$

and $G_1(x) = \exp\{-1/x\}$, which shows that the extremal index of the ARMAX process is $\theta = 1 - \beta$ with $0 < \theta \leq 1$. When $\beta = 0$, the ARMAX process is independent and the $D'(u_n)$ condition holds. On the other hand, $D'(u_n)$ fails if $\beta > 0$ when $u_n = nx$ and $x > 0$.

In this work we compare some classical estimators of the extremal index with an estimator defined as the average of a given number of estimators calculated for different high thresholds. This study is carried out by simulation.

A SIMPLE REDUCED BIAS "MAXIMUM LILELIHOOD" TAIL INDEX ESTIMATOR[8]

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Abstract: We shall here deal with bias reduction techniques for heavy tails, trying to improve the performance of classical tail index estimators. Recently, new interesting classes of reduced bias' γ-estimators have been introduced in the literature. In those classes, the second order parameters in the bias are estimated at a level k_1 of a larger order than that of the level k at which we compute the tail index estimators. Doing this, it is possible to keep the asymptotic variance of the new estimators equal to the asymptotic variance of the Hill estimator. We now introduce a similar class of γ-estimators. Asymptotic and finite sample distributional properties of those estimators are obtained. An illustration of the behaviour of the new estimators, for a set of real data in the field of insurance, is also provided.

1. Introduction.

Heavy-tailed models have revealed to be quite useful in the most diversified areas, ranging from insurance, economics and finance till telecommunications and biostatistics. In a context of *Extreme Value Theory*, a model F is said to have a heavy right tail whenever the tail function $\overline{F} := 1 - F \in RV_{-1/\gamma}$, where RV_α stands for the class of regularly varying functions with index α, i.e., positive measurable functions g such that $\lim_{t\to\infty} g(tx)/g(t) = x^\alpha$, for all $x > 0$. Equivalently, denoting $U(t) := F^{\leftarrow}(1 - 1/t) = \inf\{x : F(x) \geq 1 - 1/t\}$, we have $U \in RV_\gamma$.

[8]Partially supported by FCT/POCTI and POCI/FEDER.

We are going to base inference on the k top order statistics (o.s.), and as usual in semi-parametric estimation of parameters of extreme or even rare events, we shall assume that k is an *intermediate* sequence of integers, i.e., $k = k_n \to \infty$, $k = o(n)$, as $n \to \infty$. For heavy tails, the classical tail index estimator is the Hill estimator $\hat{\gamma} = \hat{\gamma}(k) \equiv H(k)$ (Hill, 1975).

Second order refinements. In order to derive the asymptotic non-degenerate behaviour of semi-parametric estimators of extreme events' parameters, we need more than the first order condition. A convenient condition is the following second order condition, which guarantees that $\lim_{t\to\infty} (\ln U(tx) - \ln U(t) - \gamma \ln x)/A(t) = (x^\rho - 1)/\rho$ for all $x > 0$, being $\rho \le 0$ a second order parameter. The previous limit function is necessarily of this given form, and $|A| \in RV_\rho$ (Geluk and de Haan, 1987). We shall assume to be working in a wide sub-class of Hall's class of models (Hall and Welsh, 1985), where there exist $\gamma > 0$, $\rho < 0$, $C > 0$ and β, $\beta' \neq 0$ such that $U(t) = Ct^\gamma \left(1 + \gamma\, \beta\, t^\rho/\rho + \beta' t^{2\rho} + o(t^{2\rho})\right)$, as $t \to \infty$. Typical heavy-tailed models, like the Fréchet(γ), the Generalized Pareto and the Student-t_ν model belong to such a class. Then, this second order condition holds, with $A(t) = \gamma\, \beta\, t^\rho$, $\rho < 0$.

Under the second order framework, and for intermediate k, we may guarantee the asymptotic normality of the Hill estimator $H(k)$, for an adequate k. Indeed, we may write (de Haan and Peng, 1998), $H(k) := \sum_{i=1}^k U_i/k \overset{d}{=} \gamma + \gamma\, P_k/\sqrt{k} + A(n/k)(1 + o_p(1))/(1-\rho)$, P_k asymptotically standard normal. Consequently, if we choose a level k such that $\sqrt{k}\; A(n/k) \to \lambda \neq 0$, finite, as $n \to \infty$, $\sqrt{k}\,(H(k) - \gamma)$ is asymptotically normal, with a non-null bias given by $\lambda/(1-\rho)$ and a variance equal to γ^2. Most of the times, this type of estimates exhibit a strong bias for moderate k and sample paths with very short stability regions around the target value γ. This problem has been recently addressed by several authors, among whom we mention Peng (1988), Beirlant *et al.* (1999), Feuerverger and Hall (1999) and Gomes *et al.* (2000). All these researchers consider the possibility of dealing with the bias term in an appropriate way, building different new estimators, $\hat{\gamma}_{_R}(k)$ say, the so-called second order reduced bias' estimators. Then, for k intermediate, and under the second order framework, we may write, with $P_k^{^R}$ an asymptotically standard normal r.v., $\hat{\gamma}_{_R}(k) \overset{d}{=} \gamma + \sigma_{_R}\, P_k^{^R}/\sqrt{k} + o_p(A(n/k))$, where $\sigma_{_R} > 0$. Consequently, $\sqrt{k}\,(\hat{\gamma}_{_R}(k) - \gamma)$ is asymptotically normal with a null mean value even when $\sqrt{k}\; A(n/k) \to \lambda$, finite and possibly non-null, as $n \to \infty$. Indeed, under mild additional conditions, we may even guarantee the asymptotic normality of these estimators for levels k such that $\sqrt{k}\; A(n/k) \to \infty$, as $n \to \infty$ (see, for instance, Gomes *et al.*, 2004a).

Among the above mentioned classes of second order reduced bias tail index estimators, the class of "maximum likelihood" estimators in Gomes and Martins (2002) will be the one we consider and introduce in section 2, together with two possible alternatives, studied in or suggested by recent papers on reduced bias' tail index estimation (Gomes *et al.*, 2004b; Caeiro *et al.*, 2004; Gomes and Pestana, 2004; Gomes *et al.*, 2005). In section 3, we provide an illustration of the behaviour of these estimators for a set of real data in the field of insurance, the automobile claims from an European car insurance portfolio.

2. Second order reduced bias' tail index estimation

Among the "asymptotically unbiased" or second order reduced bias' tail index estimators with the minimal asymptotic variance in Drees class of functionals (Drees, 1998), we shall select here the "maximum likelihood" estimator introduced in Gomes and Martins (2002), with the functional expression $M(k;\hat{\rho}) := D_0(k) - D_{\hat{\rho}}(k) \times T_{\hat{\rho}}(k)$, with $T_{\hat{\rho}}(k) := (d_{\hat{\rho}}(k) \times D_0(k) - D_{\hat{\rho}}(k))/(d_{\hat{\rho}}(k) \times D_{\hat{\rho}}(k) - D_{2\hat{\rho}}(k))$, where, for $\alpha \le 0$, $d_\alpha(k) := \sum_{i=1}^k (i/k)^{-\alpha}/k$, $D_\alpha(k) := \sum_{i=1}^k (i/k)^{-\alpha}\ U_i/k$. Note that $D_0(k)$ is the Hill estimator $H(k) := \sum_{i=1}^k U_i/k$. This estimator depends thus on the estimation of the second order parameter ρ, and we suggest the use of the class of estimators in Fraga Alves *et al.* (2003).

Remark 3. *An estimator of this same type, but implicit, was first introduced in* Beirlant *et al.* (1999) *and* Feuerverger *and* Hall (1999)*, and has been studied, with a misspecification of* ρ *at* $\rho = -1$*, in* Gomes *and* Martins (2004).

Estimation of second order parameters. We shall denote generically $\hat{\rho}_\tau$ any of the estimators $\hat{\rho}_\tau(k)$ introduced in Fraga Alves *al.* (2003), computed at the level $k_1 = [n^{0.995}]$. For the estimation of β, we shall consider the β-estimator, $\hat{\beta}(k;\ \hat{\rho}) := (k/n)^{\hat{\rho}} T_{\hat{\rho}}(k)$, provided in Gomes and Martins (2002). We use the simple notation $\hat{\beta}_\tau$ to denote $\hat{\beta}(k_1; \hat{\rho}(k_1;\tau))$. We use a subscript j, writing $\hat{\rho}_j$ and $\hat{\beta}_j = \hat{\beta}_{\hat{\rho}_j}$ when we want to specify that $\tau \equiv j$, $j = 0$ or 1.

Remark 4. *For models in Hall's class, when we consider* $\hat{\beta} \equiv \hat{\beta}(k_1;\ \hat{\rho})$*, with* $\hat{\rho}$ *any of the above mentioned estimators, computed also at the same level* k_1*, we get, under mild restrictions,* $\hat{\rho} - \rho = o_p(1/\ln n)$*.* $\{\hat{\beta} - \beta\}$ *is also of smaller order than* $1/\ln n$*.*

Alternative tail index estimators. The estimator $M(k;\hat{\rho})$ attains the minimal asymptotic variance in Drees' class of functionals (Drees, 1998), given by $(\gamma(1-\rho)/\rho)^2$. Notice now that we may write, $M(k;\hat{\rho}) := D_0(k) - \hat{\beta}(k;\ \hat{\rho})\ (n/k))^{\hat{\rho}} D_{\hat{\rho}}(k)$. In a spirit similar to the one in Gomes *et al.* (2004b) and Caeiro *et al.* (2005), Gomes *et al.* (2005) have considered, for a suitable ρ-estimator, $\hat{\rho}$, the β-estimator, $\hat{\beta}(k;\hat{\rho})$, but computed at an intermediate higher level k_1, like the one mentioned before. The estimate $\hat{\beta} := \hat{\beta}(k_1;\hat{\rho})$ is then incorporated in $M(k;\hat{\rho})$, and it is there suggested the consideration of the estimator, $\overline{M}(k;\hat{\beta},\hat{\rho}) := D_0(k) - \hat{\beta}\ \ (n/k))^{\hat{\rho}}\ \ D_{\hat{\rho}}(k)$. Note that $M(k;\hat{\rho}) = \overline{M}(k;\hat{\beta}(k;\hat{\rho}),\hat{\rho})$.

Here, apart from $\overline{M}(k;\hat{\beta},\hat{\rho})$, we shall also consider, now in a spirit similar to the one used in Gomes and Pestana (2004), the computation of $D_{\hat{\rho}}(k)$, a consistent estimator of $\gamma/(1-\rho)$, at its estimated optimal level. Indeed, from Gomes and Martins (2004), we know that, if the second order condition holds, if $k = k_n$ is a sequence of intermediate positive integers, then the asymptotic optimal level for $D_\rho(k)$ is provided by $k_0^{(D)}(n) = \left((1-2\ \rho)\ n^{-2\ \rho}/(-2\ \rho\ \beta^2)\right)^{1/(1-2\ \rho)}$. With the obvious notation $\hat{k}_0^{(D)}$, we thus define $\overline{\overline{M}}(k;\hat{\beta},\hat{\rho}) := D_0(k) - \hat{\beta}\ \ (n/k))^{\hat{\rho}}\ \ D_{\hat{\rho}}(\hat{k}_0^{(D)})$, again with ρ, $\hat{\rho} = \hat{\rho}(k_1;\tau)$, and $\hat{\beta} := \hat{\beta}(k_1;\hat{\rho})$. We may state:

Proposition 3. *If the second order condition holds, if $k = k_n$ is a sequence of intermediate positive integers, and if $\sqrt{k}\, A(n/k) \to \lambda$, finite and non necessarily null, as $n \to \infty$, then, with $\widetilde{M}$ denoting either $\overline{M}$ or $\overline{\overline{M}}$, $\sqrt{k}\left(\widetilde{M}_{\beta,\rho}(k) - \gamma\right) \longrightarrow^{d}_{n\to\infty} Normal\left(0,\ \gamma^2\right)$. This same limiting behaviour holds true if we replace $\widetilde{M}_{\beta,\ \rho}$ by $\widetilde{M}_{\hat\beta,\ \hat\rho}$, provided that $(\hat\rho - \rho)\ln n = o_p(1)$ for every k-value on which we base the tail index estimation, and we choose $\hat\beta := \hat\beta(k_1; \hat\rho)$.*

A case-study. We shall consider an illustration of the performance of the above mentioned estimators, through the analysis of automobile claim amounts exceeding 1,200.000 Euro over the period 1988-2001 and gathered from several European insurance companies co-operating with the same re-insurer (Secura Belgian Re). This data set was studied both in Beirlant *et al.* (2004) and Vandewalle and Beirlant (2005). In Figure 2, working with the $n = 371$ automobile claims exceeding 1,200.000 Euro, we present the sample path of the $\hat\rho_\tau$ (*left*), $\hat\beta_\tau \equiv \hat\beta_{\hat\rho_\tau}$ (*center*) estimators, as function of k, for $\tau = 0$ and $\tau = 1$, together with the sample paths of estimates of the tail index γ, provided by the Hill estimator, H, the $\overline{M}$-estimator and the $\overline{\overline{M}}$ estimator (*right*).

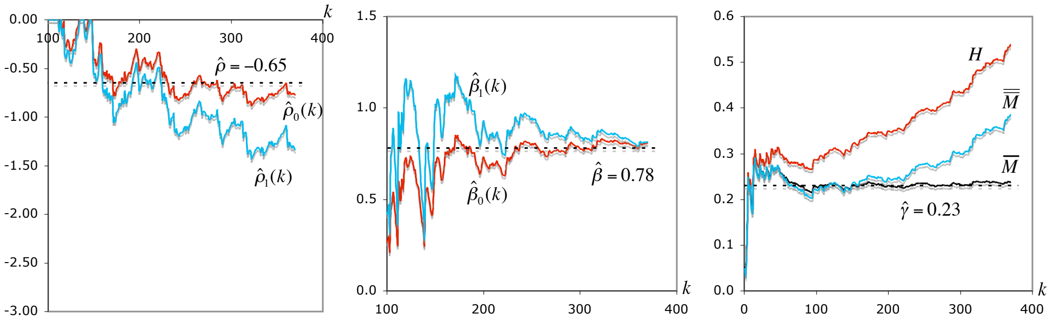

Figure 2: Estimates of the tail index γ (*left*) and of the quantile χ_p, associated to $p = 0.001$ (*right*) for the Secura Belgian Re data.

Remark 5. *Note that the sample paths of the ρ-estimates associated to $\tau = 0$ and $\tau = 1$ lead us to choose, on the basis of any stability criterion for large k, the estimate associated to $\tau = 0$. From previous experience with this type of estimates, we conclude that the underlying ρ-value is larger than or equal to -1, and the consideration of $\tau = 0$ is then advisable. The estimate of ρ is in this case $\hat\rho_0 = -0.65$, obtained at the level $k_1 = 360$. The associated β-estimator is $\hat\beta_0 = 0.78$.*

Remark 6. *The relative behaviour of the sample paths of $\overline{M}$ and $\overline{\overline{M}}$ estimators immediately suggest a possible choice of the threshold, provided by $k_{01} :=$ $\max\left\{k : \overline{M}(k) - \overline{\overline{M}}(k) \le \epsilon_{crit}\right\}$. Such a choice led to $\hat\gamma = 0.23$.*

AN EMPIRICAL TAIL INDEX and *VaR* ANALYSIS[9]

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Abstract: In many areas of application, like *statistical quality control*, *insurance* and *finance*, a typical requirement is to estimate a *high quantile*, i.e., the *Value at Risk* at a level p (VaR_p), high enough, so that the chance of an exceedance of that value is equal to p, small. In this paper we provide an empirical data analysis of log-returns associated to five sets of financial data, collected over the same period, through the use of reduced bias tail index and associated high quantile estimators. These tail index estimators depend on two second order parameters, and in order to achieve a reduction in bias without any inflation of the asymptotic variance, the second order parameters in the bias are both estimated at a higher level than that used for the estimation of the tail index.

1. Introduction and preliminaries

An important situation in risk management is the risk of a big loss that occurs very rarely. The risk is generally expressed as the *Value at Risk* (VaR_p), that is, the size of the loss occurred with fixed small probability p. In other words, we are dealing with a *high quantile* $\chi_{1-p} := F^{\leftarrow}(1-p)$ of a probability distribution function (d.f.) F, with $F^{\leftarrow}(y) = \inf\{x : F(x) \geq y\}$, the generalized inverse function of F. Let us denote $U(t)$ the inverse function of $1/(1-F)$. Then, for small p, we want to estimate the parameter $VaR_p = U(1/p),\ p = p_n \to 0,\ n\,p_n \leq 1$. Since in financial applications one encounters generally heavy tails, we shall assume that the d.f. underlying the data satisfies $1 - F(x) \ \sim\ c\, x^{-1/\gamma}$, as $x \to \infty$, for some $c > 0$.

For the semi-parametric estimation of a *high quantile* (i.e., the *Value-at-Risk*), Weissman (1978) proposed the statistic $Q_{\hat\gamma}^{(p)}(k) \ := \ X_{n-k+1:n}(k/(np))^{\hat\gamma}$, where $X_{n-k+1:n}$ is the k-th top order statistic (o.s.), $\hat\gamma$ any consistent estimator for γ and Q stands for quantile function. As usual in semi-parametric estimation of parameters of extreme events, we need that $k = k_n \to \infty,\ k \in [1, n),\ k = o(n),$ as $n \to \infty$. We then say that k is an *intermediate* sequence of integers. For heavy tails, the classical

[9]Partially supported by FCT / POCTI / FEDER.

tail index estimator, usually the one which is used for a semi-parametric quantile estimation, is the Hill estimator $\hat{\gamma} = \hat{\gamma}(k) =: H(k)$ (Hill, 1975), with the functional expression, $H(k) := \frac{1}{k}\sum_{i=1}^{k} U_i,\ U_i := i(\ln X_{n-i+1:n} - \ln X_{n-i:n}),\ 1 \le i \le k$.

Since $Q_{H}^{(p)}(k)$ is skewed (Gomes and Pestana, 2005), we shall here work with the ln-*VaR* estimator, $\ln Q_{\hat{\gamma}}^{(p)}(k) := \ln X_{n-k+1:n} + \hat{\gamma}(k)\ \ln(k/(np))$, being $\hat{\gamma}(k)$ any consistent estimator of γ. We shall thus work with the so-called classical ln-VaR_p estimator, denoted $\ln Q_{H}^{(p)}(k)$.

In order to be able to study the asymptotic behavior of $\ln Q_{H}^{(p)}(k)$, as well as of alternative $\ln VaR_p$-estimators, it is useful to impose a second order expansion on $1 - F$ or on U. Here we shall assume that we are working in Hall's class of models (Hall and Welsh, 1985), where $U(t) = Ct^{\gamma}(1 + \gamma\ \beta\ t^{\rho}/\rho + o(t^{\rho}))$, as $t \to \infty$, with $C,\ \gamma > 0,\ \rho < 0$ and β non-zero. We shall further use the notation $A(t) := \gamma\ \beta\ t^{\rho}$. From the results of de Haan and Peng (1998), it follows that in Hall's class of models, $\sqrt{k}\ (H(k) - \gamma) \ \stackrel{d}{=}\ Normal(0,\ \gamma^2) + \sqrt{k}\ (\gamma\ \beta\ (n/k)^{\rho}/(1-\rho))(1 + o_p(1))$. Under the above mentioned conditions, the asymptotic behavior of $\ln Q_{H}^{(p)}(k)$ is also well-known, i.e., $\sqrt{k}(\ln Q_{H}^{(p)}(k) - \ln VaR_p)/\ln(k/(np)) \ \longrightarrow^{d}_{n\to\infty}\ Normal(\lambda/(1-\rho),\ \gamma^2)$, provided the intermediate sequence $k = k_n$ satisfies $\lim_{n\to\infty} \sqrt{k}\ A(n/k) = \lambda \in \mathbb{R}$, finite.

We shall work with the reduced bias tail index estimators in Caeiro *et al.* (2005) and Gomes *et al.* (2005). The estimator in Caeiro *et al.* (2005) has the functional expression, $\overline{H}_{\hat{\beta},\ \hat{\rho}}(k) := H(k)(1 - \hat{\beta}(n/k))^{\hat{\rho}}/(1-\hat{\rho}))$, where $(\hat{\beta},\ \hat{\rho})$ is an adequate consistent estimator of $(\beta,\ \rho)$, with both $\hat{\beta}$ and $\hat{\rho}$ based on a number of top o.s. k_1 of a larger order than the number of top o.s. k used for the tail index estimation. This is thus a bias-corrected Hill estimator. The class of estimators in Gomes *et al.* (2005) is similar to the previous one, but it has been inspired in the tail index estimator provided in Gomes and Martins (2002), i.e., it is based on the maximum likelihood approach associated to the scaled log-spacings U_i. With the same notation as before, we shall work also with the tail index estimator $ML_{\hat{\beta},\ \hat{\rho}}(k) := H(k) - \hat{\beta}\ (n/k)^{\hat{\rho}}\ D_k(1-\hat{\rho})$, where $D_k(\alpha) = \frac{1}{k}\sum_{i=1}^{k}(i/k))^{\alpha-1}U_i$. This is another example of a bias-corrected Hill estimator, where we are using $D_k(1-\hat{\rho})$ as an estimator of $\gamma/(1-\rho)$. We shall here consider the alternative ln-VaR_p estimators, $\ln Q_{\overline{H}}^{(p)}$ and $\ln Q_{ML}^{(p)}$. Under the same conditions as before, if we work with $\overline{H}$ or ML, generally denoted T, we get $\sqrt{k}(\ln Q_{T}^{(p)}(k) - \ln VaR_p)/\ln(k/(np)) \ \longrightarrow^{d}_{n\to\infty}\ Normal(0,\ \gamma^2)$, even when $\lambda \ne 0$ (see Gomes and Pestana (2005), for a proof related to the use of $\overline{H}$).

Asymptotic confidence intervals (CI's) for γ based on the Hill estimator. Since, $\sqrt{k}\ \{H(k)/\gamma - 1 - \beta(n/k)^{\rho}/(1-\rho)\} \approx Normal(0,1)$, whenever $\sqrt{k}\ (n/k)^{\rho} \to \lambda$, finite, we may get approximate 95% CI's for γ, given by $(H(k)/(1 + \beta(n/k)^{\rho}/(1-\rho) + 1.96/\sqrt{k}),\ H(k)/(1 + \beta(n/k)^{\rho}/(1-\rho) - 1.96/\sqrt{k})) =: (LCL_{H}(k),\ UCL_{H}(k))$.

Asymptotic CI's for γ based on second order reduced bias tail index estimation. We may state the following:

Proposition 4 (Caeiro *et al.*, 2005; Gomes *et al.*, 2005)**.** *For models in Hall's class, let us assume that k is intermediate and that $\sqrt{k}\ A(n/k) \to \lambda$, finite and* *non necessarily null, as* $n \to \infty$. *Then, with* T *denoting either* $\overline{H}$ *or* ML, $\sqrt{k}\ (T_{\beta,\rho}(k) - \gamma) \ \stackrel{d}{\longrightarrow}_{n\to\infty}\ Normal(0,\ \gamma^2)$. *This same limiting behaviour holds if we replace* $T_{\beta,\rho}$ *by* $T_{\hat\beta,\hat\rho}$, *provided we consider, for instance,* $\hat\rho_\tau(k)$ *and* $\hat\beta_{\hat\rho}(k)$, *the estimators in Fraga Alves et al.* (2003) *and Gomes and Martins* (2002), *respectively, and* $\hat\rho = \hat\rho_\tau(k_1)$, $\hat\beta := \hat\beta_{\hat\rho}(k_1)$, *for any* k_1 *such that* $\hat\rho - \rho = o_p(1/\ln n)$.

On the basis of the statistics $\overline{H}$ and ML, and for levels k such that $\sqrt{k}\ (n/k)^\rho \to \lambda$, possibly different from zero, we get a 95% approximate CI for γ, given by $(LCL_T(k), UCL_T(k)) = (T(k)/(1 + 1.96/\sqrt{k}),\ T(k)/(1 - 1.96/\sqrt{k}))$, again with T denoting any of the estimators $\overline{H}$ and ML.

An adaptive choice of the level for reduced bias estimators. Here, we have decided to use a heuristic adaptive choice of k, similar to the one suggested in Gomes and Pestana (2005). We do not have simple techniques to estimate the optimal threshold of second order reduced bias' estimators, but we know that such a k should be larger than $k_0^H = ((1-\rho)\ n^{-\rho}/(\beta\ \sqrt{-2\rho}))^{2/(1-2\rho)}$, the optimal level for the Hill estimator. If we plot the 95% approximate confidence region, as a function of k, the Hill estimate is sooner or later going to cross it. We shall use such a k-value for the tail index estimation through the second order reduced bias' tail index estimator $\overline{H}(k)$ and $ML(k)$, as well as for the associated $\ln VaR$ estimation. Such a crossing level is solution of the equation $|\beta|\ (n/k)^\rho/(1-\rho) = 1.96/\sqrt{k}$, i.e., we get $k_{01} \equiv k_{01}(n;\beta,\rho) = (1.96(1-\rho)n^{-\rho}/|\beta|)^{2/(1-2\rho)}$. Levels of this type are still levels such that $\sqrt{k}\ A(n/k) \to \lambda$, finite, and are not yet optimal for the tail index estimation through second order reduced bias' tail index estimators. However, with the tail index estimators $\overline{H}$ and ML, that behave better than the Hill for all k, we are always safe.

Asymptotic CI's for ln- VaR_p**.** We may also easily estimate, now numerically, the "optimal" threshold for the ln-VaR estimation through $\ln Q_H$, i.e., the level $k_0^{Q_H} \equiv k_0^{Q_H}(n,p;\beta,\rho) := \arg\min_k\{\ln^2(k/(np))(1/k + \beta^2(n/k)^{2\rho}/(1-\rho)^2)\}$. We may also find approximate CI's for $\ln VaR_p$ on the basis of $\ln Q_H^{(p)}(k)$ and for any level k such that $\sqrt{k}\ A(n/k) \to \lambda$, finite. We get a 95% CI, dependent on γ, and with bounds $\ln Q_H(k) \mp \gamma \ln(k/(np))(1.96/\sqrt{k} \pm \beta(n/k)^\rho/(1-\rho))$. In order to have the guarantee of a coverage probability at least equal to 95%, we shall work with $(LCL_{Q_T},\ UCL_{Q_T}) = (\ln Q_T - 1.96 \times UCL_T \ln(k/(np))/\sqrt{k}, \ln Q_T + 1.96 \times UCL_T \ln(k/(np))/\sqrt{k})$, with T denoting either $\overline{H}$ or ML.

2. An algorithm for the empirical VaR analysis

We shall provide an empirical data analysis of log-returns associated to five sets of financial data, collected over the same period: from January 4, 1999, until November 17, 2005. Those sets of data are: the Euro-USA Dolar (EUSD), the EU-UK Pound (EGBP) daily exchange rates and the daily close values of Dow Jones Industrial Average In (DJI), Microsoft Corp. (MSFT) and International Business Machines Corp. (IBM) stocks. We propose the following **Algorithm**:

1. Given a sample $(X_1, X_2, \cdots, X_n)$, plot, for $\tau = 0,\ 1$, the estimates $\hat{\rho}_\tau(k)$ in Fraga Alves *et al.* (2003);
2. Consider $\{\hat{\rho}_\tau(k)\}_{k\in\mathcal{K}}$, for $k \in \mathcal{K} = ([n^{0.995}],\ [n^{.999}])$, compute χ_τ, their median. Choose the *tuning parameter* $\tau^* := \arg\min_\tau \sum_{k\in\mathcal{K}}(\hat{\rho}_\tau(k) - \chi_\tau)^2$;
3. Work with $\hat{\rho}_{\tau^*} = \hat{\rho}_{\tau^*}(k_1)$, $\hat{\beta}_{\tau^*} := \hat{\beta}_{\hat{\rho}_{\tau^*}}(k_1)$, $k_1 = \min(n-1, 2\ n^{0.995}/\ln(\ln n))$, being $\hat{\beta}_{\hat{\rho}}(k)$ the β-estimator in Gomes and Martins (2002);
4. Plot $H(k)$, and adaptively consider $H(\hat{k}_0^H)$, together with the 95% approximate CI, $(LCL_H(\hat{k}_0^H),\ UCL_H(\hat{k}_0^H))$.
5. Plot the reduced bias estimates $\overline{H}_{\tau^*}(k)$ and $ML_{\tau^*}(k)$, associated to $(\hat{\rho}_{\tau^*},\ \hat{\beta}_{\tau^*})$ obtained in step 3. Adaptively consider $\overline{H}(\hat{k}_{01})$ and $ML(\hat{k}_{01})$, together with the 95% CI's $(LCL_T(\hat{k}_{01}),\ UCL_T(\hat{k}_{01}))$, with T standing either for $\overline{H}_{\tau^*}$ or ML_{τ^*}.
6. Choose the tail index estimate providing the smallest 95% confidence size.
7. Plot the ln-*VaR* estimates, $\ln Q_H(k)$, and adaptively consider $\ln Q_H(\hat{k}_0^{Q_H})$. Consider also the approximate CI, $(LCL_{Q_H}(\hat{k}_0^{Q_H}),\ UCL_{Q_H}(\hat{k}_0^{Q_H}))$;
8. Plot the reduced bias ln-VaR_p estimates, $\ln Q^{(p)}_{T_{\tau^*}}(k)$, with T replaced by $\overline{H}$ and ML, and associated to the estimates $(\hat{\rho}_{\tau^*},\ \hat{\beta}_{\tau^*})$ obtained in step 3. Adaptively consider $\ln Q^{(p)}_{T_{\tau^*}}(\hat{k}_{01})$, together with the associated CI;
9. Choose the ln-VaR_p estimate providing the smallest confidence size.

Remark 7. *Theoreticaly, the chosen estimate in step 6. should be* $\overline{H}_0(\hat{k}_{01})$ *or* $ML_0(\hat{k}_{01})$*. Indeed, for all data sets considered, we have been led to* $ML_0(\hat{k}_{01})$*.*

3. Data analysis

In the following table, where we use the notation n_0 for the number of positive log-returns in any of the series, we summarize the tail characteristics of the five data sets.

	DJI	EGBP	EUSD	IBM	MSFT
n_0	867	835	867	881	882
$(\hat{\rho}_0,\ \hat{\beta}_0)$	$(-0.72, 1.02)$	$(-0.72, 1.02)$	$(-0.70, 1.03)$	$(-0.74, 1.02)$	$(-0.72, 1.02)$
$(\hat{k}_0^H,\ H(\hat{k}_0^H))$ $(LCL_H,\ UCL_H)$	(73, 0.270) (0.203,0.311)	(71, 0.303) (0.227, 0.350)	(68, 0.269) (0.200, 0.311)	(76, 0.386) (0.292, 0.445)	(72, 0.392) (0.294, 0.452)
$(\hat{k}_{01}, \overline{H}_0(\hat{k}_{01}))$ $(LCL_{\overline{H}},\ UCL_{\overline{H}})$	(146, 0.302) (0.260, 0.361)	(142, 0.297) (0.255, 0.356)	(137, 0.253) (0.216, 0.305)	(153, 0.376) (0.324, 0.447)	(145, 0.317) (0.272, 0.379)
$ML_0(\hat{k}_{01})$ $(LCL_{ML},\ UCL_{ML})$	$\underline{0.296}$ (0.254, 0.354)	$\underline{0.293}$ (0.251, 0.350)	$\underline{0.249}$ (0.213, 0.300)	$\underline{0.370}$ (0.319, 0.440)	$\underline{0.316}$ (0.271, 0.378)
$(\hat{k}_0^{Q_H},\ Q_H(\hat{k}_0^{Q_H}))$ (LCL_{Q_H}, UCL_{Q_H})	(48, 1.899) (1.403, 2.192)	(47, 1.031) (0.546, 1.317)	(44, 1.275) (0.805, 1.552)	(52, 2.972) (2.302, 3.365)	(48, 2.738) (2.204, 3.051)
$Q_{\overline{H}}(\hat{k}_{01})$ $(LCL_{Q_{\overline{H}}}, UCL_{Q_{\overline{H}}})$	1.917 (1.617, 2.218)	1.066 (0.764, 1.367)	1.206 (0.947, 1.464)	2.858 (2.493, 3.224)	2.689 (2.374, 3.004)
$Q_{ML}(\hat{k}_{01})$ $(LCL_{Q_{ML}}, UCL_{Q_{\overline{H}}})$	1.887 (1.593, 2.182)	1.040 (0.744, 1.336)	1.185 (0.931, 1.440)	2.830 (2.470, 3.190)	2.684 (2.370, 2.998)

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