Asymptotic Comparison of Alternative Estimators of a Shape Second-Order Parameter[*]

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Abstract

Under a third-order framework, and for heavy right tails, or equivalently a positive extreme value index, we proceed to an asymptotic comparison of three alternative estimators of the most common shape second-order parameter associated to the type of models under consideration.

Keywords. Asymptotic behaviour; extreme value theory; semi-parametric estimation; shape second-order parameter; third-order frameworks.

1 Introduction and scope of the paper

In *statistics of extremes*, a model F is said to have a *heavy right tail* whenever the right tail-function $\overline{F} := 1 - F$ is a *regularly varying function* at infinity, with a negative index of regular variation denoted by $-1/\gamma$, $\gamma > 0$, or equivalently, and using the notation $F^{\leftarrow}(x) := \inf\{y : F(y) \geq x\}$ for the generalised inverse function of F, the

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(reciprocal) quantile function $U(t) := F^{\leftarrow}(1-1/t)$, $t \geq 1$, is of regular variation with an index γ. Recall that this means that for all $x > 0$,

$$\lim_{t\to\infty} \overline{F}(tx)/\overline{F}(t) = x^{-1/\gamma} \qquad \Longleftrightarrow \qquad \lim_{t\to\infty} U(tx)/U(t) = x^{\gamma} \tag{1}$$

(see Bingham *et al.*, 1987, for details on regular variation).

The second-order parameter ρ (≤ 0) measures the rate of convergence in the first-order condition in (1), and it is the non-positive parameter in the limiting relation,

$$\lim_{t\to\infty} \frac{\ln U(tx) - \ln U(t) - \gamma \ln x}{A(t)} = \begin{cases} \frac{x^{\rho}-1}{\rho} & \text{if} \quad \rho < 0 \\ \ln x & \text{if} \quad \rho = 0 \end{cases} =: \psi_{\rho}(x), \tag{2}$$

which we assume to hold for all $x > 0$, and where $|A|$ must be of regular variation with an index ρ (Geluk and de Haan, 1987). Under a framework of a heavy right tail-function $\overline{F} = 1 - F$, with F the model underlying the available data, we proceed to an asymptotic comparison of three alternative estimators of the shape second-order parameter ρ, in (2), the implicit estimator in Feuerverger and Hall (1999), one of the classes of estimators in Fraga Alves *et al.* (2003), also included and found competitive in the more recent papers by Goegebeur *et al.* (2008; 2010), Ciuperca and Mercadier (2010) and Caeiro and Gomes (2012a,b), and the estimator introduced and studied in Caeiro and Gomes (2012a), all on the ρ-estimation.

In order to get full information on the asymptotic non-degenerate behaviour of ρ-estimators, it is often necessary to further assume a third-order condition, ruling the rate of convergence in (2). We shall assume that for all $x > 0$,

$$\lim_{t\to\infty} \frac{\frac{\ln U(tx) - \ln U(t) - \gamma \ln x}{A(t)} - \psi_{\rho}(x)}{B(t)} = \begin{cases} \frac{x^{\rho+\rho'}-1}{\rho+\rho'} & \text{if} \quad \rho \vee \rho' < 0 \\ \ln x & \text{if} \quad \rho = \rho' = 0, \end{cases} \tag{3}$$

where $|B(t)|$ must then be of regular variation with an index ρ'. We are then confronted with this third-order parameter $\rho' \leq 0$. In this article, we shall more restrictively assume that we are working with a Paretian type class of models, with a right tail-function given by

$$\overline{F}(x) = 1 - F(x) = Cx^{-1/\gamma}\big\{1 + D_1 x^{\rho/\gamma} + D_2 x^{2\rho/\gamma} + o\big(x^{2\rho/\gamma}\big)\big\} \tag{4}$$

as $x \to \infty$, with $C > 0$, D_1, $D_2 \neq 0$, $\rho < 0$. Note that to assume such a kind of right tail-function is equivalent to say that (3) holds with $\rho = \rho' < 0$ and that we can choose the same parameterisation as in Caeiro and Gomes (2011),

$$A(t) = \alpha\ t^{\rho} =: \gamma\ \beta\ t^{\rho}, \quad B(t) = \beta'\ t^{\rho} = \frac{\beta' A(t)}{\beta\gamma} =: \frac{\xi\ A(t)}{\gamma}, \quad \xi = \frac{\beta'}{\beta}, \tag{5}$$

with $\beta \neq 0$ and $\beta' \neq 0$ “scale” second and third-order parameters, respectively.

In Section 2 of this paper, we shall introduce the ρ-estimators under analysis. Section 3 is dedicated to the description of the asymptotic behaviour of such estimators under the third-oder framework in (4). Finally, in Section 4, we deal with an asymptotic comparison of the ρ-estimators under consideration for a fixed k, the number of top order statistics involved in the estimation, and at optimal levels, i.e., levels k_1 where the asymptotic mean square error (MSE) of the ρ-estimators is minimised.

2 The ρ-estimators under study

Given a random sample, $(X_1, \ldots, X_n)$, and with the notation $(X_{1:n} \leq \cdots \leq X_{n:n})$ for the sample of the associated ascending order statistics (o.s.’s), one of the classes of ρ-estimators under analysis is the simplest class introduced and studied both asymptotically and for finite samples in Fraga Alves *et al.* (2003). As shown in Caeiro and Gomes (2006), such a class of estimators can be parameterised in a tuning or control parameter τ not necessarily non-negative, but real, and it is given by

$$\hat{\rho}_n^{\mathrm{FAGH}}(k) \equiv \hat{\rho}_n^{\mathrm{FAGH}(\tau)}(k) := -\left| \frac{3(T_n^{(\tau)}(k) - 1)}{T_n^{(\tau)}(k) - 3} \right|, \tag{6}$$

with

$$T_n^{(\tau)}(k) := \frac{\left(M_n^{(1)}(k)\right)^{\tau} - \left(M_n^{(2)}(k)/2\right)^{\tau/2}}{\left(M_n^{(2)}(k)/2\right)^{\tau/2} - \left(M_n^{(3)}(k)/6\right)^{\tau/3}},$$

with the notation $a^{\tau} = \ln a$ whenever $\tau = 0$ and where, with V_{ik} the scaled log-excesses, i.e.,

$$V_{ik} := \ln X_{n-i+1:n} - \ln X_{n-k:n}, \quad 1 \leq i \leq k,$$

we have

$$M_n^{(j)}(k) := \frac{1}{k} \sum_{i=1}^{k} V_{ik}^{j}, \quad j \geq 1.$$

Remark 1. *Note that, in the notation used in* Fraga Alves *et al.* (2003), $\hat{\rho}_n^{\mathrm{FAGH}(\tau)}(k) = \hat{\rho}_{n|T}^{(\alpha,\theta_1,\theta_2,\tau)}(k)$*, for the tuning parameters* $(\alpha, \theta_1, \theta_2) = (1, 2, 3)$.

More recently, Caeiro and Gomes (2012a) considered consistent estimators of $\gamma > 0$ defined by adequate linear combinations of the scaled log-spacings

$$U_i := i\{\ln X_{n-i+1:n} - \ln X_{n-i:n}\}, \quad 1 \le i \le k < n, \tag{7}$$

given by

$$N_{n,k}^{(\alpha)} := \frac{\alpha}{k} \sum_{i=1}^{k} \left(\frac{i}{k}\right)^{\alpha-1} U_i, \quad \alpha \ge 1,$$

and proposed a new class of ρ-estimators with the functional expression,

$$\hat{\rho}_n^{CG}(k) = \hat{\rho}_n^{CG(\tau)}(k) := \min\left\{0\,,\, 1 + \frac{1}{1 - R_{n,k}^{(\tau)}}\right\}, \quad \tau \in \mathbb{R}, \tag{8}$$

where

$$R_{n,k}^{(\tau)} = \frac{\left(N_{n,k}^{(1)}\right)^{\tau} - \left(N_{n,k}^{(3/2)}\right)^{\tau}}{\left(N_{n,k}^{(3/2)}\right)^{\tau} - \left(N_{n,k}^{(2)}\right)^{\tau}}, \quad \tau \in \mathbb{R},$$

again with the notation $a^\tau = \ln a$ whenever $\tau = 0$.

Remark 2. *As mentioned in* Caeiro and Gomes (2012a)*, note that we could also have worked with*

$$R_{n,k}^{(\tau,\alpha_1,\alpha_2,\alpha_3)} = \frac{\left(N_{n,k}^{(\alpha_1)}\right)^{\tau} - \left(N_{n,k}^{(\alpha_2)}\right)^{\tau}}{\left(N_{n,k}^{(\alpha_2)}\right)^{\tau} - \left(N_{n,k}^{(\alpha_3)}\right)^{\tau}}, \quad \tau \in \mathbb{R},$$

$\alpha_i \ne \alpha_j$*,* $1 \le i < j \le 3$ *and* $\min_{1 \le i \le 3}(\alpha_i) \ge 1$*. But then we had to deal with the choice of the values of additional tuning parameters.*

The third class of ρ-estimators is based on the fact that in Hall-Welsh's class of models (Hall, 1982; Hall and Welsh, 1985), with a right tail function

$$\overline{F}(x) = C x^{-1/\gamma}\big\{1 + D x^{\rho/\gamma} + o(x^\rho)\big\},$$

as $x \to \infty$, $C > 0$, $D \ne 0$, $\rho < 0$, the log-spacings $U_i, 1 \le i \le k$, in (7), are approximately exponential with mean value

$$\mu_i = \gamma\ e^{\beta(i/n)^{-\rho}}, \quad 1 \le i \le k.$$

Feuerverger and Hall (1999) considered the joint maximization, in γ, β and ρ, of the log-likelihood of the scaled log-spacings, U_i, $1 \le i \le k$, in (7), given by

$$\ln L(\gamma, \beta, \rho; U_i, 1 \le i \le k) = -k \ln \gamma - \beta \sum_{i=1}^{k} \left(\frac{i}{n}\right)^{-\rho} - \frac{1}{\gamma} \sum_{i=1}^{k} e^{-\beta(i/n)^{-\rho}} U_i.$$

Such a maximization leads to implicit estimators $\hat{\beta}$ and $\hat{\rho}$, such that

$$(\hat{\beta}, \hat{\rho}) := \arg\min_{(\beta,\rho)} \left\{ \ln\left(\frac{1}{k}\sum_{i=1}^{k} e^{-\beta(i/n)^{-\rho}} U_i \right) + \beta\left(\frac{1}{k}\sum_{i=1}^{k} \left(\frac{i}{n}\right)^{-\rho} \right) \right\}. \tag{9}$$

Such a class of implicit ρ-estimators will be denoted $\hat{\rho}_n^{\mathrm{FH}}(k)$.

3 Asymptotic behaviour of the ρ-estimators

Let us consider the usual notation $\mathrm{Normal}\big(\mu, \sigma^2\big)$ for a random variable (r.v.) with mean value μ and variance σ^2. On the basis of the research performed in Fraga Alves *et al.* (2003), for the ρ-estimators $\hat{\rho}_n^{\mathrm{FAGH}}(k)$, in (6), in Caeiro and Gomes (2012a), for the ρ-estimators $\hat{\rho}_n^{\mathrm{CG}}(k)$, in (8), and in Caeiro and Gomes (2011), regarding $\hat{\rho}_n^{\mathrm{FH}}(k)$, the implicit ρ-estimator in Feuerverger and Hall (1999) obtained through (9), but with some further computations related to the asymptotic bias of the estimators under analysis, performed in Caeiro *et al.* (2009) and in this article, we now state the following theorem.

Theorem 1. *Under the validity of the second-order condition, in* (2), *with* $\rho < 0$, *and for intermediate values of* k, *i.e.,* k*-values such that*

$$k = k_n \to \infty, \quad k/n \to 0, \quad \textit{as } n \to \infty,$$

if we further assume that $\sqrt{k}\; A(n/k) \to \infty$, *as* $n \to \infty$, *any of the classes of estimators,* $\hat{\rho}_n^{\mathrm{FAGH}}(k) \equiv \hat{\rho}_n^{\mathrm{FAGH}(\tau)}(k)$, $\hat{\rho}_n^{\mathrm{CG}}(k) \equiv \hat{\rho}_n^{\mathrm{CG}(\tau)}(k)$, $\tau \in \mathbb{R}$ *and* $\hat{\rho}_n^{\mathrm{FH}}(k)$, *given in* (6), (8) *and* (9), *respectively, are consistent for the estimation of* ρ.

If we further assume the validity of the third-order condition in (3), A *and* B *given in* (5), *and with* U_k^{FH}, U_k^{FAGH} *and* U_k^{CG} *asymptotically standard normal r.v.'s, we can guarantee that*

$$\hat{\rho}_n^{\bullet}(k) - \rho \;\overset{d}{=}\; \frac{\sigma_{\bullet}\; U_k^{\bullet}}{\sqrt{k}\; A(n/k)} + b_{\bullet}\; A(n/k)(1 + o_p(1)), \tag{10}$$

where

$$\sigma_{\mathrm{FH}} = \gamma\; (1-\rho)(1-2\rho)\sqrt{1-2\rho}/|\rho|, \tag{11}$$

$$\sigma_{\mathrm{FAGH}} = \gamma(1-\rho)^3\sqrt{2\rho^2 - 2\rho + 1}/|\rho|, \tag{12}$$

$$\sigma_{\text{CG}} = \gamma(1-\rho)(2-\rho)(3-2\rho)\sqrt{4\rho^2-4\rho+7}/\big(\sqrt{120}|\rho|\big), \tag{13}$$

$$b_{\text{FH}} = \frac{(2\xi-1)(1-\rho)^2}{2\gamma(1-2\rho)(1-3\rho)^2}, \tag{14}$$

$$b_{\text{FAGH}} = \frac{\tau\rho(3-\rho)(3-2\rho)(1-2\rho)^2 - 6\rho^2(4\rho^3-16\rho^2+20\rho-7)}{12\gamma(1-\rho)^2(1-2\rho)^2} + \frac{2\rho\xi(1-\rho)^3}{\gamma(1-2\rho)^3}, \tag{15}$$

$$b_{\text{CG}} = \frac{(\tau-1)\rho}{2\gamma} + \frac{\rho\xi(2-\rho)(3-2\rho)}{\gamma(1-2\rho)(3-4\rho)}. \tag{16}$$

Consequently, for levels k such that $\sqrt{k}\ A^2(n/k) \to \lambda_{_A}$, finite, there exist real constants b_{FH}, b_{FAGH}, b_{CG}, defined in (14), (15) *and* (16), *respectively, and generally denoted $b_\bullet$, and positive real constants σ_{FH}, σ_{FAGH}, σ_{CG}, defined in* (11), (12), (13), *respectively, generally denoted $\sigma_\bullet$, such that*

$$\sqrt{k}\ A(n/k)\big(\hat\rho_n^\bullet(k)-\rho\big) \underset{n\to\infty}{\xrightarrow{\text{d}}} \text{Normal}\big(\lambda_{_A} b_\bullet, \sigma_\bullet^2\big). \tag{17}$$

Remark 3. *The asymptotic variance σ_{FH}, in* (11), *was computed in* Caeiro and Gomes (2011). *The asymptotic variance σ_{FAGH}, in* (12), *was explicitly computed in* Fraga Alves *et al.* (2003) *and the asymptotic variance σ_{CG}, in* (13), *was explicitly computed in* Caeiro and Gomes (2012a). *For models such that the third-order condition in* (3) *holds, with A and B given in* (5), *the asymptotic bias b_{FH}, in* (14), *was also derived in* Caeiro and Gomes (2011). *The asymptotic bias b_{FAGH}, in* (15), *was given in* Fraga Alves *et al.* (2003), *and explicitly computed in* Caeiro *et al.* (2009), *for models with ρ possibly different from ρ'. Finally, the asymptotic bias b_{CG}, in* (16), *was explicitly computed in* Caeiro and Gomes(2012a), *also for models with ρ possibly different from ρ', with (ρ,ρ') the second-order parameters in* (3).

4 Asymptotic comparison of the ρ-estimators

We shall now proceed to an asymptotic comparison of the estimators under study, generally denoted $\hat\rho_n^\bullet(k)$, not only for any general k, but also at optimal levels, i.e., at levels $k = k_1^\bullet$ where the asymptotic mean square error of $\hat\rho_n^\bullet(k)$ is minimum.

4.1 Asymptotic comparison at a level k

Regarding the asymptotic standard deviations σ_{FH} and σ_{FAGH}, in (11) and (12), respectively, note that since $\rho < 0$, we have $\sigma_{\text{FH}} < \sigma_{\text{FAGH}}$, $\forall\ \rho < 0$, with

$$\frac{\sigma_{\text{FH}}}{\sigma_{\text{FAGH}}} = \frac{(1-2\rho)\sqrt{1-2\rho}}{(1-\rho)\sqrt{2\rho^2-2\rho+1}}$$

approaching zero, as $|\rho| \to \infty$, and equal to 1 for $\rho = 0$. Regarding the asymptotic standard deviations σ_{FH} and σ_{CG}, in (11) and (13), respectively, we have

$$\frac{\sigma_{\text{FH}}}{\sigma_{\text{CG}}} = \frac{(1-2\rho)\sqrt{120(1-2\rho)}}{(2-\rho)(3-2\rho)\sqrt{4\rho^2-4\rho+7}},$$

smaller than one for all $\rho \le 0$, approaching zero, as $|\rho| \to \infty$, with a maximum close to one at $\rho = -0.725$ and approaching the value $\sqrt{210}/21$ as ρ approaches zero. In Figure 1 we illustrate such a behaviour.

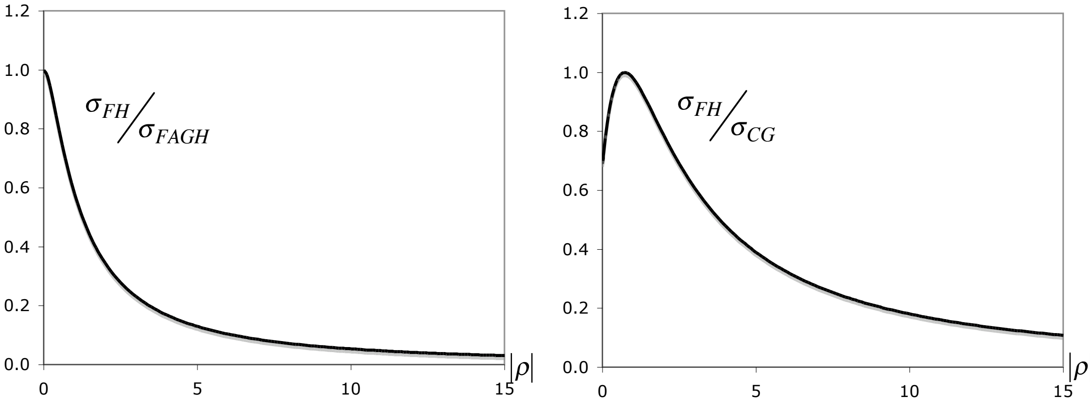


Figure 1: Values of $\sigma_{\text{FH}}/\sigma_{\text{FAGH}}$ (*left*) and $\sigma_{\text{FH}}/\sigma_{\text{CG}}$ (*right*) , as a function of $|\rho|$

The other way round, if we think on the bias b_{FH} and b_{FAGH}, given in (14) an (15), respectively, we get the quotient

$$\frac{b_{\text{FAGH}}}{b_{\text{FH}}} = \frac{(1-2\rho)(1-3\rho)^2}{3(2\xi-1)}\left(\frac{\tau\rho(3-\rho)(3-2\rho)}{2(1-\rho)^4} - \frac{3\rho^2(4\rho^3-16\rho^2+20\rho-7)}{(1-\rho)^4(1-2\rho)^2} + \frac{12\rho\xi(1-\rho)}{(1-2\rho)^3}\right),$$

and due to the fat that $\rho < 0$, if $\xi \neq 1/2$, there is always a value of τ that leads us to $b_{\text{FAGH}}/b_{\text{FH}} = 0$. Such a value, given by

$$\tau_0^{\text{FAGH}} \equiv \tau_0^{\text{FAGH}}(\xi,\rho) = \frac{6\rho(4\rho^3-16\rho^2+20\rho-7)(1-2\rho) - 24\xi(1-\rho)^5}{(1-2\rho)^3(3-\rho)(3-2\rho)}, \tag{18}$$

approaches $3(2-\xi)/2$, as $\rho \to -\infty$, and $-8\xi/3$, as $\rho \to 0$. If we think on the bias b_{FH} and b_{CG}, given in (14) an (16), respectively, we get the quotient

$$\frac{b_{\text{CG}}}{b_{\text{FH}}} = \left(\frac{(\tau-1)\rho}{2} + \frac{\rho\xi(2-\rho)(3-2\rho)}{(1-2\rho)(3-4\rho)}\right)\frac{2(1-2\rho)(1-3\rho)^2}{(2\xi-1)(1-\rho)^2}$$

and due to the fat that $\rho < 0$, if $\xi \neq 1/2$, there is also always a value of τ that leads us to $b_{\text{CG}}/b_{\text{FH}} = 0$, given by

$$\tau_0^{\text{CG}} \equiv \tau_0^{\text{CG}}(\xi,\rho) = 1 - \frac{2\xi(2-\rho)(3-2\rho)}{(1-2\rho)(3-4\rho)}, \tag{19}$$

a value that approaches $-\xi/2$, as $\rho \to -\infty$, and $1-4\xi$, as ρ approaches zero. These facts were already noticed in Caeiro and Gomes (2012b), where bias reduction is considered.

One of the most typical values of ξ is $\xi = 1$, associated to models like the generalised Pareto (GP) model, with d.f. $F(x) = 1-(1+\gamma x)^{-1/\gamma}$, $x \geq 0$, $\gamma > 0$, and the Burr model, with d.f. $F(x) = 1-(1+x^{-\rho/\gamma})^{1/\rho}$, $x \geq 0$, $\gamma > 0$. For a Fréchet parent, another typical heavy-tailed model, we get $\xi = 10/6$. Finally, for the Student's t_ν-model with ν degrees of freedom, with a probability density function

$$f_{t_\nu}(t) = \Gamma((\nu+1)/2)\left(1+t^2/\nu\right)^{-(\nu+1)/2}/(\sqrt{\pi\nu}\,\Gamma(\nu/2)),\ t \in \mathbb{R} \quad (\nu > 0),$$

we get $\gamma = 1/\nu$ and $\rho' = \rho = -2/\nu$. For an explicit expression of β and β' as a function of ν, see Caeiro and Gomes (2008). We have

$$\xi = \beta'/\beta = (\nu^2+4\nu+2)/((\nu+1)(\nu+4)) \in (0.5, 1)$$

(see Caeiro and Gomes, 2011). In Figure 2 we represent graphically $\tau_0^{\text{FAGH}}(\xi,\rho)$, in (18), for a few values of ξ. Similarly, in Figure 3 we represent graphically $\tau_0^{\text{CG}}(\xi,\rho)$, in (19), for a few values of ξ.

The same values $\tau_0^{FAGH}(\xi,\rho)$ and $\tau_0^{CG}(\xi,\rho)$ are next presented at Figures 4 and 5, respectively, in the (ξ,ρ)-plane.

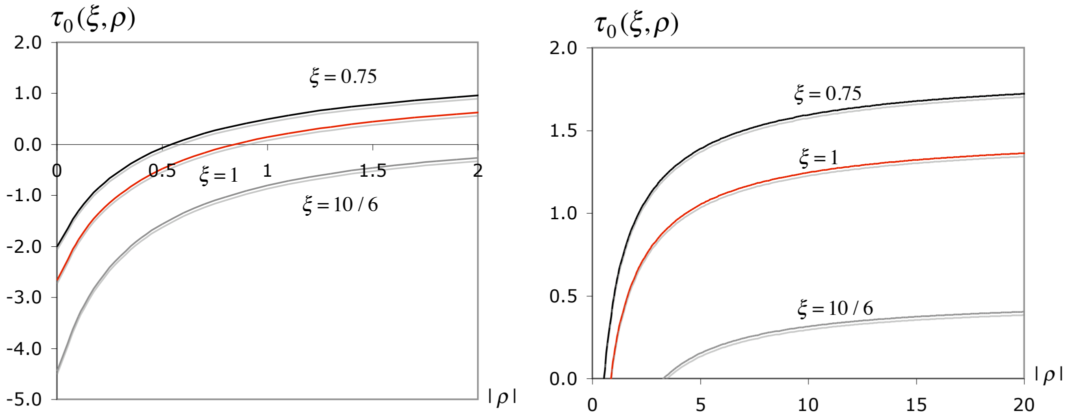


Figure 2: Values of $\tau_0(\xi, \rho) = \tau_0^{FAGH}(\xi, \rho)$, as a function of $|\rho|$ for $\xi = 0.75, 1$ and $10/6$, in two different scales

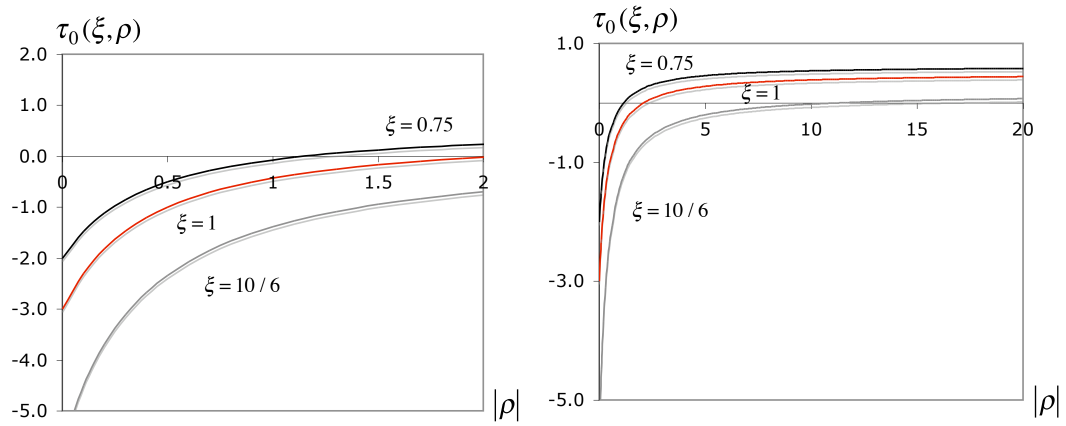
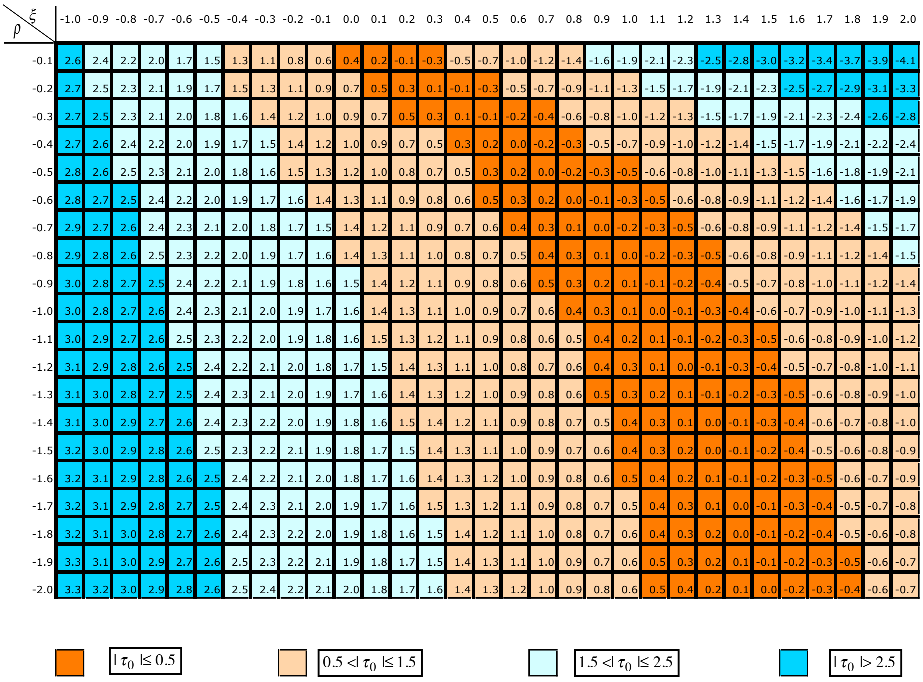


Figure 3: Values of $\tau_0(\xi, \rho) = \tau_0^{CG}(\xi, \rho)$, as a function of $|\rho|$ for $\xi = 0.75, 1$ and $10/6$, in two different scales



ρ \ ξ	-1.0	-0.9	-0.8	-0.7	-0.6	-0.5	-0.4	-0.3	-0.2	-0.1	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
-0.1	2.6	2.4	2.2	2.0	1.7	1.5	1.3	1.1	0.8	0.6	0.4	0.2	-0.1	-0.3	-0.5	-0.7	-1.0	-1.2	-1.4	-1.6	-1.9	-2.1	-2.3	-2.5	-2.8	-3.0	-3.2	-3.4	-3.7	-3.9	-4.1
-0.2	2.7	2.5	2.3	2.1	1.9	1.7	1.5	1.3	1.1	0.9	0.7	0.5	0.3	0.1	-0.1	-0.3	-0.5	-0.7	-0.9	-1.1	-1.3	-1.5	-1.7	-1.9	-2.1	-2.3	-2.5	-2.7	-2.9	-3.1	-3.3
-0.3	2.7	2.5	2.3	2.1	2.0	1.8	1.6	1.4	1.2	1.0	0.9	0.7	0.5	0.3	0.1	-0.1	-0.2	-0.4	-0.6	-0.8	-1.0	-1.2	-1.3	-1.5	-1.7	-1.9	-2.1	-2.3	-2.4	-2.6	-2.8
-0.4	2.7	2.6	2.4	2.2	2.0	1.9	1.7	1.5	1.4	1.2	1.0	0.9	0.7	0.5	0.3	0.2	0.0	-0.2	-0.3	-0.5	-0.7	-0.9	-1.0	-1.2	-1.4	-1.5	-1.7	-1.9	-2.1	-2.2	-2.4
-0.5	2.8	2.6	2.5	2.3	2.1	2.0	1.8	1.6	1.5	1.3	1.2	1.0	0.8	0.7	0.5	0.3	0.2	0.0	-0.2	-0.3	-0.5	-0.6	-0.8	-1.0	-1.1	-1.3	-1.5	-1.6	-1.8	-1.9	-2.1
-0.6	2.8	2.7	2.5	2.4	2.2	2.0	1.9	1.7	1.6	1.4	1.3	1.1	0.9	0.8	0.6	0.5	0.3	0.2	0.0	-0.1	-0.3	-0.5	-0.6	-0.8	-0.9	-1.1	-1.2	-1.4	-1.6	-1.7	-1.9
-0.7	2.9	2.7	2.6	2.4	2.3	2.1	2.0	1.8	1.7	1.5	1.4	1.2	1.1	0.9	0.7	0.6	0.4	0.3	0.1	0.0	-0.2	-0.3	-0.5	-0.6	-0.8	-0.9	-1.1	-1.2	-1.4	-1.5	-1.7
-0.8	2.9	2.8	2.6	2.5	2.3	2.2	2.0	1.9	1.7	1.6	1.4	1.3	1.1	1.0	0.8	0.7	0.5	0.4	0.3	0.1	0.0	-0.2	-0.3	-0.5	-0.6	-0.8	-0.9	-1.1	-1.2	-1.4	-1.5
-0.9	3.0	2.8	2.7	2.5	2.4	2.2	2.1	1.9	1.8	1.6	1.5	1.4	1.2	1.1	0.9	0.8	0.6	0.5	0.3	0.2	0.1	-0.1	-0.2	-0.4	-0.5	-0.7	-0.8	-1.0	-1.1	-1.2	-1.4
-1.0	3.0	2.8	2.7	2.6	2.4	2.3	2.1	2.0	1.9	1.7	1.6	1.4	1.3	1.1	1.0	0.9	0.7	0.6	0.4	0.3	0.1	0.0	-0.1	-0.3	-0.4	-0.6	-0.7	-0.9	-1.0	-1.1	-1.3
-1.1	3.0	2.9	2.7	2.6	2.5	2.3	2.2	2.0	1.9	1.8	1.6	1.5	1.3	1.2	1.1	0.9	0.8	0.6	0.5	0.4	0.2	0.1	-0.1	-0.2	-0.3	-0.5	-0.6	-0.8	-0.9	-1.0	-1.2
-1.2	3.1	2.9	2.8	2.6	2.5	2.4	2.2	2.1	2.0	1.8	1.7	1.5	1.4	1.3	1.1	1.0	0.8	0.7	0.6	0.4	0.3	0.1	0.0	-0.1	-0.3	-0.4	-0.5	-0.7	-0.8	-1.0	-1.1
-1.3	3.1	3.0	2.8	2.7	2.5	2.4	2.3	2.1	2.0	1.9	1.7	1.6	1.4	1.3	1.2	1.0	0.9	0.8	0.6	0.5	0.3	0.2	0.1	-0.1	-0.2	-0.3	-0.5	-0.6	-0.8	-0.9	-1.0
-1.4	3.1	3.0	2.9	2.7	2.6	2.4	2.3	2.2	2.0	1.9	1.8	1.6	1.5	1.4	1.2	1.1	0.9	0.8	0.7	0.5	0.4	0.3	0.1	0.0	-0.1	-0.3	-0.4	-0.6	-0.7	-0.8	-1.0
-1.5	3.2	3.0	2.9	2.8	2.6	2.5	2.3	2.2	2.1	1.9	1.8	1.7	1.5	1.4	1.3	1.1	1.0	0.9	0.7	0.6	0.4	0.3	0.2	0.0	-0.1	-0.2	-0.4	-0.5	-0.6	-0.8	-0.9
-1.6	3.2	3.1	2.9	2.8	2.6	2.5	2.4	2.2	2.1	2.0	1.8	1.7	1.6	1.4	1.3	1.2	1.0	0.9	0.8	0.6	0.5	0.4	0.2	0.1	-0.1	-0.2	-0.3	-0.5	-0.6	-0.7	-0.9
-1.7	3.2	3.1	2.9	2.8	2.7	2.5	2.4	2.3	2.1	2.0	1.9	1.7	1.6	1.5	1.3	1.2	1.1	0.9	0.8	0.7	0.5	0.4	0.3	0.1	0.0	-0.1	-0.3	-0.4	-0.5	-0.7	-0.8
-1.8	3.2	3.1	3.0	2.8	2.7	2.6	2.4	2.3	2.2	2.0	1.9	1.8	1.6	1.5	1.4	1.2	1.1	1.0	0.8	0.7	0.6	0.4	0.3	0.2	0.0	-0.1	-0.2	-0.4	-0.5	-0.6	-0.8
-1.9	3.3	3.1	3.0	2.9	2.7	2.6	2.5	2.3	2.2	2.1	1.9	1.8	1.7	1.5	1.4	1.3	1.1	1.0	0.9	0.7	0.6	0.5	0.3	0.2	0.1	-0.1	-0.2	-0.3	-0.5	-0.6	-0.7
-2.0	3.3	3.2	3.0	2.9	2.8	2.6	2.5	2.4	2.2	2.1	2.0	1.8	1.7	1.6	1.4	1.3	1.2	1.0	0.9	0.8	0.6	0.5	0.4	0.2	0.1	0.0	-0.2	-0.3	-0.4	-0.6	-0.7

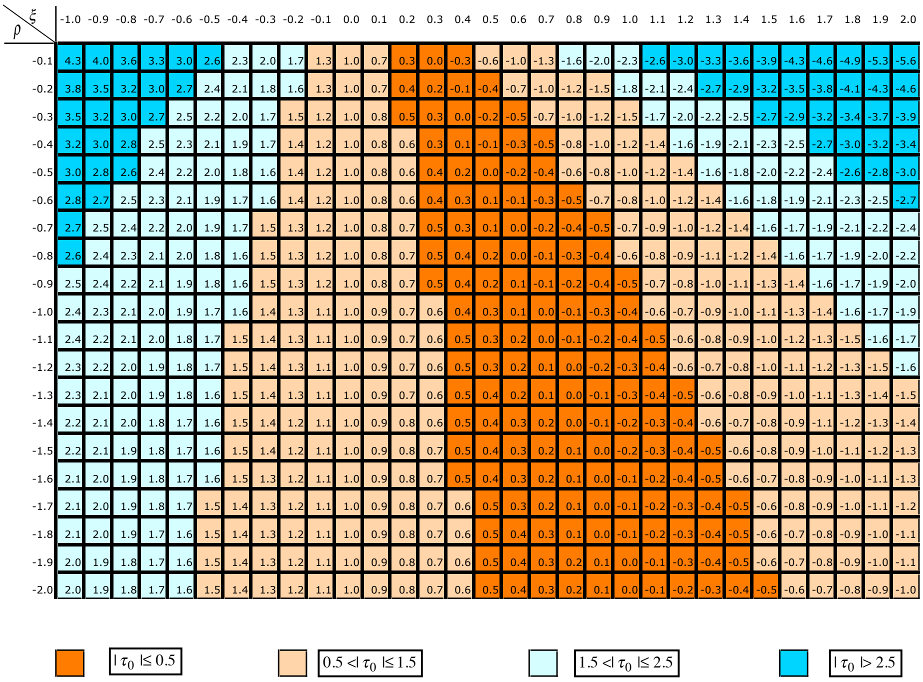
🟧 $|\tau_0| \leq 0.5$

🟨 $0.5 < |\tau_0| \leq 1.5$

⬜ $1.5 < |\tau_0| \leq 2.5$

🟦 $|\tau_0| > 2.5$

Figure 4: Values $\tau_0 = \tau_0^{FAGH}(\xi, \rho)$ such that $b_{\text{FAGH}} = 0$



ρ \ ξ	-1.0	-0.9	-0.8	-0.7	-0.6	-0.5	-0.4	-0.3	-0.2	-0.1	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
-0.1	4.3	4.0	3.6	3.3	3.0	2.6	2.3	2.0	1.7	1.3	1.0	0.7	0.3	0.0	-0.3	-0.6	-1.0	-1.3	-1.6	-2.0	-2.3	-2.6	-3.0	-3.3	-3.6	-3.9	-4.3	-4.6	-4.9	-5.3	-5.6
-0.2	3.8	3.5	3.2	3.0	2.7	2.4	2.1	1.8	1.6	1.3	1.0	0.7	0.4	0.2	-0.1	-0.4	-0.7	-1.0	-1.2	-1.5	-1.8	-2.1	-2.4	-2.7	-2.9	-3.2	-3.5	-3.8	-4.1	-4.3	-4.6
-0.3	3.5	3.2	3.0	2.7	2.5	2.2	2.0	1.7	1.5	1.2	1.0	0.8	0.5	0.3	0.0	-0.2	-0.5	-0.7	-1.0	-1.2	-1.5	-1.7	-2.0	-2.2	-2.5	-2.7	-2.9	-3.2	-3.4	-3.7	-3.9
-0.4	3.2	3.0	2.8	2.5	2.3	2.1	1.9	1.7	1.4	1.2	1.0	0.8	0.6	0.3	0.1	-0.1	-0.3	-0.5	-0.8	-1.0	-1.2	-1.4	-1.6	-1.9	-2.1	-2.3	-2.5	-2.7	-3.0	-3.2	-3.4
-0.5	3.0	2.8	2.6	2.4	2.2	2.0	1.8	1.6	1.4	1.2	1.0	0.8	0.6	0.4	0.2	0.0	-0.2	-0.4	-0.6	-0.8	-1.0	-1.2	-1.4	-1.6	-1.8	-2.0	-2.2	-2.4	-2.6	-2.8	-3.0
-0.6	2.8	2.7	2.5	2.3	2.1	1.9	1.7	1.6	1.4	1.2	1.0	0.8	0.6	0.4	0.3	0.1	-0.1	-0.3	-0.5	-0.7	-0.8	-1.0	-1.2	-1.4	-1.6	-1.8	-1.9	-2.1	-2.3	-2.5	-2.7
-0.7	2.7	2.5	2.4	2.2	2.0	1.9	1.7	1.5	1.3	1.2	1.0	0.8	0.7	0.5	0.3	0.1	0.0	-0.2	-0.4	-0.5	-0.7	-0.9	-1.0	-1.2	-1.4	-1.6	-1.7	-1.9	-2.1	-2.2	-2.4
-0.8	2.6	2.4	2.3	2.1	2.0	1.8	1.6	1.5	1.3	1.2	1.0	0.8	0.7	0.5	0.4	0.2	0.0	-0.1	-0.3	-0.4	-0.6	-0.8	-0.9	-1.1	-1.2	-1.4	-1.6	-1.7	-1.9	-2.0	-2.2
-0.9	2.5	2.4	2.2	2.1	1.9	1.8	1.6	1.5	1.3	1.2	1.0	0.8	0.7	0.5	0.4	0.2	0.1	-0.1	-0.2	-0.4	-0.5	-0.7	-0.8	-1.0	-1.1	-1.3	-1.4	-1.6	-1.7	-1.9	-2.0
-1.0	2.4	2.3	2.1	2.0	1.9	1.7	1.6	1.4	1.3	1.1	1.0	0.9	0.7	0.6	0.4	0.3	0.1	0.0	-0.1	-0.3	-0.4	-0.6	-0.7	-0.9	-1.0	-1.1	-1.3	-1.4	-1.6	-1.7	-1.9
-1.1	2.4	2.2	2.1	2.0	1.8	1.7	1.5	1.4	1.3	1.1	1.0	0.9	0.7	0.6	0.5	0.3	0.2	0.0	-0.1	-0.2	-0.4	-0.5	-0.6	-0.8	-0.9	-1.0	-1.2	-1.3	-1.5	-1.6	-1.7
-1.2	2.3	2.2	2.0	1.9	1.8	1.7	1.5	1.4	1.3	1.1	1.0	0.9	0.7	0.6	0.5	0.3	0.2	0.1	0.0	-0.2	-0.3	-0.4	-0.6	-0.7	-0.8	-1.0	-1.1	-1.2	-1.3	-1.5	-1.6
-1.3	2.3	2.1	2.0	1.9	1.8	1.6	1.5	1.4	1.3	1.1	1.0	0.9	0.7	0.6	0.5	0.4	0.2	0.1	0.0	-0.1	-0.3	-0.4	-0.5	-0.6	-0.8	-0.9	-1.0	-1.1	-1.3	-1.4	-1.5
-1.4	2.2	2.1	2.0	1.8	1.7	1.6	1.5	1.4	1.2	1.1	1.0	0.9	0.8	0.6	0.5	0.4	0.3	0.2	0.0	-0.1	-0.2	-0.3	-0.4	-0.6	-0.7	-0.8	-0.9	-1.1	-1.2	-1.3	-1.4
-1.5	2.2	2.1	1.9	1.8	1.7	1.6	1.5	1.4	1.2	1.1	1.0	0.9	0.8	0.7	0.5	0.4	0.3	0.2	0.1	0.0	-0.2	-0.3	-0.4	-0.5	-0.6	-0.8	-0.9	-1.0	-1.1	-1.2	-1.3
-1.6	2.1	2.0	1.9	1.8	1.7	1.6	1.5	1.3	1.2	1.1	1.0	0.9	0.8	0.7	0.5	0.4	0.3	0.2	0.1	0.0	-0.1	-0.2	-0.4	-0.5	-0.6	-0.7	-0.8	-0.9	-1.0	-1.1	-1.3
-1.7	2.1	2.0	1.9	1.8	1.7	1.5	1.4	1.3	1.2	1.1	1.0	0.9	0.8	0.7	0.6	0.5	0.3	0.2	0.1	0.0	-0.1	-0.2	-0.3	-0.4	-0.5	-0.6	-0.8	-0.9	-1.0	-1.1	-1.2
-1.8	2.1	2.0	1.9	1.7	1.6	1.5	1.4	1.3	1.2	1.1	1.0	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.1	0.0	-0.1	-0.2	-0.3	-0.4	-0.5	-0.6	-0.7	-0.8	-0.9	-1.0	-1.1
-1.9	2.0	1.9	1.8	1.7	1.6	1.5	1.4	1.3	1.2	1.1	1.0	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1	0.0	-0.1	-0.3	-0.4	-0.5	-0.6	-0.7	-0.8	-0.9	-1.0	-1.1
-2.0	2.0	1.9	1.8	1.7	1.6	1.5	1.4	1.3	1.2	1.1	1.0	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1	0.0	-0.1	-0.2	-0.3	-0.4	-0.5	-0.6	-0.7	-0.8	-0.9	-1.0

🟧 $|\tau_0| \leq 0.5$

🟨 $0.5 < |\tau_0| \leq 1.5$

⬜ $1.5 < |\tau_0| \leq 2.5$

🟦 $|\tau_0| > 2.5$

Figure 5: Values $\tau_0 = \tau_0^{CG}(\xi, \rho)$ such that $b_{\text{CG}} = 0$

4.2 Asymptotic comparison at optimal levels

We shall next proceed to the comparison of the estimators under study at their optimal levels. This is done in a way similar to the one used in de Haan and Peng (1998), Gomes and Martins (2001), Gomes *et al.* (2005, 2007b), Gomes and Neves (2008) and Gomes and Henriques-Rodrigues (2010) for the classical EVI-estimators, in Gomes *et al.* (2007a) for MVRB maximum likelihood EVI-estimators, and in Caeiro and Gomes (2011) for a larger set of MVRB EVI-estimators.

On the basis of the asymptotic distribution of $\hat\rho_n^\bullet(k)$, in (10), its asymptotic MSE (AMSE) is given by

$$\mathrm{AMSE}_\bullet(k) = \frac{\sigma_\bullet^2}{kA^2(n/k)} + b_\bullet^2\ A^2(n/k).$$

With the parameterisation in (5), we can write

$$\begin{aligned}\mathrm{AMSE}_\bullet(k) &= \frac{\sigma_\bullet^2}{k\gamma^2\beta^2(n/k)^{2\rho}} + b_\bullet^2\ \gamma^2\beta^2(n/k)^{2\rho}\\ &= \frac{1}{\gamma^2\beta^2}\left(\sigma_\bullet^2 n^{-2\rho}k^{2\rho-1} + b_\bullet^2\gamma^4\beta^4 n^{2\rho}k^{-2\rho}\right).\end{aligned}$$

The value $k_1^\bullet := \arg\min_k \mathrm{AMSE}_\bullet(k)$ is solution in k of the equation,

$$-(1-2\rho)\sigma_\bullet^2 n^{-2\rho}k^{2\rho-2} + (-2\rho)b_\bullet^2\gamma^4\beta^4 n^{2\rho}k^{-2\rho-1} = 0.$$

We thus get

$$\frac{(1-2\rho)\sigma_\bullet^2}{(-2\rho)b_\bullet^2\gamma^4\beta^4}\ n^{-4\rho} = (k_1^\bullet)^{1-4\rho}$$

i.e. $\mathrm{AMSE}_\bullet(k)$ is minised at a level

$$k_1^\bullet := \arg\inf_k \mathrm{AMSE}_\bullet(k)\ = \left(\frac{\sigma_\bullet^2(1-2\rho)}{b_\bullet^2\gamma^4\beta^4(-2\rho)}\right)^{1/(1-4\rho)} n^{-4\rho/(1-4\rho)}. \tag{20}$$

Let us use the notation $\hat\rho_{n0}^\bullet := \hat\rho_n^\bullet(k_1^\bullet)$, with $k_1^\bullet$ given in (20).

On the basis of (20), we can write

$$\left(\frac{n}{k_1^\bullet}\right)^{2\rho} = \left(\frac{\sigma_\bullet^2(1-2\rho)}{b_\bullet^2\gamma^4\beta^4(-2\rho)}\right)^{-2\rho/(1-4\rho)} n^{2\rho/(1-4\rho)}.$$

Let us use the notation

$$\varphi(\rho) := \left(\frac{1-2\rho}{-2\rho}\right)^{-(1-2\rho)/(1-4\rho)} + \left(\frac{1-2\rho}{-2\rho}\right)^{-2\rho/(1-4\rho)}.$$

We are now interested in the computation of

$$\begin{aligned}\mathrm{AMSE}_\bullet(k_1^\bullet) &= \frac{\sigma_\bullet^2}{k_1\gamma^2\beta^2(n/k_1^\bullet)^{2\rho}} + b_\bullet^2\ \gamma^2\beta^2(n/k_1^\bullet)^{2\rho} \\ &= \left(\gamma^2\beta^2\right)^{1/(1-4\rho)} \left(\sigma_\bullet^2\right)^{-2\rho/(1-4\rho)} \left(b_\bullet^2\right)^{(1-2\rho)/(1-4\rho)} \varphi(\rho) n^{2\rho/(1-4\rho)}.\end{aligned}$$

Consequently, and with the notation $\mathrm{AMSE}(\hat\rho_{n0}^\bullet) := \mathrm{AMSE}_\bullet(k_1^\bullet)$, we can guarantee that whenever $b_\bullet \neq 0$, there exists a function $\psi(n) = \psi(n, \gamma, \beta, \rho)$, such that

$$\lim_{n\to\infty} \psi(n)\ \mathrm{AMSE}(\hat\rho_{n0}^\bullet) = \left(\sigma_\bullet^2\right)^{-\frac{2\rho}{1-4\rho}} \left(b_\bullet^2\right)^{\frac{1-2\rho}{1-4\rho}} =: \mathrm{LMSE}\left(\hat\rho_{n0}^\bullet\right).$$

It is then sensible to consider the following definition of *asymptotic root efficiency* (AREFF).

Definition 1. *Given two biased estimators* $\hat\rho_n^{(1)}(k)$ *and* $\hat\rho_n^{(2)}(k)$*, for which a distributional representation of the type of the one in* (10) *holds, with constants* (σ_1, b_1) *and* (σ_2, b_2)*,* $b_1, b_2 \neq 0$*, respectively, both computed at their optimal levels, the* AREFF *of* $\hat\rho_{n0}^{(1)}$ *relatively to* $\hat\rho_{n0}^{(2)}$ *is*

$$\mathrm{AREFF}_{1|2} \equiv \mathrm{AREFF}_{\hat\rho_{n0}^{(1)}|\hat\rho_{n0}^{(2)}} := \sqrt{\frac{\mathrm{LMSE}\big(\hat\rho_{n0}^{(2)}\big)}{\mathrm{LMSE}\big(\hat\rho_{n0}^{(1)}\big)}} = \left(\left(\frac{\sigma_2}{\sigma_1}\right)^{-4\rho}\left|\frac{b_2}{b_1}\right|^{2(1-2\rho)}\right)^{\frac{1}{1-4\rho}}.$$

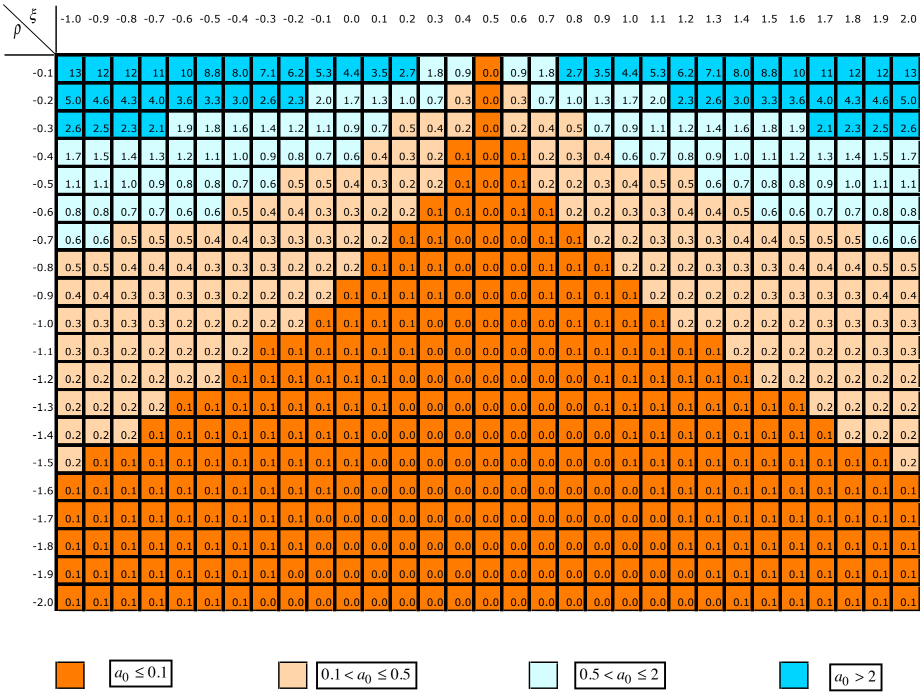
Let us think on $\mathrm{AREFF}_{\mathrm{FAGH}(\tau)|\mathrm{FH}}$. Using the value τ_0^{FAGH}, in (18), we have $\mathrm{AREFF}_{\mathrm{FAGH}(\tau)|\mathrm{FH}} > 1$ in a region $\tau \in \tau_0^{\mathrm{FAGH}} \pm a_0^{\mathrm{FAGH}}(\xi, \rho)$, with

$$a_0^{\mathrm{FAGH}}(\xi,\rho) = \frac{6|2\xi-1|}{|\rho|(1-3\rho)^2(3-\rho)(3-2\rho)} \left(\frac{(1-\rho)^{4(2-3\rho)}(1-2\rho)^{-2(1+\rho)}}{(2\rho^2-2\rho+1)^{-2\rho}}\right)^{1/(2(1-2\rho))}. \tag{21}$$

The values $a_0^{\mathrm{FAGH}}(\xi, \rho)$ are next presented at Figure 6, in the (ξ, ρ)-plane. See also Figure 7, where we present the regions of the (ξ, ρ)-plane where $\tau_0^{\mathrm{FAGH}} - a_0^{\mathrm{FAGH}}$ and $\tau_0^{\mathrm{FAGH}} + a_0^{\mathrm{FAGH}}$ are both non-negative, non-positive or with different signs. Note that for values of ρ close to zero and $\xi \neq 1/2$, the interval $(\tau_0^{\mathrm{FAGH}} - a_0^{\mathrm{FAGH}}, \tau_0^{\mathrm{FAGH}} + a_0^{\mathrm{FAGH}})$ covers the value $\tau = 0$, a value commonly used in practice for the region $|\rho \leq 1$.

SImilarly, let us now think on $\mathrm{AREFF}_{\mathrm{CG}(\tau)|\mathrm{FH}}$. Using the value τ_0^{CG}, in (19), we have $\mathrm{AREFF}_{\mathrm{CG}(\tau)|\mathrm{FH}} > 1$ in a region $\tau \in \tau_0^{\mathrm{CG}} \pm a_0^{\mathrm{CG}}(\xi, \rho)$, with

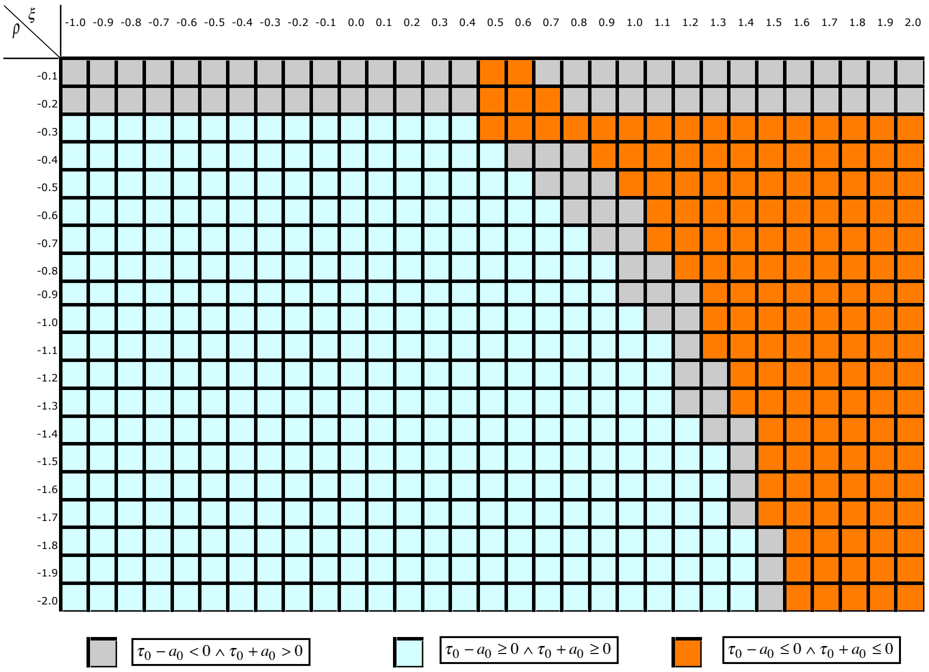
$$a_0^{\mathrm{CG}}(\xi,\rho) = \frac{|2\xi-1|(1-\rho)^2}{|\rho|(1-3\rho)^2} \left(\frac{(1-2\rho)^{-2(1+\rho)}120^{-2\rho}}{(2-\rho)^{-4\rho}(3-2\rho)^{-4\rho}(4\rho^2-4\rho+7)^{-2\rho}}\right)^{1/(2(1-2\rho))}. \tag{22}$$



ρ \ ξ	-1.0	-0.9	-0.8	-0.7	-0.6	-0.5	-0.4	-0.3	-0.2	-0.1	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
-0.1	13	12	12	11	10	8.8	8.0	7.1	6.2	5.3	4.4	3.5	2.7	1.8	0.9	0.0	0.9	1.8	2.7	3.5	4.4	5.3	6.2	7.1	8.0	8.8	10	11	12	12	13
-0.2	5.0	4.6	4.3	4.0	3.6	3.3	3.0	2.6	2.3	2.0	1.7	1.3	1.0	0.7	0.3	0.0	0.3	0.7	1.0	1.3	1.7	2.0	2.3	2.6	3.0	3.3	3.6	4.0	4.3	4.6	5.0
-0.3	2.6	2.5	2.3	2.1	1.9	1.8	1.6	1.4	1.2	1.1	0.9	0.7	0.5	0.4	0.2	0.0	0.2	0.4	0.5	0.7	0.9	1.1	1.2	1.4	1.6	1.8	1.9	2.1	2.3	2.5	2.6
-0.4	1.7	1.5	1.4	1.3	1.2	1.1	1.0	0.9	0.8	0.7	0.6	0.4	0.3	0.2	0.1	0.0	0.1	0.2	0.3	0.4	0.6	0.7	0.8	0.9	1.0	1.1	1.2	1.3	1.4	1.5	1.7
-0.5	1.1	1.1	1.0	0.9	0.8	0.8	0.7	0.6	0.5	0.5	0.4	0.3	0.2	0.2	0.1	0.0	0.1	0.2	0.2	0.3	0.4	0.5	0.5	0.6	0.7	0.8	0.8	0.9	1.0	1.1	1.1
-0.6	0.8	0.8	0.7	0.7	0.6	0.6	0.5	0.4	0.4	0.3	0.3	0.2	0.2	0.1	0.1	0.0	0.1	0.1	0.2	0.2	0.3	0.3	0.4	0.4	0.5	0.6	0.6	0.7	0.7	0.8	0.8
-0.7	0.6	0.6	0.5	0.5	0.5	0.4	0.4	0.3	0.3	0.3	0.2	0.2	0.1	0.1	0.0	0.0	0.0	0.1	0.1	0.2	0.2	0.3	0.3	0.3	0.4	0.4	0.5	0.5	0.5	0.6	0.6
-0.8	0.5	0.5	0.4	0.4	0.4	0.3	0.3	0.3	0.2	0.2	0.2	0.1	0.1	0.1	0.0	0.0	0.0	0.1	0.1	0.1	0.2	0.2	0.2	0.3	0.3	0.3	0.4	0.4	0.4	0.5	0.5
-0.9	0.4	0.4	0.3	0.3	0.3	0.3	0.2	0.2	0.2	0.2	0.1	0.1	0.1	0.1	0.0	0.0	0.0	0.1	0.1	0.1	0.1	0.2	0.2	0.2	0.2	0.3	0.3	0.3	0.3	0.4	0.4
-1.0	0.3	0.3	0.3	0.3	0.2	0.2	0.2	0.2	0.2	0.1	0.1	0.1	0.1	0.0	0.0	0.0	0.0	0.0	0.1	0.1	0.1	0.1	0.2	0.2	0.2	0.2	0.2	0.3	0.3	0.3	0.3
-1.1	0.3	0.3	0.2	0.2	0.2	0.2	0.2	0.1	0.1	0.1	0.1	0.1	0.1	0.0	0.0	0.0	0.0	0.0	0.1	0.1	0.1	0.1	0.1	0.1	0.2	0.2	0.2	0.2	0.2	0.3	0.3
-1.2	0.2	0.2	0.2	0.2	0.2	0.2	0.1	0.1	0.1	0.1	0.1	0.1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.1	0.1	0.1	0.1	0.1	0.1	0.2	0.2	0.2	0.2	0.2	0.2
-1.3	0.2	0.2	0.2	0.2	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.2	0.2	0.2	0.2
-1.4	0.2	0.2	0.2	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.2	0.2	0.2
-1.5	0.2	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.2
-1.6	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1
-1.7	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1
-1.8	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1
-1.9	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1
-2.0	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.1	0.1	0.1	0.1	0.1	0.1	0.1

■ $a_0 \leq 0.1$ ▨ $0.1 < a_0 \leq 0.5$ □ $0.5 < a_0 \leq 2$ ▩ $a_0 > 2$

Figure 6: Values a_0 such that $\text{AREFF}_{\text{FAGH}(\tau)|\text{FH}} > 1$ for $\tau \in (\tau_0^{\text{FAGH}} - a_0, \tau_0^{\text{FAGH}} + a_0)$

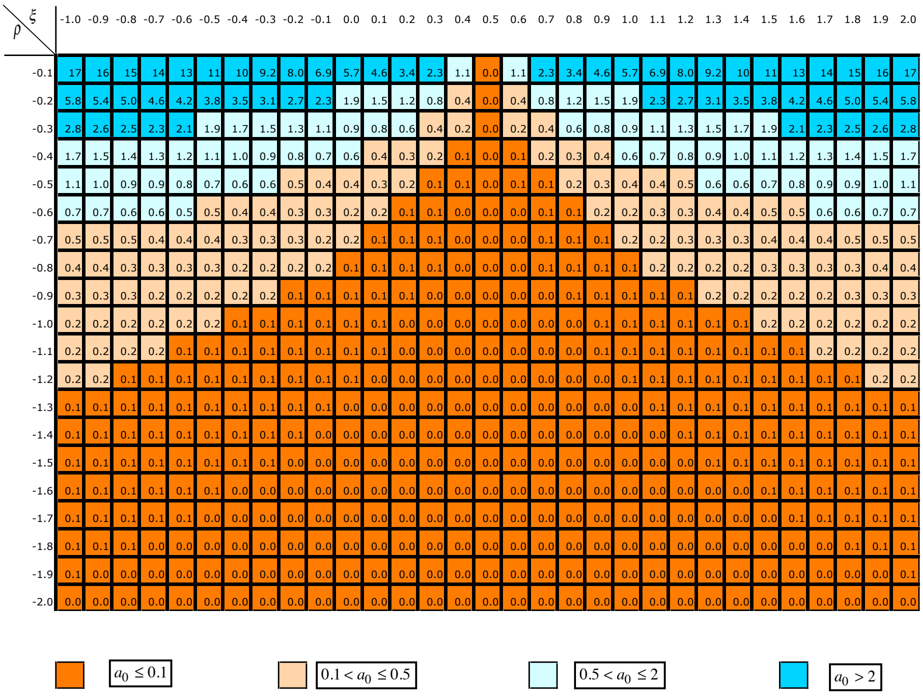


ρ \ ξ	-1.0	-0.9	-0.8	-0.7	-0.6	-0.5	-0.4	-0.3	-0.2	-0.1	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
-0.1	▨	▨	▨	▨	▨	▨	▨	▨	▨	▨	▨	▨	▨	▨	▨	■	■	▨	▨	▨	▨	▨	▨	▨	▨	▨	▨	▨	▨	▨	▨
-0.2	▨	▨	▨	▨	▨	▨	▨	▨	▨	▨	▨	▨	▨	▨	▨	■	■	■	▨	▨	▨	▨	▨	▨	▨	▨	▨	▨	▨	▨	▨
-0.3	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	■	■	■	■	■	■	■	■	■	■	■	■	■	■	■	■
-0.4	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	▨	▨	▨	■	■	■	■	■	■	■	■	■	■	■	■
-0.5	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	▨	▨	▨	■	■	■	■	■	■	■	■	■	■	■
-0.6	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	▨	▨	▨	■	■	■	■	■	■	■	■	■	■
-0.7	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	▨	▨	■	■	■	■	■	■	■	■	■	■
-0.8	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	▨	▨	■	■	■	■	■	■	■	■	■
-0.9	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	▨	▨	▨	■	■	■	■	■	■	■	■
-1.0	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	▨	▨	■	■	■	■	■	■	■	■
-1.1	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	▨	■	■	■	■	■	■	■	■
-1.2	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	▨	▨	■	■	■	■	■	■	■
-1.3	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	▨	▨	■	■	■	■	■	■	■
-1.4	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	▨	▨	■	■	■	■	■	■
-1.5	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	▨	■	■	■	■	■	■
-1.6	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	▨	■	■	■	■	■	■
-1.7	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	▨	■	■	■	■	■	■
-1.8	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	▨	■	■	■	■	■
-1.9	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	▨	■	■	■	■	■
-2.0	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	▨	■	■	■	■	■

▨ $\tau_0 - a_0 < 0 \wedge \tau_0 + a_0 > 0$ □ $\tau_0 - a_0 \geq 0 \wedge \tau_0 + a_0 \geq 0$ ■ $\tau_0 - a_0 \leq 0 \wedge \tau_0 + a_0 \leq 0$

Figure 7: Signs of $\tau_0^{\text{FAGH}} - a_0$ and $\tau_0^{\text{FAGH}} + a_0$

The values $a_0^{\mathrm{CG}}(\xi, \rho)$ are next presented at Figure 8, in the (ξ, ρ)-plane. See also Figure 9, where we present the signs of $\tau_0^{\mathrm{CG}} - a_0^{\mathrm{CG}}$ and $\tau_0^{\mathrm{CG}} + a_0^{\mathrm{CG}}$.



ρ ξ	-1.0	-0.9	-0.8	-0.7	-0.6	-0.5	-0.4	-0.3	-0.2	-0.1	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
-0.1	17	16	15	14	13	11	10	9.2	8.0	6.9	5.7	4.6	3.4	2.3	1.1	0.0	1.1	2.3	3.4	4.6	5.7	6.9	8.0	9.2	10	11	13	14	15	16	17
-0.2	5.8	5.4	5.0	4.6	4.2	3.8	3.5	3.1	2.7	2.3	1.9	1.5	1.2	0.8	0.4	0.0	0.4	0.8	1.2	1.5	1.9	2.3	2.7	3.1	3.5	3.8	4.2	4.6	5.0	5.4	5.8
-0.3	2.8	2.6	2.5	2.3	2.1	1.9	1.7	1.5	1.3	1.1	0.9	0.8	0.6	0.4	0.2	0.0	0.2	0.4	0.6	0.8	0.9	1.1	1.3	1.5	1.7	1.9	2.1	2.3	2.5	2.6	2.8
-0.4	1.7	1.5	1.4	1.3	1.2	1.1	1.0	0.9	0.8	0.7	0.6	0.4	0.3	0.2	0.1	0.0	0.1	0.2	0.3	0.4	0.6	0.7	0.8	0.9	1.0	1.1	1.2	1.3	1.4	1.5	1.7
-0.5	1.1	1.0	0.9	0.9	0.8	0.7	0.6	0.6	0.5	0.4	0.4	0.3	0.2	0.1	0.1	0.0	0.1	0.1	0.2	0.3	0.4	0.4	0.5	0.6	0.6	0.7	0.8	0.9	0.9	1.0	1.1
-0.6	0.7	0.7	0.6	0.6	0.5	0.5	0.4	0.4	0.3	0.3	0.2	0.2	0.1	0.1	0.0	0.0	0.0	0.1	0.1	0.2	0.2	0.3	0.3	0.4	0.4	0.5	0.5	0.6	0.6	0.7	0.7
-0.7	0.5	0.5	0.5	0.4	0.4	0.4	0.3	0.3	0.3	0.2	0.2	0.1	0.1	0.1	0.0	0.0	0.0	0.1	0.1	0.1	0.2	0.2	0.3	0.3	0.3	0.4	0.4	0.4	0.5	0.5	0.5
-0.8	0.4	0.4	0.3	0.3	0.3	0.3	0.2	0.2	0.2	0.2	0.1	0.1	0.1	0.1	0.0	0.0	0.0	0.1	0.1	0.1	0.1	0.2	0.2	0.2	0.2	0.3	0.3	0.3	0.3	0.4	0.4
-0.9	0.3	0.3	0.3	0.2	0.2	0.2	0.2	0.2	0.1	0.1	0.1	0.1	0.1	0.0	0.0	0.0	0.0	0.0	0.1	0.1	0.1	0.1	0.1	0.2	0.2	0.2	0.2	0.2	0.3	0.3	0.3
-1.0	0.2	0.2	0.2	0.2	0.2	0.2	0.1	0.1	0.1	0.1	0.1	0.1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.1	0.1	0.1	0.1	0.1	0.1	0.2	0.2	0.2	0.2	0.2	0.2
-1.1	0.2	0.2	0.2	0.2	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.2	0.2	0.2	0.2
-1.2	0.2	0.2	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.2	0.2
-1.3	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1
-1.4	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1
-1.5	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1
-1.6	0.1	0.1	0.1	0.1	0.1	0.1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.1	0.1	0.1	0.1	0.1	0.1
-1.7	0.1	0.1	0.1	0.1	0.1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.1	0.1	0.1	0.1	0.1
-1.8	0.1	0.1	0.1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.1	0.1	0.1
-1.9	0.1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.1
-2.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0

🟧 $a_0 \leq 0.1$ ▨ $0.1 < a_0 \leq 0.5$ □ $0.5 < a_0 \leq 2$ 🟦 $a_0 > 2$

Figure 8: Values a_0 such that $\mathrm{AREFF}_{\mathrm{CG}(\tau)|\mathrm{FH}} > 1$ for $\tau \in (\tau_0^{\mathrm{CG}} - a_0, \tau_0^{\mathrm{CG}} + a_0)$

For models with $\xi = 1/2$, $\hat{\rho}_{n0}^{\mathrm{FH}}$ beats $\hat{\rho}_{n0}^{\mathrm{FAGH}(\tau)}$ $\left(\hat{\rho}_{n0}^{\mathrm{CG}(\tau)}\right)$ unless we consider the tuning parameter τ_0^{FAGH} $\left(\tau_0^{\mathrm{CG}}\right)$ in (18) ((19)), i.e. $\hat{\rho}_{n0}^{\mathrm{FAGH}(\tau_0)}$ $\left(\hat{\rho}_{n0}^{\mathrm{CG}(\tau_0)}\right)$. For the asymptotic comparison between $\hat{\rho}_{n0}^{\mathrm{FH}}$, $\hat{\rho}_{n0}^{\mathrm{FAGH}(\tau_0^{\mathrm{FAGH}})}$ and $\hat{\rho}_{n0}^{\mathrm{CG}(\tau_0^{\mathrm{CG}})}$ in the region $\xi = 1/2$ we should consider a fourth-order framework, a topic out of the scope of this article. The same comment applies to the asymptotic comparison at optimal levels, in the whole (ξ, ρ)-plane of the estimators $\hat{\rho}_{n0}^{\mathrm{FAGH}(\tau_0^{\mathrm{FAGH}})}$ and $\hat{\rho}_{n0}^{\mathrm{CG}(\tau_0^{\mathrm{CG}})}$.

As mentined before, $\sigma_{_{\mathrm{FAGH}}} > \sigma_{_{\mathrm{FH}}}$, for all $\rho < 0$. For any fixed $\tau \notin (\tau_0^{\mathrm{FAGH}} - a_0^{\mathrm{FAGH}}, \tau_0^{\mathrm{FAGH}} + a_0^{\mathrm{FAGH}})$, with τ_0^{FAGH} and a_0^{FAGH} given in (18) and (21), respectively, $\hat{\rho}_{n0}^{\mathrm{FH}}$ beats $\hat{\rho}_{n0}^{\mathrm{FAGH}(\tau)}$ in a wide region of the (ξ, ρ)-plane, only with the exclusion of values of ρ close to zero and a region around the line $b_{_{\mathrm{FAGH}}} = 0$. See the region in Figure 10 associated with $\tau = 0$, a value commonly used in practice for models with $|\rho| \leq 1$. A similar comment applies to the comparative behaviour of $\hat{\rho}_{n0}^{\mathrm{FH}}$ and $\hat{\rho}_{n0}^{\mathrm{CG}(\tau)}$ for any fixed $\tau \notin (\tau_0^{\mathrm{CG}} - a_0^{\mathrm{FAGH}}, \tau_0^{\mathrm{CG}} + a_0^{\mathrm{CG}})$, with τ_0^{CG} and a_0^{CG} given in (19) and (22),

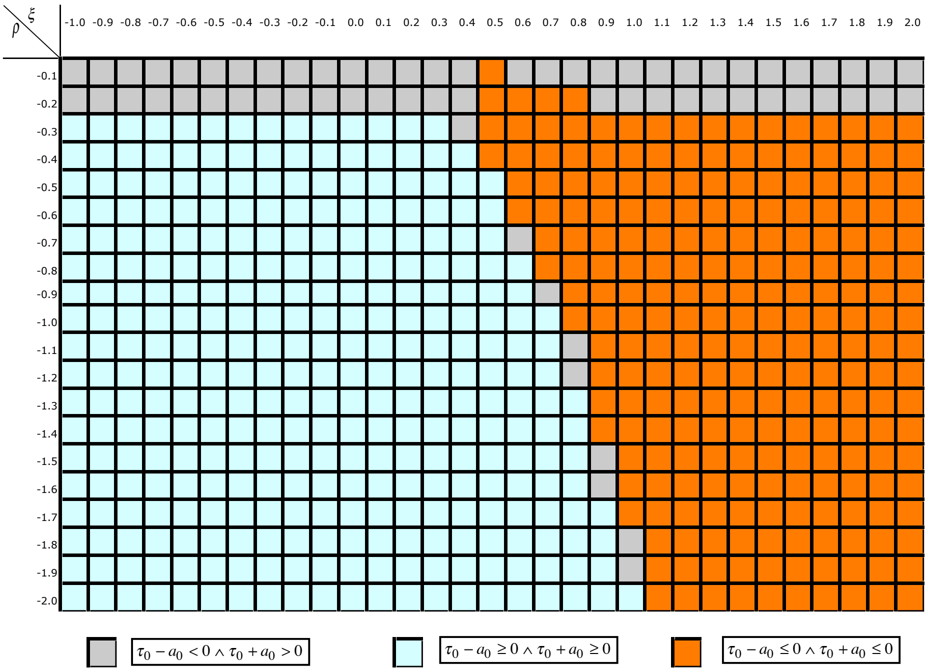
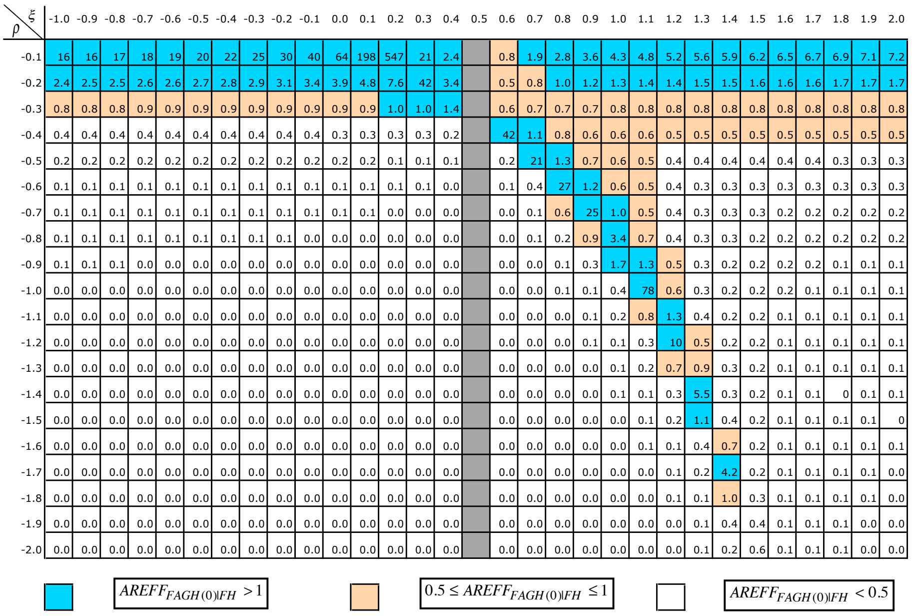


Figure 9: Values the signs of $\tau_0^{\mathrm{CG}} - a_0$ and $\tau_0^{\mathrm{CG}} + a_0$

respectively

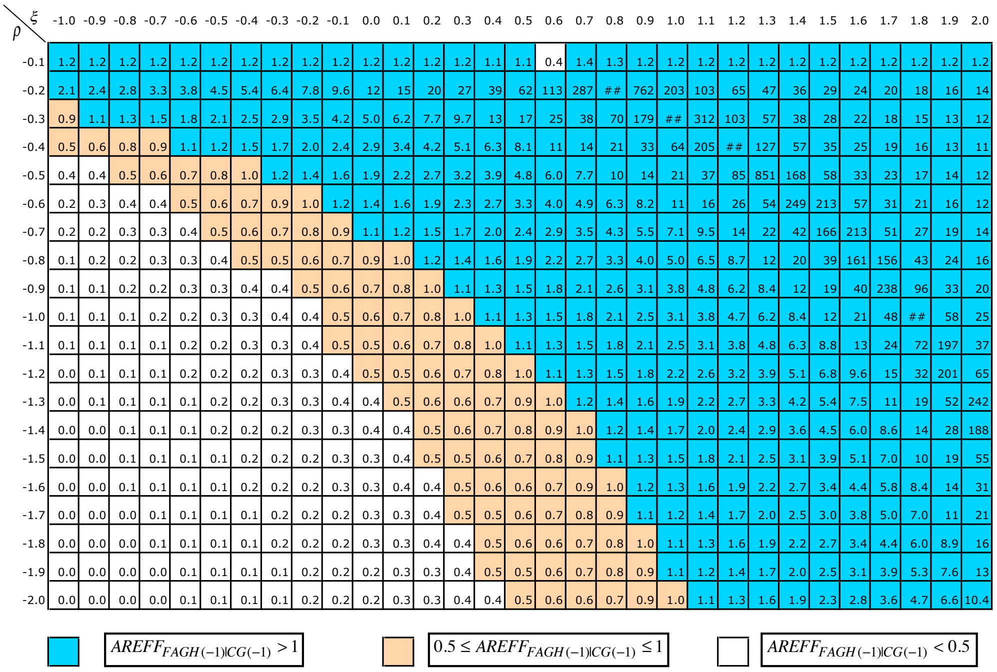


ρ \ ξ	-1.0	-0.9	-0.8	-0.7	-0.6	-0.5	-0.4	-0.3	-0.2	-0.1	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
-0.1	16	16	17	18	19	20	22	25	30	40	64	198	547	21	2.4		0.8	1.9	2.8	3.6	4.3	4.8	5.2	5.6	5.9	6.2	6.5	6.7	6.9	7.1	7.2
-0.2	2.4	2.5	2.5	2.6	2.6	2.7	2.8	2.9	3.1	3.4	3.9	4.8	7.6	42	3.4		0.5	0.8	1.0	1.2	1.3	1.4	1.4	1.5	1.5	1.6	1.6	1.6	1.7	1.7	1.7
-0.3	0.8	0.8	0.8	0.9	0.9	0.9	0.9	0.9	0.9	0.9	0.9	0.9	1.0	1.0	1.4		0.6	0.7	0.7	0.7	0.8	0.8	0.8	0.8	0.8	0.8	0.8	0.8	0.8	0.8	0.8
-0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.3	0.3	0.3	0.3	0.2		42	1.1	0.8	0.6	0.6	0.6	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5
-0.5	0.2	0.2	0.2	0.2	0.2	0.2	0.2	0.2	0.2	0.2	0.2	0.2	0.1	0.1	0.1		0.2	21	1.3	0.7	0.6	0.5	0.4	0.4	0.4	0.4	0.4	0.4	0.3	0.3	0.3
-0.6	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.0		0.1	0.4	27	1.2	0.6	0.5	0.4	0.3	0.3	0.3	0.3	0.3	0.3	0.3	0.3
-0.7	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.0	0.0	0.0		0.0	0.1	0.6	25	1.0	0.5	0.4	0.3	0.3	0.3	0.2	0.2	0.2	0.2	0.2
-0.8	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.0	0.0	0.0	0.0	0.0	0.0		0.0	0.1	0.2	0.9	3.4	0.7	0.4	0.3	0.3	0.2	0.2	0.2	0.2	0.2	0.2
-0.9	0.1	0.1	0.1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0		0.0	0.0	0.1	0.3	1.7	1.3	0.5	0.3	0.2	0.2	0.2	0.2	0.1	0.1	0.1
-1.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0		0.0	0.0	0.1	0.1	0.4	78	0.6	0.3	0.2	0.2	0.2	0.1	0.1	0.1	0.1
-1.1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0		0.0	0.0	0.0	0.1	0.2	0.8	1.3	0.4	0.2	0.2	0.1	0.1	0.1	0.1	0.1
-1.2	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0		0.0	0.0	0.0	0.1	0.1	0.3	10	0.5	0.2	0.2	0.1	0.1	0.1	0.1	0.1
-1.3	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0		0.0	0.0	0.0	0.0	0.1	0.2	0.7	0.9	0.3	0.2	0.1	0.1	0.1	0.1	0.1
-1.4	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0		0.0	0.0	0.0	0.0	0.1	0.1	0.3	5.5	0.3	0.2	0.1	0.1	0	0.1	0.1
-1.5	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0		0.0	0.0	0.0	0.0	0.0	0.1	0.2	1.1	0.4	0.2	0.1	0.1	0.1	0.1	0
-1.6	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0		0.0	0.0	0.0	0.0	0.0	0.1	0.1	0.4	0.7	0.2	0.1	0.1	0.1	0.1	0.1
-1.7	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0		0.0	0.0	0.0	0.0	0.0	0.0	0.1	0.2	4.2	0.2	0.1	0.1	0.1	0.1	0.0
-1.8	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0		0.0	0.0	0.0	0.0	0.0	0.0	0.1	0.1	1.0	0.3	0.1	0.1	0.1	0.1	0.0
-1.9	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0		0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.1	0.4	0.4	0.1	0.1	0.1	0.0	0.0
-2.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0		0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.1	0.2	0.6	0.1	0.1	0.1	0.0	0.0

Figure 10: Values of $\mathrm{AREFF}_{\mathrm{FAGH}(\tau)|\mathrm{FH}}$ for $\tau = 0$

For models with $\xi \neq 1/2$, $\hat{\rho}_{n0}^{\mathrm{FAGH}(\tau_0^{\mathrm{FAGH}})}$, as well as $\hat{\rho}_{n0}^{\mathrm{CG}(\tau_0^{\mathrm{CG}})}$, beat $\hat{\rho}_{n0}^{\mathrm{FH}}$ for all (ξ, ρ). Indeed, the same happens with $\hat{\rho}_{n0}^{\mathrm{FAGH}(\tau)}$, $\tau \in (\tau_0^{\mathrm{FAGH}} - a_0^{\mathrm{FAGH}}, \tau_0^{\mathrm{FAGH}} + a_0^{\mathrm{FAGH}})$, with τ_0^{FAGH} and a_0^{FAGH} given in (18) and (21), respectively, and with $\hat{\rho}_{n0}^{\mathrm{CG}(\tau)}$, $\tau \in (\tau_0^{\mathrm{CG}} - a_0^{\mathrm{CG}}, \tau_0^{\mathrm{CG}} + a_0^{\mathrm{CG}})$, with τ_0^{CG} and a_0^{CG} given in (19) and (22), respectively.

Regarding the asymptotic comparison at optimal levels of $\hat{\rho}_{n0}^{\mathrm{FAGH}(\tau)}$ and $\hat{\rho}_{n0}^{\mathrm{CG}(\tau)}$ for a fixed τ, we next present in Figures 11, 12, 13, 14 and 15, the values of $\mathrm{AREFF}_{\mathrm{FAGH}(\tau)|\mathrm{G}(\tau)}$ for $\tau = -1, -0.5, 0, 0.5$ and 1, respectively. Due to the fact that for the most common models in practice $\xi > 0.5$, we can say that for a fixed τ, $\hat{\rho}_{n0}^{\mathrm{FAGH}(\tau)}$ beats $\hat{\rho}_{n0}^{\mathrm{CG}(\tau)}$, at optimal levels, in a wide region of the (ξ, ρ)-plane.



$\rho \backslash \xi$	-1.0	-0.9	-0.8	-0.7	-0.6	-0.5	-0.4	-0.3	-0.2	-0.1	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
-0.1	1.2	1.2	1.2	1.2	1.2	1.2	1.2	1.2	1.2	1.2	1.2	1.2	1.2	1.2	1.1	1.1	0.4	1.4	1.3	1.2	1.2	1.2	1.2	1.2	1.2	1.2	1.2	1.2	1.2	1.2	1.2
-0.2	2.1	2.4	2.8	3.3	3.8	4.5	5.4	6.4	7.8	9.6	12	15	20	27	39	62	113	287	##	762	203	103	65	47	36	29	24	20	18	16	14
-0.3	0.9	1.1	1.3	1.5	1.8	2.1	2.5	2.9	3.5	4.2	5.0	6.2	7.7	9.7	13	17	25	38	70	179	##	312	103	57	38	28	22	18	15	13	12
-0.4	0.5	0.6	0.8	0.9	1.1	1.2	1.5	1.7	2.0	2.4	2.9	3.4	4.2	5.1	6.3	8.1	11	14	21	33	64	205	##	127	57	35	25	19	16	13	11
-0.5	0.4	0.4	0.5	0.6	0.7	0.8	1.0	1.2	1.4	1.6	1.9	2.2	2.7	3.2	3.9	4.8	6.0	7.7	10	14	21	37	85	851	168	58	33	23	17	14	12
-0.6	0.2	0.3	0.4	0.4	0.5	0.6	0.7	0.9	1.0	1.2	1.4	1.6	1.9	2.3	2.7	3.3	4.0	4.9	6.3	8.2	11	16	26	54	249	213	57	31	21	16	12
-0.7	0.2	0.2	0.3	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.1	1.2	1.5	1.7	2.0	2.4	2.9	3.5	4.3	5.5	7.1	9.5	14	22	42	166	213	51	27	19	14
-0.8	0.1	0.2	0.2	0.3	0.3	0.4	0.5	0.5	0.6	0.7	0.9	1.0	1.2	1.4	1.6	1.9	2.2	2.7	3.3	4.0	5.0	6.5	8.7	12	20	39	161	156	43	24	16
-0.9	0.1	0.1	0.2	0.2	0.3	0.3	0.4	0.4	0.5	0.6	0.7	0.8	1.0	1.1	1.3	1.5	1.8	2.1	2.6	3.1	3.8	4.8	6.2	8.4	12	19	40	238	96	33	20
-1.0	0.1	0.1	0.1	0.2	0.2	0.3	0.3	0.4	0.4	0.5	0.6	0.7	0.8	1.0	1.1	1.3	1.5	1.8	2.1	2.5	3.1	3.8	4.7	6.2	8.4	12	21	48	##	58	25
-1.1	0.1	0.1	0.1	0.1	0.2	0.2	0.3	0.3	0.4	0.5	0.5	0.6	0.7	0.8	1.0	1.1	1.3	1.5	1.8	2.1	2.5	3.1	3.8	4.8	6.3	8.8	13	24	72	197	37
-1.2	0.0	0.1	0.1	0.1	0.2	0.2	0.2	0.3	0.3	0.4	0.5	0.5	0.6	0.7	0.8	1.0	1.1	1.3	1.5	1.8	2.2	2.6	3.2	3.9	5.1	6.8	9.6	15	32	201	65
-1.3	0.0	0.1	0.1	0.1	0.1	0.2	0.2	0.3	0.3	0.4	0.4	0.5	0.6	0.6	0.7	0.9	1.0	1.2	1.4	1.6	1.9	2.2	2.7	3.3	4.2	5.4	7.5	11	19	52	242
-1.4	0.0	0.0	0.1	0.1	0.1	0.2	0.2	0.2	0.3	0.3	0.4	0.4	0.5	0.6	0.7	0.8	0.9	1.0	1.2	1.4	1.7	2.0	2.4	2.9	3.6	4.5	6.0	8.6	14	28	188
-1.5	0.0	0.0	0.1	0.1	0.1	0.1	0.2	0.2	0.2	0.3	0.3	0.4	0.5	0.5	0.6	0.7	0.8	0.9	1.1	1.3	1.5	1.8	2.1	2.5	3.1	3.9	5.1	7.0	10	19	55
-1.6	0.0	0.0	0.1	0.1	0.1	0.1	0.2	0.2	0.2	0.3	0.3	0.4	0.4	0.5	0.6	0.6	0.7	0.9	1.0	1.2	1.3	1.6	1.9	2.2	2.7	3.4	4.4	5.8	8.4	14	31
-1.7	0.0	0.0	0.0	0.1	0.1	0.1	0.1	0.2	0.2	0.2	0.3	0.3	0.4	0.5	0.5	0.6	0.7	0.8	0.9	1.1	1.2	1.4	1.7	2.0	2.5	3.0	3.8	5.0	7.0	11	21
-1.8	0.0	0.0	0.0	0.1	0.1	0.1	0.1	0.2	0.2	0.2	0.3	0.3	0.4	0.4	0.5	0.6	0.6	0.7	0.8	1.0	1.1	1.3	1.6	1.9	2.2	2.7	3.4	4.4	6.0	8.9	16
-1.9	0.0	0.0	0.0	0.0	0.1	0.1	0.1	0.1	0.2	0.2	0.2	0.3	0.3	0.4	0.5	0.5	0.6	0.7	0.8	0.9	1.1	1.2	1.4	1.7	2.0	2.5	3.1	3.9	5.3	7.6	13
-2.0	0.0	0.0	0.0	0.0	0.1	0.1	0.1	0.1	0.2	0.2	0.2	0.3	0.3	0.4	0.4	0.5	0.6	0.6	0.7	0.9	1.0	1.1	1.3	1.6	1.9	2.3	2.8	3.6	4.7	6.6	10.4

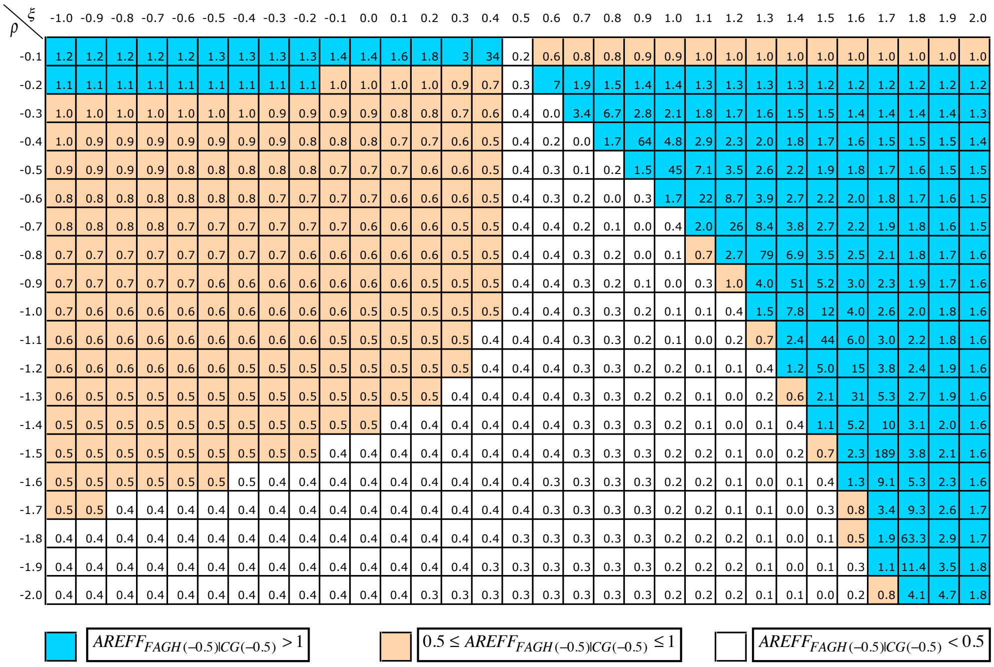
🟦 $AREFF_{FAGH(-1)|CG(-1)} > 1$

🟧 $0.5 \leq AREFF_{FAGH(-1)|CG(-1)} \leq 1$

□ $AREFF_{FAGH(-1)|CG(-1)} < 0.5$

Figure 11: Values of $\mathrm{AREFF}_{\mathrm{FAGH}(\tau)|\mathrm{CG}}$ for $\tau = -1$

In practice, it is however sensible the adequate choice of a value of τ close to either τ_0^{FAGH} or τ_0^{CG}. To make such a choice we can use any heuristic algorithm based on sample stability, like the one used in Caeiro and Gomes (2012b), similar in spirit to the ones in Figueiredo *et al.* (2012) for value-at-risk-estimation and in Gomes *et al.* (2011a,b) for estimation of the extreme value index.



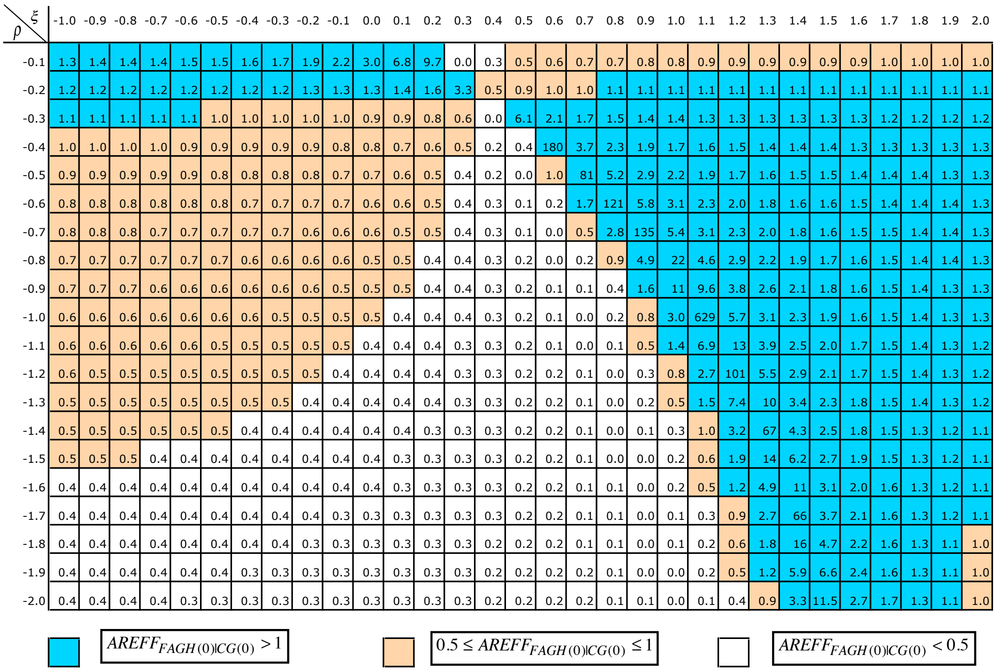
$\rho \backslash \xi$	-1.0	-0.9	-0.8	-0.7	-0.6	-0.5	-0.4	-0.3	-0.2	-0.1	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
-0.1	1.2	1.2	1.2	1.2	1.2	1.3	1.3	1.3	1.3	1.4	1.4	1.6	1.8	3	34	0.2	0.6	0.8	0.8	0.9	0.9	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
-0.2	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.0	1.0	1.0	1.0	0.9	0.7	0.3	7	1.9	1.5	1.4	1.4	1.3	1.3	1.3	1.3	1.2	1.2	1.2	1.2	1.2	1.2
-0.3	1.0	1.0	1.0	1.0	1.0	1.0	1.0	0.9	0.9	0.9	0.9	0.8	0.8	0.7	0.6	0.4	0.0	3.4	6.7	2.8	2.1	1.8	1.7	1.6	1.5	1.5	1.4	1.4	1.4	1.4	1.3
-0.4	1.0	0.9	0.9	0.9	0.9	0.9	0.9	0.9	0.8	0.8	0.8	0.7	0.7	0.6	0.5	0.4	0.2	0.0	1.7	64	4.8	2.9	2.3	2.0	1.8	1.7	1.6	1.5	1.5	1.5	1.4
-0.5	0.9	0.9	0.9	0.9	0.8	0.8	0.8	0.8	0.8	0.7	0.7	0.7	0.6	0.6	0.5	0.4	0.3	0.1	0.2	1.5	45	7.1	3.5	2.6	2.2	1.9	1.8	1.7	1.6	1.5	1.5
-0.6	0.8	0.8	0.8	0.8	0.8	0.8	0.8	0.7	0.7	0.7	0.7	0.6	0.6	0.6	0.5	0.4	0.3	0.2	0.0	0.3	1.7	22	8.7	3.9	2.7	2.2	2.0	1.8	1.7	1.6	1.5
-0.7	0.8	0.8	0.8	0.8	0.7	0.7	0.7	0.7	0.7	0.7	0.6	0.6	0.6	0.5	0.5	0.4	0.4	0.2	0.1	0.0	0.4	2.0	26	8.4	3.8	2.7	2.2	1.9	1.8	1.6	1.5
-0.8	0.7	0.7	0.7	0.7	0.7	0.7	0.7	0.7	0.6	0.6	0.6	0.6	0.6	0.5	0.5	0.4	0.4	0.3	0.2	0.0	0.1	0.7	2.7	79	6.9	3.5	2.5	2.1	1.8	1.7	1.6
-0.9	0.7	0.7	0.7	0.7	0.7	0.6	0.6	0.6	0.6	0.6	0.6	0.6	0.5	0.5	0.5	0.4	0.4	0.3	0.2	0.1	0.0	0.3	1.0	4.0	51	5.2	3.0	2.3	1.9	1.7	1.6
-1.0	0.7	0.6	0.6	0.6	0.6	0.6	0.6	0.6	0.6	0.6	0.5	0.5	0.5	0.5	0.5	0.4	0.4	0.3	0.3	0.2	0.1	0.1	0.4	1.5	7.8	12	4.0	2.6	2.0	1.8	1.6
-1.1	0.6	0.6	0.6	0.6	0.6	0.6	0.6	0.6	0.6	0.5	0.5	0.5	0.5	0.5	0.4	0.4	0.4	0.3	0.3	0.2	0.1	0.0	0.2	0.7	2.4	44	6.0	3.0	2.2	1.8	1.6
-1.2	0.6	0.6	0.6	0.6	0.6	0.6	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.4	0.4	0.4	0.3	0.3	0.2	0.2	0.1	0.1	0.4	1.2	5.0	15	3.8	2.4	1.9	1.6
-1.3	0.6	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.4	0.4	0.4	0.4	0.3	0.3	0.2	0.2	0.1	0.0	0.2	0.6	2.1	31	5.3	2.7	1.9	1.6
-1.4	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.4	0.4	0.4	0.4	0.4	0.4	0.3	0.3	0.3	0.2	0.1	0.0	0.1	0.4	1.1	5.2	10	3.1	2.0	1.6
-1.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.3	0.3	0.3	0.2	0.2	0.1	0.0	0.2	0.7	2.3	189	3.8	2.1	1.6
-1.6	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.3	0.3	0.3	0.3	0.2	0.2	0.1	0.0	0.1	0.4	1.3	9.1	5.3	2.3	1.6
-1.7	0.5	0.5	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.3	0.3	0.3	0.3	0.2	0.2	0.1	0.1	0.0	0.3	0.8	3.4	9.3	2.6	1.7
-1.8	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.3	0.3	0.3	0.3	0.3	0.2	0.2	0.2	0.1	0.0	0.1	0.5	1.9	63.3	2.9	1.7
-1.9	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.3	0.3	0.3	0.3	0.3	0.3	0.2	0.2	0.2	0.1	0.0	0.1	0.3	1.1	11.4	3.5	1.8
-2.0	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.3	0.3	0.3	0.3	0.3	0.3	0.3	0.3	0.2	0.2	0.2	0.1	0.1	0.0	0.2	0.8	4.1	4.7	1.8

🟦 $AREFF_{FAGH(-0.5)|CG(-0.5)} > 1$

🟧 $0.5 \leq AREFF_{FAGH(-0.5)|CG(-0.5)} \leq 1$

☐ $AREFF_{FAGH(-0.5)|CG(-0.5)} < 0.5$

Figure 12: Values of $\mathrm{AREFF}_{\mathrm{FAGH}(\tau)|\mathrm{CG}}$ for $\tau = -0.5$



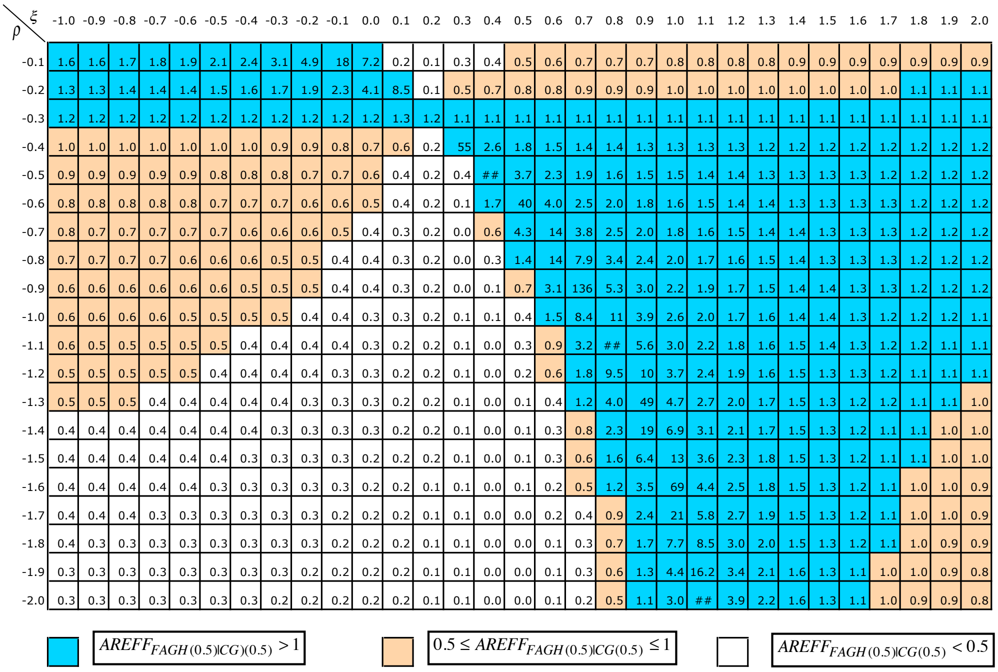
$\rho \backslash \xi$	-1.0	-0.9	-0.8	-0.7	-0.6	-0.5	-0.4	-0.3	-0.2	-0.1	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
-0.1	1.3	1.4	1.4	1.4	1.5	1.5	1.6	1.7	1.9	2.2	3.0	6.8	9.7	0.0	0.3	0.5	0.6	0.7	0.7	0.8	0.8	0.9	0.9	0.9	0.9	0.9	0.9	1.0	1.0	1.0	1.0
-0.2	1.2	1.2	1.2	1.2	1.2	1.2	1.2	1.2	1.3	1.3	1.3	1.4	1.6	3.3	0.5	0.9	1.0	1.0	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1
-0.3	1.1	1.1	1.1	1.1	1.1	1.0	1.0	1.0	1.0	1.0	0.9	0.9	0.8	0.6	0.0	6.1	2.1	1.7	1.5	1.4	1.4	1.3	1.3	1.3	1.3	1.3	1.3	1.2	1.2	1.2	1.2
-0.4	1.0	1.0	1.0	1.0	0.9	0.9	0.9	0.9	0.9	0.8	0.8	0.7	0.6	0.5	0.2	0.4	180	3.7	2.3	1.9	1.7	1.6	1.5	1.4	1.4	1.4	1.3	1.3	1.3	1.3	1.3
-0.5	0.9	0.9	0.9	0.9	0.9	0.8	0.8	0.8	0.8	0.7	0.7	0.6	0.5	0.4	0.2	0.0	1.0	81	5.2	2.9	2.2	1.9	1.7	1.6	1.5	1.5	1.4	1.4	1.4	1.3	1.3
-0.6	0.8	0.8	0.8	0.8	0.8	0.8	0.7	0.7	0.7	0.7	0.6	0.6	0.5	0.4	0.3	0.1	0.2	1.7	121	5.8	3.1	2.3	2.0	1.8	1.6	1.6	1.5	1.4	1.4	1.4	1.3
-0.7	0.8	0.8	0.8	0.7	0.7	0.7	0.7	0.7	0.6	0.6	0.6	0.5	0.5	0.4	0.3	0.1	0.0	0.5	2.8	135	5.4	3.1	2.3	2.0	1.8	1.6	1.5	1.5	1.4	1.4	1.3
-0.8	0.7	0.7	0.7	0.7	0.7	0.7	0.6	0.6	0.6	0.6	0.5	0.5	0.4	0.4	0.3	0.2	0.0	0.2	0.9	4.9	22	4.6	2.9	2.2	1.9	1.7	1.6	1.5	1.4	1.4	1.3
-0.9	0.7	0.7	0.7	0.6	0.6	0.6	0.6	0.6	0.6	0.5	0.5	0.5	0.4	0.4	0.3	0.2	0.1	0.1	0.4	1.6	11	9.6	3.8	2.6	2.1	1.8	1.6	1.5	1.4	1.3	1.3
-1.0	0.6	0.6	0.6	0.6	0.6	0.6	0.6	0.5	0.5	0.5	0.5	0.4	0.4	0.4	0.3	0.2	0.1	0.0	0.2	0.8	3.0	629	5.7	3.1	2.3	1.9	1.6	1.5	1.4	1.3	1.3
-1.1	0.6	0.6	0.6	0.6	0.6	0.5	0.5	0.5	0.5	0.5	0.4	0.4	0.4	0.3	0.3	0.2	0.1	0.0	0.1	0.5	1.4	6.9	13	3.9	2.5	2.0	1.7	1.5	1.4	1.3	1.2
-1.2	0.6	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.4	0.4	0.4	0.4	0.3	0.3	0.2	0.2	0.1	0.0	0.3	0.8	2.7	101	5.5	2.9	2.1	1.7	1.5	1.4	1.3	1.2
-1.3	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.4	0.4	0.4	0.4	0.4	0.3	0.3	0.2	0.2	0.1	0.0	0.2	0.5	1.5	7.4	10	3.4	2.3	1.8	1.5	1.4	1.3	1.2
-1.4	0.5	0.5	0.5	0.5	0.5	0.5	0.4	0.4	0.4	0.4	0.4	0.4	0.3	0.3	0.3	0.2	0.2	0.1	0.0	0.1	0.3	1.0	3.2	67	4.3	2.5	1.8	1.5	1.3	1.2	1.1
-1.5	0.5	0.5	0.5	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.3	0.3	0.3	0.3	0.2	0.2	0.1	0.0	0.0	0.2	0.6	1.9	14	6.2	2.7	1.9	1.5	1.3	1.2	1.1
-1.6	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.3	0.3	0.3	0.3	0.2	0.2	0.1	0.1	0.0	0.2	0.5	1.2	4.9	11	3.1	2.0	1.6	1.3	1.2	1.1
-1.7	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.3	0.3	0.3	0.3	0.3	0.3	0.2	0.2	0.1	0.1	0.0	0.1	0.3	0.9	2.7	66	3.7	2.1	1.6	1.3	1.2	1.1
-1.8	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.3	0.3	0.3	0.3	0.3	0.3	0.2	0.2	0.2	0.1	0.1	0.0	0.1	0.2	0.6	1.8	16	4.7	2.2	1.6	1.3	1.1	1.0
-1.9	0.4	0.4	0.4	0.4	0.4	0.4	0.3	0.3	0.3	0.3	0.3	0.3	0.3	0.3	0.2	0.2	0.2	0.2	0.1	0.0	0.0	0.2	0.5	1.2	5.9	6.6	2.4	1.6	1.3	1.1	1.0
-2.0	0.4	0.4	0.4	0.4	0.3	0.3	0.3	0.3	0.3	0.3	0.3	0.3	0.3	0.3	0.2	0.2	0.2	0.2	0.1	0.1	0.0	0.1	0.4	0.9	3.3	11.5	2.7	1.7	1.3	1.1	1.0

🟦 $AREFF_{FAGH(0)|CG(0)} > 1$

🟧 $0.5 \leq AREFF_{FAGH(0)|CG(0)} \leq 1$

☐ $AREFF_{FAGH(0)|CG(0)} < 0.5$

Figure 13: Values of $\mathrm{AREFF}_{\mathrm{FAGH}(\tau)|\mathrm{CG}}$ for $\tau = 0$



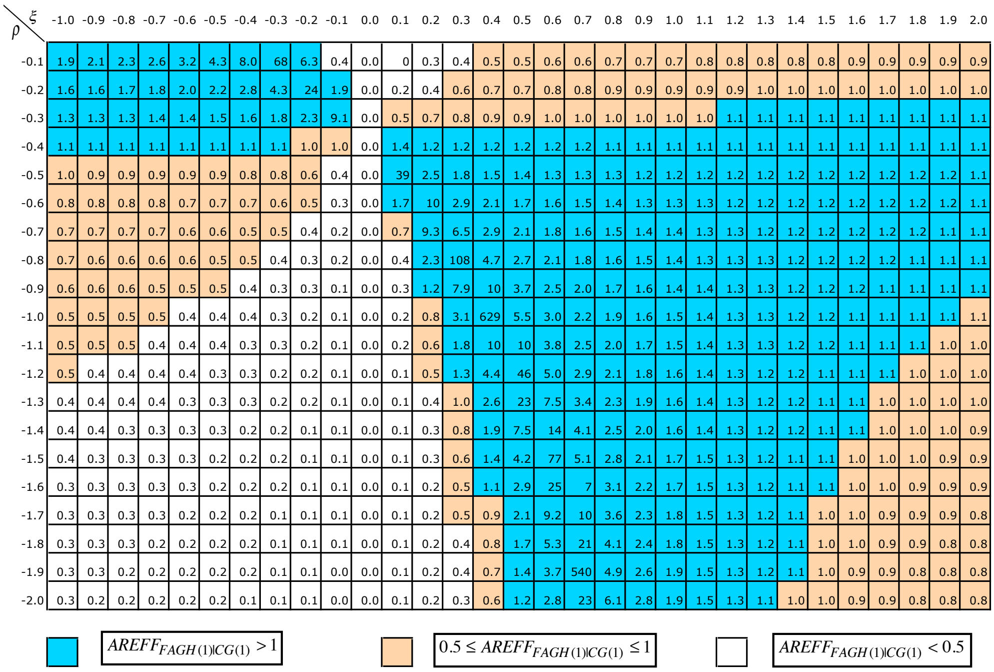
ρ \ ξ	-1.0	-0.9	-0.8	-0.7	-0.6	-0.5	-0.4	-0.3	-0.2	-0.1	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
-0.1	1.6	1.6	1.7	1.8	1.9	2.1	2.4	3.1	4.9	18	7.2	0.2	0.1	0.3	0.4	0.5	0.6	0.7	0.7	0.7	0.8	0.8	0.8	0.8	0.9	0.9	0.9	0.9	0.9	0.9	0.9
-0.2	1.3	1.3	1.4	1.4	1.4	1.5	1.6	1.7	1.9	2.3	4.1	8.5	0.1	0.5	0.7	0.8	0.8	0.9	0.9	0.9	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.1	1.1	1.1
-0.3	1.2	1.2	1.2	1.2	1.2	1.2	1.2	1.2	1.2	1.2	1.2	1.3	1.2	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1
-0.4	1.0	1.0	1.0	1.0	1.0	1.0	1.0	0.9	0.9	0.8	0.7	0.6	0.2	55	2.6	1.8	1.5	1.4	1.4	1.3	1.3	1.3	1.3	1.2	1.2	1.2	1.2	1.2	1.2	1.2	1.2
-0.5	0.9	0.9	0.9	0.9	0.9	0.8	0.8	0.8	0.7	0.7	0.6	0.4	0.2	0.4	##	3.7	2.3	1.9	1.6	1.5	1.5	1.4	1.4	1.3	1.3	1.3	1.3	1.2	1.2	1.2	1.2
-0.6	0.8	0.8	0.8	0.8	0.8	0.7	0.7	0.7	0.6	0.6	0.5	0.4	0.2	0.1	1.7	40	4.0	2.5	2.0	1.8	1.6	1.5	1.4	1.4	1.3	1.3	1.3	1.3	1.2	1.2	1.2
-0.7	0.8	0.7	0.7	0.7	0.7	0.7	0.6	0.6	0.6	0.5	0.4	0.3	0.2	0.0	0.6	4.3	14	3.8	2.5	2.0	1.8	1.6	1.5	1.4	1.4	1.3	1.3	1.3	1.2	1.2	1.2
-0.8	0.7	0.7	0.7	0.7	0.6	0.6	0.6	0.5	0.5	0.4	0.4	0.3	0.2	0.0	0.3	1.4	14	7.9	3.4	2.4	2.0	1.7	1.6	1.5	1.4	1.3	1.3	1.3	1.2	1.2	1.2
-0.9	0.6	0.6	0.6	0.6	0.6	0.6	0.5	0.5	0.5	0.4	0.4	0.3	0.2	0.0	0.1	0.7	3.1	136	5.3	3.0	2.2	1.9	1.7	1.5	1.4	1.4	1.3	1.3	1.2	1.2	1.2
-1.0	0.6	0.6	0.6	0.6	0.5	0.5	0.5	0.5	0.4	0.4	0.3	0.3	0.2	0.1	0.1	0.4	1.5	8.4	11	3.9	2.6	2.0	1.7	1.6	1.4	1.4	1.3	1.2	1.2	1.2	1.1
-1.1	0.6	0.5	0.5	0.5	0.5	0.5	0.4	0.4	0.4	0.4	0.3	0.2	0.2	0.1	0.0	0.3	0.9	3.2	##	5.6	3.0	2.2	1.8	1.6	1.5	1.4	1.3	1.2	1.2	1.1	1.1
-1.2	0.5	0.5	0.5	0.5	0.5	0.4	0.4	0.4	0.4	0.3	0.3	0.2	0.2	0.1	0.0	0.2	0.6	1.8	9.5	10	3.7	2.4	1.9	1.6	1.5	1.3	1.3	1.2	1.1	1.1	1.1
-1.3	0.5	0.5	0.5	0.4	0.4	0.4	0.4	0.4	0.3	0.3	0.3	0.2	0.2	0.1	0.0	0.1	0.4	1.2	4.0	49	4.7	2.7	2.0	1.7	1.5	1.3	1.2	1.2	1.1	1.1	1.0
-1.4	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.3	0.3	0.3	0.3	0.2	0.2	0.1	0.0	0.1	0.3	0.8	2.3	19	6.9	3.1	2.1	1.7	1.5	1.3	1.2	1.1	1.1	1.0	1.0
-1.5	0.4	0.4	0.4	0.4	0.4	0.4	0.3	0.3	0.3	0.3	0.2	0.2	0.2	0.1	0.0	0.1	0.3	0.6	1.6	6.4	13	3.6	2.3	1.8	1.5	1.3	1.2	1.1	1.1	1.0	1.0
-1.6	0.4	0.4	0.4	0.4	0.4	0.3	0.3	0.3	0.3	0.3	0.2	0.2	0.1	0.1	0.0	0.1	0.2	0.5	1.2	3.5	69	4.4	2.5	1.8	1.5	1.3	1.2	1.1	1.0	1.0	0.9
-1.7	0.4	0.4	0.4	0.3	0.3	0.3	0.3	0.3	0.3	0.2	0.2	0.2	0.1	0.1	0.0	0.0	0.2	0.4	0.9	2.4	21	5.8	2.7	1.9	1.5	1.3	1.2	1.1	1.0	1.0	0.9
-1.8	0.4	0.3	0.3	0.3	0.3	0.3	0.3	0.3	0.3	0.2	0.2	0.2	0.1	0.1	0.0	0.0	0.1	0.3	0.7	1.7	7.7	8.5	3.0	2.0	1.5	1.3	1.2	1.1	1.0	0.9	0.9
-1.9	0.3	0.3	0.3	0.3	0.3	0.3	0.3	0.3	0.2	0.2	0.2	0.2	0.1	0.1	0.0	0.0	0.1	0.3	0.6	1.3	4.4	16.2	3.4	2.1	1.6	1.3	1.1	1.0	1.0	0.9	0.8
-2.0	0.3	0.3	0.3	0.3	0.3	0.3	0.3	0.2	0.2	0.2	0.2	0.2	0.1	0.1	0.0	0.0	0.1	0.2	0.5	1.1	3.0	##	3.9	2.2	1.6	1.3	1.1	1.0	0.9	0.9	0.8

🟦 $AREFF_{FAGH(0.5)|CG)(0.5)} > 1$

🟧 $0.5 \leq AREFF_{FAGH(0.5)|CG(0.5)} \leq 1$

☐ $AREFF_{FAGH(0.5)|CG(0.5)} < 0.5$

Figure 14: Values of $\mathrm{AREFF}_{\mathrm{FAGH}(\tau)|\mathrm{CG}}$ for $\tau = 0.5$



ρ \ ξ	-1.0	-0.9	-0.8	-0.7	-0.6	-0.5	-0.4	-0.3	-0.2	-0.1	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
-0.1	1.9	2.1	2.3	2.6	3.2	4.3	8.0	68	6.3	0.4	0.0	0	0.3	0.4	0.5	0.5	0.6	0.6	0.7	0.7	0.7	0.8	0.8	0.8	0.8	0.8	0.9	0.9	0.9	0.9	0.9
-0.2	1.6	1.6	1.7	1.8	2.0	2.2	2.8	4.3	24	1.9	0.0	0.2	0.4	0.6	0.7	0.7	0.8	0.8	0.9	0.9	0.9	0.9	0.9	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
-0.3	1.3	1.3	1.3	1.4	1.4	1.5	1.6	1.8	2.3	9.1	0.0	0.5	0.7	0.8	0.9	0.9	1.0	1.0	1.0	1.0	1.0	1.0	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1
-0.4	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.0	1.0	0.0	1.4	1.2	1.2	1.2	1.2	1.2	1.2	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1
-0.5	1.0	0.9	0.9	0.9	0.9	0.9	0.8	0.8	0.6	0.4	0.0	39	2.5	1.8	1.5	1.4	1.3	1.3	1.3	1.2	1.2	1.2	1.2	1.2	1.2	1.2	1.2	1.2	1.2	1.2	1.1
-0.6	0.8	0.8	0.8	0.8	0.7	0.7	0.7	0.6	0.5	0.3	0.0	1.7	10	2.9	2.1	1.7	1.6	1.5	1.4	1.3	1.3	1.3	1.2	1.2	1.2	1.2	1.2	1.2	1.2	1.2	1.1
-0.7	0.7	0.7	0.7	0.7	0.6	0.6	0.5	0.5	0.4	0.2	0.0	0.7	9.3	6.5	2.9	2.1	1.8	1.6	1.5	1.4	1.4	1.3	1.3	1.2	1.2	1.2	1.2	1.2	1.2	1.1	1.1
-0.8	0.7	0.6	0.6	0.6	0.6	0.5	0.5	0.4	0.3	0.2	0.0	0.4	2.3	108	4.7	2.7	2.1	1.8	1.6	1.5	1.4	1.3	1.3	1.3	1.2	1.2	1.2	1.2	1.1	1.1	1.1
-0.9	0.6	0.6	0.6	0.5	0.5	0.5	0.4	0.3	0.3	0.1	0.0	0.3	1.2	7.9	10	3.7	2.5	2.0	1.7	1.6	1.4	1.4	1.3	1.3	1.2	1.2	1.2	1.1	1.1	1.1	1.1
-1.0	0.5	0.5	0.5	0.5	0.4	0.4	0.4	0.3	0.2	0.1	0.0	0.2	0.8	3.1	629	5.5	3.0	2.2	1.9	1.6	1.5	1.4	1.3	1.3	1.2	1.2	1.1	1.1	1.1	1.1	1.1
-1.1	0.5	0.5	0.5	0.4	0.4	0.4	0.3	0.3	0.2	0.1	0.0	0.2	0.6	1.8	10	10	3.8	2.5	2.0	1.7	1.5	1.4	1.3	1.3	1.2	1.2	1.1	1.1	1.1	1.0	1.0
-1.2	0.5	0.4	0.4	0.4	0.4	0.3	0.3	0.2	0.2	0.1	0.0	0.1	0.5	1.3	4.4	46	5.0	2.9	2.1	1.8	1.6	1.4	1.3	1.2	1.2	1.1	1.1	1.1	1.0	1.0	1.0
-1.3	0.4	0.4	0.4	0.4	0.3	0.3	0.3	0.2	0.2	0.1	0.0	0.1	0.4	1.0	2.6	23	7.5	3.4	2.3	1.9	1.6	1.4	1.3	1.2	1.2	1.1	1.1	1.0	1.0	1.0	1.0
-1.4	0.4	0.4	0.3	0.3	0.3	0.3	0.2	0.2	0.1	0.1	0.0	0.1	0.3	0.8	1.9	7.5	14	4.1	2.5	2.0	1.6	1.4	1.3	1.2	1.2	1.1	1.1	1.0	1.0	1.0	0.9
-1.5	0.4	0.3	0.3	0.3	0.3	0.2	0.2	0.2	0.1	0.1	0.0	0.1	0.3	0.6	1.4	4.2	77	5.1	2.8	2.1	1.7	1.5	1.3	1.2	1.1	1.1	1.0	1.0	1.0	0.9	0.9
-1.6	0.3	0.3	0.3	0.3	0.3	0.2	0.2	0.2	0.1	0.1	0.0	0.1	0.2	0.5	1.1	2.9	25	7	3.1	2.2	1.7	1.5	1.3	1.2	1.1	1.1	1.0	1.0	0.9	0.9	0.9
-1.7	0.3	0.3	0.3	0.3	0.2	0.2	0.2	0.1	0.1	0.1	0.0	0.1	0.2	0.5	0.9	2.1	9.2	10	3.6	2.3	1.8	1.5	1.3	1.2	1.1	1.0	1.0	0.9	0.9	0.9	0.8
-1.8	0.3	0.3	0.3	0.2	0.2	0.2	0.2	0.1	0.1	0.1	0.0	0.1	0.2	0.4	0.8	1.7	5.3	21	4.1	2.4	1.8	1.5	1.3	1.2	1.1	1.0	1.0	0.9	0.9	0.8	0.8
-1.9	0.3	0.3	0.2	0.2	0.2	0.2	0.2	0.1	0.1	0.0	0.0	0.1	0.2	0.4	0.7	1.4	3.7	540	4.9	2.6	1.9	1.5	1.3	1.2	1.1	1.0	0.9	0.9	0.8	0.8	0.8
-2.0	0.3	0.2	0.2	0.2	0.2	0.2	0.1	0.1	0.1	0.0	0.0	0.1	0.2	0.3	0.6	1.2	2.8	23	6.1	2.8	1.9	1.5	1.3	1.1	1.0	1.0	0.9	0.9	0.8	0.8	0.8

🟦 $AREFF_{FAGH(1)|CG(1)} > 1$

🟧 $0.5 \leq AREFF_{FAGH(1)|CG(1)} \leq 1$

☐ $AREFF_{FAGH(1)|CG(1)} < 0.5$

Figure 15: Values of $\mathrm{AREFF}_{\mathrm{FAGH}(\tau)|\mathrm{CG}}$ for $\tau = 1$

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