Limiting crossing probabilities of random fields

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Abstract: Random fields on $\mathbb{Z}_+^2$, with long range weak dependence for each coordinate at a time, usually present clustering of high values. For each one of the eight directions in $\mathbb{Z}_+^2$, we can restrict the local occurrence of two or more crossings of high levels. These smooth oscillation conditions enable to compute a clustering measure, called extremal index, from the limiting mean number of crossings. In fact, only four directions must be inspected since for opposite directions we find the same local path crossing behaviour and the same limiting mean number of crossings. The general theory is illustrated with several 1-dependent non stationary random fields.

Key words: random field, dependence, non-stationarity, extremal index

1 Introduction

Let $\mathbf{X} = \{X_\mathbf{n}\}_{\mathbf{n}\geq\mathbf{1}}$ be a random field on $\mathbb{Z}_+^d$, where $\mathbb{Z}_+$ is the set of all positive integers and $d \geq 2$. We shall consider the conditions and results for $d = 2$ since it is notationally simplest and the proofs for higher dimensions follow analogous arguments.

For a family of real levels $\{u_{\mathbf{n},\mathbf{i}} : \mathbf{i} \leq \mathbf{n}\}_{\mathbf{n}\geq\mathbf{1}}$ and a subset $\mathbf{I}$ of the rectangle of points $\mathbf{R_n} = \{1,\ldots,n_1\}\times\{1,\ldots,n_2\}$, we will denote the event $\{\bigcap_{\mathbf{i}\in\mathbf{I}} X_\mathbf{i} \leq u_{\mathbf{n},\mathbf{i}}\}$ by $\{M_\mathbf{n}(\mathbf{I}) \leq u\}$ or simply by $\{M_\mathbf{n} \leq u\}$ when $\mathbf{I} = \mathbf{R_n}$.

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For each $i = 1, 2$, we say the pair $\mathbf{I} \subset \mathbb{Z}_+^2$ and $\mathbf{J} \subset \mathbb{Z}_+^2$ is in $\mathcal{S}_i(l)$ if the distance between $\Pi_i(\mathbf{I})$ and $\Pi_i(\mathbf{J})$ is great or equal to l, where Π_i, $i = 1, 2$ denote the cartesian projections. The distance $d(\mathbf{I}, \mathbf{J})$ between sets $\mathbf{I}$ and $\mathbf{J}$ of $\mathbb{Z}_+^d$, $d \geq 1$, is the minimum of distances $d(\mathbf{i}, \mathbf{j}) = max\{|i_s - j_s|, s = 1, \ldots, d\}$, $\mathbf{i} \in \mathbf{I}$ and $\mathbf{j} \in \mathbf{J}$.

Suppose that $\mathbf{X}$ satisfies a coordinatewise-mixing type condition as the $\Delta(u_{\mathbf{n}})$-condition introduced in [9], which exploits the past and future separation one coordinate at a time. Let $\mathcal{F}$ be a family of indexes sets in $\mathbf{R_n}$. We shall assume that there exist sequences of integer valued constants $\{k_{n_i}\}$, $\{l_{n_i}\}$, $i = 1, 2$, such that, as $\mathbf{n} = (n_1, n_2) \to \boldsymbol{\infty}$, we have

$$(k_{n_1}, k_{n_2}) \to \boldsymbol{\infty}, \; (\frac{k_{n_1} l_{n_1}}{n_1}, \frac{k_{n_2} l_{n_2}}{n_2}) \to \mathbf{0} \tag{1.1}$$

and $(k_{n_1}\Delta_1, k_{n_1}k_{n_2}\Delta_2) \to \mathbf{0}$, where Δ_i are the components of the mixing coefficient defined as follows:

$$\Delta_1 = sup|P(M_{\mathbf{n}}(\mathbf{I}_1) \leq u, M_{\mathbf{n}}(\mathbf{I}_2) \leq u) - P(M_{\mathbf{n}}(\mathbf{I}_1) \leq u)P(M_{\mathbf{n}}(\mathbf{I}_2) \leq u)|,$$

where the supremum is taken over pairs $\mathbf{I}_1$ and $\mathbf{I}_2$ in $\mathcal{S}_1(l_{n_1}) \cap \mathcal{F}$,

$$\Delta_2 = sup|P(M_{\mathbf{n}}(\mathbf{I}_1) \leq u, M_{\mathbf{n}}(\mathbf{I}_2) \leq u) - P(M_{\mathbf{n}}(\mathbf{I}_1) \leq u)P(M_{\mathbf{n}}(\mathbf{I}_2) \leq u)|,$$

where the supremum is taken over pairs $\mathbf{I}_1$ and $\mathbf{I}_2$ in $\mathcal{S}_2(l_{n_2}) \cap \mathcal{F}$. We say then $\mathbf{X}$ satisfies the $D(u_{\mathbf{n},\mathbf{i}})$ condition over $\mathcal{F}$.

In fact, we could consider a slightly weaker condition, as in [9], if we where concerned only with stationary random fields.

We prove, in the next section, that the maxima over disjoint rectangles behave asymptotically as independent maxima.

Restrictions on clustering of high values for stationary and non-stationary time series have been considered in the form of D' condition introduced in [6] (see also [4]). In [11] we introduced a D' condition tailored for random fields not necessarily stationary. That condition and the coordinatewise long range dependence lead to a Poisson approximation for the probability of no exceedances over $\mathbf{R_n}$ and the result can be applied to nonstationary Gaussian random fields.

Here, in section 3, we discuss the behaviour of the maxima when clustering of high values of $\mathbf{X}$ is allowed but we restrict the local occurrence of two or more crossings of the high levels $u_{\mathbf{n},\mathbf{i}}$. For each one of the eight directions in $\mathbb{Z}_+^2$, we can restrict the local occurrence of two or more crossings. These smooth oscillation conditions enable to compute a clustering measure, called extremal index, from the limiting mean number of crossings. We prove that, in fact, only four directions must be inspected since for opposite directions we find the same local path crossing behaviour and the same limiting mean number of crossings.

We illustrate these results with several 1-dependent non stationary random fields, which satisfy different local crossing conditions.

2 Asymptotic independence of maxima

Under the coordinatewise-mixing $D(u_{\mathbf{n},\mathbf{i}})$-condition we have the asymptotic independence for maxima over disjoint rectangles of indexes. In the following $\overline{F}_{max}$ denotes $max\{P(X_{\mathbf{i}} > u_{\mathbf{n},\mathbf{i}}) : \mathbf{i} \leq \mathbf{n}\}$.

Proposition 2.1 *Suppose that the random field* $\mathbf{X}$ *satisfies the condition* $D(u_{\mathbf{n},\mathbf{i}})$ *over* $\mathcal{F}$ *such that* $(\mathbf{I} \subset \mathbf{J} \wedge \mathbf{J} \in \mathcal{F}) \Rightarrow \mathbf{J} \in \mathcal{F}$ *and for* $\{u_{\mathbf{n},\mathbf{i}} : \mathbf{i} \leq \mathbf{n}\}_{\mathbf{n}\geq\mathbf{1}}$ *such that*

$$\{n_1 n_2 \overline{F}_{max}\}_{\mathbf{n}\geq\mathbf{1}} \text{ is bounded.} \tag{2.2}$$

If $\mathbf{V}_{r,p} = I_r \times J_{r,p}$, $r = 1, \ldots, k_{n_1}, p = 1, \ldots, k_{n_2}$, *are disjoint rectangles in* $\mathcal{F}$, *then, as* $\mathbf{n} \to \infty$,

$$P\Big(\bigcap_{r,p} M_{\mathbf{n}}(\mathbf{V}_{r,p}) \leq u\Big) - \prod_{r,p} P(M_{\mathbf{n}}(\mathbf{V}_{r,p}) \leq u) \to 0.$$

Proof: From (1.1) and (2.2), for the purpose of the above convergence we can assume that $\Pi_1(\mathbf{V}_{r,p}) > l_{n_1}$ or $\Pi_2(\mathbf{V}_{r,p}) > l_{n_2}$. If all the pairs of rectangles $\mathbf{V}_{r,p}$ are in $\mathcal{S}_1(l_{n_1}) \cup \mathcal{S}_2(l_{n_2})$ then the result follows inductively from the condition $D(u_{\mathbf{n},\mathbf{i}})$. On the contrary, we can eliminate l_{n_1} columns or l_{n_2} rows of indexes in $\mathbf{V}_{r,p}$ in order to obtain $\mathbf{V}^*_{r,p} \subset \mathbf{V}_{r,p}$, $r = 1, \ldots, k_{n_1}, p = 1, \ldots, k_{n_2}$, to which we can apply inductively the condition $D(u_{\mathbf{n},\mathbf{i}})$. □

3 Limiting crossing probabilities

We discuss now the limiting distribution of maximum when, in addition to coordinatewise-mixing condition, we restrict the local path behaviour with respect to the number of crossings of the high levels $u_{\mathbf{n},\mathbf{i}}$.

Since the natural notion of crossing at $\mathbf{i} = (i_1, i_2)$ would get in consideration the values of the random field over the eight neighbours of $\mathbf{i}$, *id est*, over the points $\mathbf{j}$ such that $d(\mathbf{i},\mathbf{j}) = 1$, then by taking $\beta(\{\mathbf{i}\}) = \{\mathbf{j} : d(\mathbf{i},\mathbf{j}) = 1\}$, we say that $\mathbf{X}$ has a crossing at $\mathbf{i}$ if occurs the event

$$B_{\mathbf{i},\mathbf{n}} = \{X_{\mathbf{i}} \leq u_{\mathbf{n},\mathbf{i}}, \bigcup_{\mathbf{j}\in\beta(\{\mathbf{i}\})} X_{\mathbf{j}} > u_{\mathbf{n},\mathbf{j}}\}.$$

Using the ideas of [8], in combination with [5] and [2], to avoid clustering of crossings by a nonstationary random field, we would assume, for each rectangle $\mathbf{I}$ satisfying

$$\sum_{\mathbf{i}\in\mathbf{I}} P(B_{\mathbf{i},\mathbf{n}}) \leq \frac{1}{k_{n_1}k_{n_2}} \sum_{\mathbf{i}\leq\mathbf{n}} P(B_{\mathbf{i},\mathbf{n}}),$$

that it holds

$$k_{n_1}k_{n_2} \sum_{\mathbf{i},\mathbf{j}\in\mathbf{I}} P(B_{\mathbf{i},\mathbf{n}}, B_{\mathbf{j},\mathbf{n}}) \xrightarrow[\mathbf{n}\to\infty]{} 0.$$

However, we verified that an i.i.d. random field doesn't satisfy the previous condition for normalized levels $\{u_{\mathbf{n}}\}_{\mathbf{n}\geq\mathbf{1}}$ such that

$$n_1 n_2 P(X_{\mathbf{1}} > u_{\mathbf{n}}) \xrightarrow[\mathbf{n}\to\infty]{} \tau. \tag{3.3}$$

For each $\mathbf{i} = (i_1, i_2) \in \mathbb{Z}^2_+$, let $b_s(\mathbf{i})$, $s = 1, \ldots, 8$, be the neighbours of $\mathbf{i}$ defined as $b_1(\mathbf{i}) = (i_1+1, i_2)$, $b_2(\mathbf{i}) = \mathbf{i}+\mathbf{1}$, $b_3(\mathbf{i}) = (i_1, i_2+1)$, $b_4(\mathbf{i}) = (i_1-1, i_2+1)$, $b_5(\mathbf{i}) = (i_1-1, i_2)$, $b_6(\mathbf{i}) = \mathbf{i}-\mathbf{1}$, $b_7(\mathbf{i}) = (i_1, i_2-1)$ and $b_8(\mathbf{i}) = (i_1+1, i_2-1)$. For each $s = 1, \ldots, 8$, we shall denote the s-crossing event $\{X_{\mathbf{i}} \leq u_{\mathbf{n},\mathbf{i}}, X_{b_s(\mathbf{i})} > u_{\mathbf{n},b_s(\mathbf{i})}\}$ by $B_{\mathbf{i},b_s(\mathbf{i}),n}$ or simply by $B_{\mathbf{i},b_s(\mathbf{i})}$, where $X_{b_s(\mathbf{i})} = -\infty$ if $b_s(\mathbf{i}) \notin \mathbb{Z}^2_+$.

In fact, for an i.i.d. random field $\mathbf{X}$, $\{u_{\mathbf{n}}\}_{\mathbf{n}\geq\mathbf{1}}$ satisfying (3.3), and $\mathbf{I} = \{1, \ldots, [\frac{n_1}{k_{n_1}}]\} \times\{1, \ldots, [\frac{n_2}{k_{n_2}}]\}$ we have

$$\sum_{\mathbf{i}\in\mathbf{I}} P(B_{\mathbf{i},n}) \leq \frac{1}{k_{n_1}k_{n_2}} \sum_{\mathbf{i}\leq\mathbf{n}} P(B_{\mathbf{i},n})$$

and

$$k_{n_1}k_{n_2}\sum_{\mathbf{i},\mathbf{j}\in\mathbf{I}}P(B_{\mathbf{i},n},B_{\mathbf{j},n})\geq$$

$$k_{n_1}k_{n_2}\sum_{s\neq t}\sum_{\mathbf{i},\mathbf{j}\in\mathbf{I}}P(B_{\mathbf{i},b_s(\mathbf{i})},B_{\mathbf{j},b_t(\mathbf{j})})\geq$$

$$k_{n_1}k_{n_2}\prod_{i=1}^{2}([\frac{n_i}{k_{n_i}}]-1)P(X_{\mathbf{1}}>u_{\mathbf{n}})P^2(X_{\mathbf{1}}\leq u_{\mathbf{n}}),$$

which tends to τ, as $\mathbf{n}\to\infty$. By an analogous reasoning with subsets of β ($\{\mathbf{i}\}$) with more than one element, we conclude that we can only restrict the number of crossings in each one of the eight directions at a time.

Since the only direction that joins the notion of past and future along both coordinate axes, simultaneously, is the diagonal direction from $\mathbf{i}$ to $\mathbf{i}+\mathbf{1}$, it was considered in [11] a condition which restricts the local occurrence of two or more of these diagonal crossings, id est, a condition that restricts the local occurrence of two or more events $\{X_{\mathbf{i}}\leq u_{\mathbf{n},\mathbf{i}},X_{\mathbf{i+1}}>u_{\mathbf{n},\mathbf{i+1}}\}$.

Here we shall consider a more general approach to crossing events of random fields through the family of the eight local conditions. Different examples can verify different conditions of this family, as we shall illustrate in the end of this section.

Let $\mathcal{C}(B_{\mathbf{i},b_s(\mathbf{i}),n})$ denote the family of indexes sets $\mathbf{I}\subset\mathbf{R_n}$ such that

$$\sum_{\mathbf{i}\in\mathbf{I}}P(B_{\mathbf{i},b_s(\mathbf{i}),n})\leq\frac{1}{k_{n_1}k_{n_2}}\sum_{\mathbf{i}\leq\mathbf{n}}P(B_{\mathbf{i},b_s(\mathbf{i}),n}).$$

Definition 3.1. *Let $s\in\{1,\ldots,8\}$. The condition $D''(B_{\mathbf{i},b_s(\mathbf{i}),n})$ holds for $\mathbf{X}$ if for each $\mathbf{I}\in\mathcal{C}$ $(B_{\mathbf{i},b_s(\mathbf{i}),n})$ we have*

$$k_{n_1}k_{n_2}\sum_{\mathbf{i},\mathbf{j}\in\mathbf{I}}P(B_{\mathbf{i},b_s(\mathbf{i}),n},B_{\mathbf{j},b_s(\mathbf{j}),n})\xrightarrow[\mathbf{n}\to\infty]{}0.$$

Each one of these eight local conditions $D''(B_{\mathbf{i},b_s(\mathbf{i}),n})$ will be a sufficient condition to compute the limit of $P(M_{\mathbf{n}}\leq u)$ from the limiting mean number of s-crossings, in Proposition 3.2. In order to apply this proposition, we only need to inspect one of four directions, as shows the next result.

Proposition 3.1 *Suppose that the random field* $\mathbf{X}$ *satisfies (2.2) and let* $s \in \{1,\ldots,4\}$. *Then*

(i) $\mathbf{X}$ *satisfies the condition* $D''(B_{\mathbf{i},b_s(\mathbf{i}),n})$ *if and only if it satisfies the condition* $D''(B_{\mathbf{i},\overline{b_s}(\mathbf{i}),n})$, *where* $\overline{b_s} = b_{s+4}$;

(ii) $\sum_{\mathbf{i}\leq\mathbf{n}} P(B_{\mathbf{i},b_s(\mathbf{i}),n}) \xrightarrow[\mathbf{n}\to\infty]{} \nu > 0$ *if and only if* $\sum_{\mathbf{i}\leq\mathbf{n}} P(B_{\mathbf{i},\overline{b_s}(\mathbf{i}),n}) \xrightarrow[\mathbf{n}\to\infty]{} \nu > 0$.

Proof: To obtain (i) we first note that

$$\sum_{\mathbf{i},\mathbf{j}\in\mathbf{I}} P(B_{\mathbf{i},b_s(\mathbf{i})}, B_{\mathbf{j},b_s(\mathbf{j})}) = \sum_{\mathbf{i},\mathbf{j}\in\mathbf{I}} P(B_{\mathbf{i},b_s(\mathbf{i})}, X_{b_s(\mathbf{j})} > u_{n,b_s(\mathbf{j})}) -$$

$$\sum_{\mathbf{i},\mathbf{j}\in\mathbf{I}} P(B_{\mathbf{i},b_s(\mathbf{i})}, X_{\mathbf{j}} > u_{n,\mathbf{j}}) + \sum_{\mathbf{i},\mathbf{j}\in\mathbf{I}} P(B_{\mathbf{i},b_s(\mathbf{i})}, X_{\mathbf{j}} > u_{n,\mathbf{j}}, X_{b_s(\mathbf{j})} \leq u_{n,b_s(\mathbf{j})}).$$

By applying the same decomposition to $B_{\mathbf{i},b_s(\mathbf{i})}$ in each of the above terms we get

$$\sum_{\mathbf{i},\mathbf{j}\in\mathbf{I}} P(B_{\mathbf{i},b_s(\mathbf{i})}, B_{\mathbf{j},b_s(\mathbf{j})}) = \sum_{\mathbf{i},\mathbf{j}\in\mathbf{I}} P(B_{\mathbf{i},\overline{b_s}(\mathbf{i})}, B_{\mathbf{j},\overline{b_s}(\mathbf{j})}) + o(k_{n_1}k_{n_2}).$$

The result in (ii) follows by an analogous argument. □

Under the conditions (1.1) and (2.2), in the proof of the proposition 3.2 we can suppose that, for each rectangle $\mathbf{V}_{r,p}$ in the partitions that arise for $\mathbf{R_n}$, the variables $X_{\mathbf{i}}$ with indexes in the boundary of $\mathbf{V}_{r,p}$ exhibit values below the correspondents levels $\mathbf{u_{n,i}}$. Asymptotically, the probability of the complementary of that event is negligible. So, for each fixed s, it occurs some event $B_{\mathbf{i},b_s(\mathbf{i})}$ with $\mathbf{i} \in \mathbf{V}_{r,p}$ if and only if we have some exceedance over $\mathbf{V}_{r,p}$.

Proposition 3.2 *Suppose that the random field* $\mathbf{X}$ *verifies conditions (2.2),* $D''(B_{\mathbf{i},b_s(\mathbf{i}),n})$ *for some* $s \in \{1,\ldots,8\}$ *and* $D(u_{\mathbf{n},\mathbf{i}})$ *over* $\mathcal{C}(B_{\mathbf{i},b_s(\mathbf{i}),n})$. *Then, as* $\mathbf{n} \longrightarrow \infty$,

$$P(\bigcap_{\mathbf{i}\leq\mathbf{n}} X_{\mathbf{i}} \leq u_{\mathbf{n},\mathbf{i}}) \to e^{-\nu},\ \nu > 0,$$

if and only if

$$\sum_{\mathbf{i}\leq\mathbf{n}} P(B_{\mathbf{i},b_s(\mathbf{i}),n}) \to \nu > 0.$$

Proof: We will built $k_{n_1}k_{n_2}$ rectangles in $\mathcal{C}(B_{\mathbf{i},b_s(\mathbf{i}),n})$ as follows. First split $\mathbf{R_n}$ in k_{n_1} quasi-rectangles $\mathbf{I'}_r = \{s_{r-1}+1\} \times \{t^*_{r-1}+1,\dots,n_2\} \cup \{s_{r-1}+2,\dots,s_r\} \times \{1,\dots,n_2\} \cup \{s_r+1\} \times \{1,\dots,t^*_r \le n_2\}$, $r = 0,\dots,k_{n_1}$, $s_0 = 0 = t^*_0$, with t^*_r maximally choosen such that

$$\sum_{\mathbf{i}\in\mathbf{I'}_r} P(B_{\mathbf{i},b_s(\mathbf{i})}) \le \frac{1}{k_{n_1}} \sum_{\mathbf{i}\le\mathbf{n}} P(B_{\mathbf{i},b_s(\mathbf{i})}).$$

Let $\mathbf{I}_r = \{s_{r-1}+2,\dots,s_r\} \times \{1,\dots,n_2\}$ and now we split each rectangle $\mathbf{I}_r$ in $k_{n_1}k_{n_2}$ quasi-rectangles $\mathbf{V'}_{r,p} = \{s^*_{r,p-1}+1,\dots,s_r\} \times \{t_{p-1}+1\} \cup \{s_{r-1}+1,\dots,s_r\} \times \{t_{p-1}+2,\dots,t_p\} \cup \{s_{r-1}+1,\dots,s^*_{r,p} \le s_r\} \times \{t_p+1\}$, $p = 1,\dots,k_{n_2}$, $t_0 = 0$, $s^*_{r,0} = s_{r-1}$, with $s^*_{r,p}$ maximally choosen such that

$$\sum_{\mathbf{i}\in\mathbf{V'}_{r,p}} P(B_{\mathbf{i},b_s(\mathbf{i})}) \le \frac{1}{k_{n_1}k_{n_2}} \sum_{\mathbf{i}\le\mathbf{n}} P(B_{\mathbf{i},b_s(\mathbf{i})}).$$

Let $\mathbf{V}_{r,p} = \{s_{r-1}+2,\dots,s_r\} \times \{t_{p-1}+2,\dots,t_p\}$ and $B(\mathbf{V}_{r,p})$ its boundary. By (1.1) and (2.2), we have

$$\sum_{r,p} P(M_{\mathbf{n}}(B(\mathbf{V}_{r,p})) > u) \le 2(k_{n_2}n_1 + k_{n_1}n_2)\overline{F}_{max} = o(1) \tag{3.4}$$

and to obtain the result it is sufficient to prove that

$$P(\bigcap_{r,p} M_{\mathbf{n}}(\mathbf{V}_{r,p}) \le u) \to e^{-\nu},\ \nu > 0,$$

if and only if

$$\sum_{r,p} \sum_{\mathbf{i}\in\mathbf{V}_{r,p}} P(B_{\mathbf{i},b_s(\mathbf{i})}) \to \nu > 0.$$

This follows from Proposition 2.1, (3.4), condition $D''(B_{\mathbf{i},b_s(\mathbf{i}),n})$ and the following relations:

$$\prod_{r,p} P(M_{\mathbf{n}}(\mathbf{V}_{r,p}) \le u) = exp\Big(-(1+o(1))\sum_{r,p}(1 - P(M_{\mathbf{n}}(\mathbf{V}_{r,p}) \le u))\Big) =$$

$$exp\Big(-(1+o(1))\sum_{r,p} P(M_{\mathbf{n}}(\mathbf{V}_{r,p}) > u, M_{\mathbf{n}}(B(\mathbf{V}_{r,p})) \le u) + o(1)\Big) =$$

$$exp\Big(-(1+o(1))\sum_{r,p} \sum_{\mathbf{i}\in\mathbf{V}_{r,p}} P(B_{\mathbf{i},b_s(\mathbf{i})}) + o(1)\Big).$$

□

If $\mathbf{X}$ is stationary the result follows by assuming $u_{\mathbf{n},\mathbf{i}} = u_{\mathbf{n}}$, $\mathbf{i} \leq \mathbf{n}$, and condition $D''(B_{\mathbf{i},b_s(\mathbf{i}),n})$ as

$$n_1 n_2 \sum_{\mathbf{i} \leq ([\frac{n_1}{k_{n_1}}],[\frac{n_2}{k_{n_2}}])} P(B_{\mathbf{1},b_s(\mathbf{1}),n}, B_{\mathbf{i},b_s(\mathbf{i}),n}) \xrightarrow[\mathbf{n}\to\infty]{} 0.$$

Weaker local dependence conditions can be considered as in [3].

Accordingly to [1], the stationary random field $\mathbf{X}$ has extremal index $\theta \in [0,1]$ if, for each $\tau > 0$, there exists $\{u_{\mathbf{n}}^{(\tau)}\}_{\mathbf{n}\geq\mathbf{1}}$ satisfying (3.3) and $P(M_{\mathbf{n}} \leq u_{\mathbf{n}}^{(\tau)}) \to exp(-\theta\tau)$, as $\mathbf{n} \to \infty$.

If $\mathbf{X}$ is an i.i.d random field or a stationary random field satisfying the conditions of the Proposition 3.1 of [11], then the extremal index equals to 1.

For nonstationary random fields the extremal index can be defined in a similar way:

$$\theta(\tau) = \frac{-log \lim_{\mathbf{n}} P(\bigcap_{\mathbf{i}\leq\mathbf{n}} X_{\mathbf{i}} \leq u_{\mathbf{n},\mathbf{i}}^{(\tau)})}{\tau}$$

where

$$\tau = \lim_{\mathbf{n}} \sum_{\mathbf{i}\leq\mathbf{n}} P(X_{\mathbf{i}} > u_{\mathbf{n},\mathbf{i}}^{(\tau)}). \tag{3.5}$$

Here the extremal index may depend on τ, as pointed out examples in [4].

The following result gives a convenient existence criterion of the extremal index and follows immediately from Proposition 3.2.

Corollary 3.1. *Suppose that the random field* $\mathbf{X}$ *satisfies (2.2),* $D''(B_{\mathbf{i},b_s(\mathbf{i}),n})$ *for some* $s \in \{1,\ldots,8\}$ *and* $D(u_{\mathbf{n},\mathbf{i}})$ *over* $\mathcal{C}(B_{\mathbf{i},b_s(\mathbf{i}),n})$ *with* $u_{\mathbf{n},\mathbf{i}} \equiv u_{\mathbf{n},\mathbf{i}}^{(\tau)}$ *satisfying (3.5). Then there exists* $\theta(\tau)$ *if and only if there exists*

$$\nu = \lim_{\mathbf{n}\to\infty} \sum_{\mathbf{i}\leq\mathbf{n}} P(B_{\mathbf{i},b_s(\mathbf{i}),n})$$

and, in this case, it holds

$$\theta(\tau) = \frac{\nu}{\tau}.$$

The clustering measure extremal index can be considered for sub-fields of $\mathbf{X}$. Let $\{\mathbf{I_n}\}_{\mathbf{n}\geq\mathbf{1}}$ be an increasing sequence of sub-sets of $\mathbf{R_n}$. If for each $\tau > 0$ there exists a family of levels $\{v_{\mathbf{n},\mathbf{i}}^{(\tau)}, \mathbf{i} \in \mathbf{I_n}\}_{\mathbf{n}\geq\mathbf{1}}$ such that

$$\sum_{\mathbf{i}\in\mathbf{I_n}} P(X_\mathbf{i} > v_{\mathbf{n},\mathbf{i}}^{(\tau)}) \xrightarrow[\mathbf{n}\to\infty]{} \tau$$

and

$$P(\bigcap_{\mathbf{i}\in\mathbf{I_n}} X_\mathbf{i} \leq v_{\mathbf{n},\mathbf{i}}^{(\tau)}) \xrightarrow[\mathbf{n}\to\infty]{} \exp(-\theta\tau),$$

we say that $\mathbf{X}$ has extremal index θ over $\bigcup_{\mathbf{n}\geq\mathbf{1}} \mathbf{I_n}$.

In general, we can't compare the extremal indexes over regions with the extremal index of the random field since the normalized levels are not, in general, coincident.

We will now illustrate the results with several 1-dependent random fields which satisfy different local dependence conditions. Let $\mathbf{Y} = \{Y_\mathbf{n}\}_{\mathbf{n}\geq\mathbf{1}}$ be an i.i.d. random field and $\{u_\mathbf{n}\}_{\mathbf{n}\geq\mathbf{1}}$ such that $n_1 n_2 P(Y_\mathbf{1} > u_\mathbf{n}) \longrightarrow \tau$. From $\mathbf{Y}$ we shall define several nonstationary and non isotrophic random fields.

Let $\mathbf{X} = \{X_\mathbf{n}\}_{\mathbf{n}\geq\mathbf{1}}$ be such that for each $\mathbf{i} = (i_1, i_2) = (2k+1, 2s+1)$, $k, s \geq 0$, it holds $X_\mathbf{i} = Y_\mathbf{i}$, $X_{b_1(\mathbf{i})} = Y_{b_1(\mathbf{i})}$, $X_{b_2(\mathbf{i})} = max\{Y_{b_1(\mathbf{i})}, Y_{b_2(\mathbf{i})}\}$ and $X_{b_3(\mathbf{i})} = max\{Y_\mathbf{i}, Y_{b_3(\mathbf{i})}\}$.
This random field only satisfies $D''(B_{\mathbf{i},b_s(\mathbf{i})})$ for $s=3$ and $s=7$ and has extremal index $\theta = 2/3$.

Let $\mathbf{W} = \{W_\mathbf{n}\}_{\mathbf{n}\geq\mathbf{1}}$ be such that for each $\mathbf{i} = (i_1, i_2) = (2k+1, 2s+1)$, $k, s \geq 0$, it holds $W_\mathbf{i} = Y_\mathbf{i}$, $W_{b_1(\mathbf{i})} = max\{Y_\mathbf{i}, Y_{b_1(\mathbf{i})}\}$, $W_{b_2(\mathbf{i})} = Y_{b_2(\mathbf{i})}$ and $W_{b_3(\mathbf{i})} = max\{Y_\mathbf{i}, Y_{b_3(\mathbf{i})}\}$.
This random field satisfies $D''(B_{\mathbf{i},b_s(\mathbf{i})})$ for $s = 1, 3, 5, 7$ and has extremal index $\theta = 5/6$.

Finally, let $\mathbf{U} = \{U_\mathbf{n}\}_{\mathbf{n}\geq\mathbf{1}}$ be such that for each $\mathbf{i} = (i_1, i_2) = (2k+1, 2s+1)$, $k, s \geq 0$, it holds $U_\mathbf{i} = Y_\mathbf{i}$, $U_{b_1(\mathbf{i})} = max\{Y_\mathbf{i}, Y_{b_1(\mathbf{i})}\}$, $U_{b_2(\mathbf{i})} = max\{Y_{b_1(\mathbf{i})}, Y_{b_2(\mathbf{i})}\}$ and $U_{b_3(\mathbf{i})} = max\{Y_{b_2(\mathbf{i})}, Y_{b_3(\mathbf{i})}\}$.
This random field does not satisfy any condition $D''(B_{\mathbf{i},b_s(\mathbf{i})})$. However it has extremal index $\theta = 2/3$ which we can easily compute directly. Let $\mathbf{I}_{1,\mathbf{n}} = \{(i_1, 2s+1) : i_1 \leq n_1 \wedge 2s+1 \leq n_2\}$ and $\mathbf{I}_{2,\mathbf{n}} = \mathbf{R_n} - \mathbf{I}_{1,\mathbf{n}}$. We have

$$P(M_\mathbf{n} \leq u) = P(M_\mathbf{n}(\mathbf{I}_{1,\mathbf{n}}) \leq u) P(M_\mathbf{n}(\mathbf{I}_{2,\mathbf{n}}) \leq u | M_\mathbf{n}(\mathbf{I}_{1,\mathbf{n}}) \leq u)$$

which converges to $\exp(-\frac{7}{6})\tau$.

Since $\sum_{\mathbf{i}\leq\mathbf{n}} P(X_{\mathbf{i}} > u_{\mathbf{n}}) \rightarrow \frac{7}{4}\tau > 0$ we find the value $\theta = 2/3$.

For the region $\bigcup_{\mathbf{n}\geq\mathbf{1}} \mathbf{I}_{1,\mathbf{n}}$ we find $\theta_{\mathbf{I},1} = \frac{2}{3}$ and for $\bigcup_{\mathbf{n}\geq\mathbf{1}} \mathbf{I}_{2,\mathbf{n}}$ we find $\theta_{\mathbf{I},2} = \frac{3}{4}$.

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