On extremal dependence: some contributions

Marta Ferreira *Department of Mathematics, University of Minho, Braga, Portugal*

Helena Ferreira *Department of Mathematics, University of Beira Interior, Covilhã, Portugal*

Abstract: The usual coefficients of tail dependence are based on exceedances of high values but other extremal events as upcrossings can also be used and provide useful information. In this context we define *upcrossings tail dependence coefficients* and analyze all types of dependence coming out. These coefficients are related with multivariate coefficients of tail dependence. Connections with measures of temporal dependence as the extremal index and the upcrossings index, as well as, with some local dependence conditions will be stated. Several illustrative examples will be exploited and a small note on inference will be given.

Keywords: Extreme values, measures of tail dependence, asymptotic independence.

1 Introduction

Measures to quantify the extremal dependence of a random pair (X, Y) started to appear in literature around the decade of 50. One natural measure is the so called "tail dependence coefficient",

$$\lambda = \lim_{x \to x_F} P(Y > x | X > x)$$

where x_F is the upper limit of the support of the common marginal distribution F, provided the limit exists. Loosely stated, λ is the probability of one variable being extreme given that the other is extreme. In the case, $\lambda = 0$, the variables are said to be asymptotically independent and if, $0 < \lambda \leq 1$, they are asymptotically dependent. Observe that, the boundary cases of total dependence and total independence corresponds, respectively, to $\lambda = 1$ and $\lambda \sim P(Y > x)$. The importance of this class was recognized as far back as Geffroy [14] (1958/59), Sibuya [22] (1960), Tiago de Oliveira [25] (1962/63) and Mardia [20] (1964). Coefficient λ is generalized to the case of X and Y non-identically distributed, with marginal d.f., respectively, F_X and F_Y, by transformation to Uniform margins, $F_X(X)$ and $F_Y(Y)$, and setting,

$$\lambda = \lim_{u \uparrow 1} P(F_Y(Y) > u | F_X(X) > u). \tag{1}$$

In order to graduate the "strength" of dependence in the case of asymptotic tail independence ($\lambda = 0$), Ledford and Tawn [18, 19] (1996, 1997) have considered the following formulation that states the rate of convergence towards zero:

$$P(X > x, Y > x) \sim P(X > x)^{1/\eta_{X,Y}} L^{\star}_{\eta_{X,Y}}(1/P(X > x)), \text{ as } x \to x_F \tag{2}$$

where $L^{\star}_{\eta_{X,Y}}$ is a slowly varying function at ∞, i.e., $L^{\star}_{\eta_{X,Y}}(tx)/L^{\star}_{\eta_{X,Y}}(x) \to 1$ as $x \to \infty$ for any fixed $t > 0$, and $\eta_{X,Y} \in (0, 1]$ is the "*Ledford and Tawn coefficient*". The coefficient $\eta_{X,Y}$ describes the type of limiting dependence between X and Y, and $L(t)$ its relative strength given a particular value of $\eta_{X,Y}$. Observe that equation (2) can also be expressed as

$$P(Y > x | X > x) \sim P(X > x)^{1/\eta_{X,Y} - 1} L^{\star}_{\eta_{X,Y}}(1/P(X > x)), \text{ as } x \to x_F$$

showing how λ changes with $\eta_{X,Y}$. Whenever $\eta_{X,Y} = 1$ and $L^{\star}_{\eta_{X,Y}}(x) \to a$ as $x \to \infty$ for some $0 < a \leq 1$, r.v.'s X and Y are asymptotically dependent with total dependence occurring if $a = 1$. The random pair is asymptotically independent when either $\eta_{X,Y} < 1$ or when $\eta_{X,Y} = 1$ with $L^{\star}_{\eta_{X,Y}}(x) \to 0$ as $x \to \infty$. The case $\eta_{X,Y} > 1/2$ corresponds to positive extremal dependence,

$\eta_{X,Y} < 1/2$ to negative dependence and $\eta_{X,Y} = 1/2$ to (almost) independence (perfect independence if $L^{\star}_{\eta_{X,Y}}(x) = 1$). We can also generalize to the case of X and Y non-identically distributed, with marginal d.f., respectively, F_X and F_Y, by transformation to Uniform margins. For convenience we take regular variation at point 0. More precisely,

$$P(F_X(X) > 1-t, F_Y(Y) > 1-t) \sim t^{1/\eta_{X,Y}} L_{\eta_{X,Y}}(t), \text{ as } t \downarrow 0 \tag{3}$$

where and $L_{\eta_{X,Y}}(t)$ is a slowly varying function at 0.

Observe that all these measures concern tail dependence based on extremal events of the type $\{X_1 > x\}$ for large x, i.e. an exceedance of a high level x. Another extremal event of interest is the upcrossing of a high level x, $\{X_1 \leq x < X_2\}$. In an analogous way, we will state tail dependence measures based on these latter. These new coefficients are related with the multivariate tail dependence coefficients defined in Schmidt and Stadtmüller [23], 2006 or in Ferreira [12], 2008, as well as, with a multivariate formulation of the coefficient of Ledford and Tawn (Section 2).

In Section 3 we will see straight connections between tail dependence coefficients and local dependence conditions, D$'$ (Leadbetter *et al.* [16]) and D$''$ (Leadbetter and Nandagopalan [17]), as well as, with the extremal index and the upcrossings index (Ferreira [11]). Section 4 is devoted to several examples illustrating the previous results, including the well-known M4 processes and Extended M4 processes (EM4) (Heffernan *et al.* [15] 2007). We also look at the sequence of levels persisting in time, which was inspired by the modeling of tidal that persist for successive time instants, under independent levels and under max-autoregressive dependence (Draisma [7]; Ferreira and Canto e Castro [10]). We end with some notes on estimation of the new coefficients in Section 5.

2 Tail dependence through upcrossings

Let (X_1, Y_1) and (X_2, Y_2) be two random pairs identically distributed as (X, Y), which has common marginal d.f. F. As already mentioned, the usual tail dependence measures are based on the extremal events "exceedances" by each margin. In an analogous way, we can state tail dependence measures based on upcrossings. More precisely, similar to the tail dependence coefficient λ in (1), we consider the *upcrossings tail dependence coefficients*,

$$\begin{gathered} \mu_{Y|X} = \lim_{x \to x_F} P(Y_1 \leq x < Y_2 | X_1 \leq x < X_2) \\ \text{and} \\ \mu_{X|Y} = \lim_{x \to x_F} P(X_1 \leq x < X_2 | Y_1 \leq x < Y_2). \end{gathered} \tag{4}$$

provided the limits exist.

The boundary cases of total upcrossings dependence and total upcrossings independence between events $\{Y_1 \leq x < Y_2\}$ and $\{X_1 \leq x < X_2\}$ corresponds to, respectively, $\mu_{Y|X} = \mu_{X|Y} = 1$, and $\mu_{Y|X} \sim P(Y_1 \leq x < Y_2)$ and $\mu_{X|Y} \sim P(X_1 \leq x < X_2)$.

We can also state coefficients $\mu_{Y|X}$ and $\mu_{X|Y}$ generalized to the case of random pairs (X_j, Y_j) ($j = 1, 2$) with marginal d.f., respectively, F_X and F_Y. Considering, for instance, $\mu_{Y|X}$, by transformation to Uniform margins, $F_X(X_j)$ and $F_Y(Y_j)$, we have,

$$\mu_{Y|X} = \lim_{u \uparrow 1} P(F_Y(Y_1) \leq u < F_Y(Y_2) | F_X(X_1) \leq u < F_X(X_2)). \tag{5}$$

In order to distinguish coefficient λ from coefficients $\mu_{Y|X}$ and $\mu_{X|Y}$, concerning the tail dependence, we give the following definitions.

Definition 2.1 *A random pair (X, Y) is called asymptotic exceedances-tail independent if $\lambda = 0$ in (1) and exceedances-tail dependent otherwise.*

Definition 2.2 *Random pairs (X_1, Y_1) and (X_2, Y_2) are called asymptotic upcrossings-tail independent if $\mu_{Y|X} = 0$ or $\mu_{X|Y} = 0$ in (5) and upcrossings-tail dependent otherwise.*

Next we relate coefficients $\mu_{Y|X}$ and $\mu_{X|Y}$ with the more general *multivariate tail dependence coefficient* (Schmidt and Stadtmüller [23], 2006; Ferreira [12], 2008),

$$\lambda_{X_I,Y_J|X_L,Y_K} = \lim_{u\uparrow 1} P\Bigg(\bigcap_{i\in I, j\in J} \{F_X(X_i) > u, F_Y(Y_j) > u\} \Bigg| \bigcap_{l\in L, k\in K} \{F_X(X_l) > u, F_Y(Y_k) > u\} \Bigg) \tag{6}$$

for non empty sets $I, J, L, K \subset \{1, ..., n\}$.

Proposition 2.1 *Coefficient $\mu_{Y|X}$ given in (5) can be stated as*

$$\mu_{Y|X} = (1-\lambda^{(X)})^{-1} \Big[\lambda - \lambda^{(Y)}\, \lambda_{X_{\{2\}}|Y_{\{1,2\}}} - \lambda^{(X)}\, \lambda_{Y_{\{2\}}|X_{\{1,2\}}} + \lambda\, \lambda_{X_{\{2\}},Y_{\{2\}}|X_{\{1\}},Y_{\{1\}}}\Big], \tag{7}$$

where $\lambda \equiv \lambda_{Y_{\{2\}}|X_{\{2\}}} = \lambda_{Y_{\{1\}}|X_{\{1\}}}$, $\lambda^{(X)} \equiv \lambda_{X_{\{2\}}|X_{\{1\}}}$ and $\lambda^{(Y)} \equiv \lambda_{Y_{\{2\}}|Y_{\{1\}}}$, provided the existence of these limits and $\lambda^{(X)} \neq 1$.

Proof. Observe that,

$$\begin{aligned}
& P(F_Y(Y_1) \le u < F_Y(Y_2), F_X(X_1) \le u < F_X(X_2)) \\
= \; & P(F_X(X_2) > u, F_Y(Y_2) > u) \\
& - P(F_X(X_2) > u, F_Y(Y_2) > u, F_Y(Y_1) > u) \\
& - P(F_X(X_2) > u, F_Y(Y_2) > u, F_X(X_1) > u) \\
& + P(F_X(X_2) > u, F_Y(Y_2) > u, F_Y(Y_1) > u, F_X(X_1) > u) \\
= \; & P(F_Y(Y_2) > u | F_X(X_2) > u) P(F_X(X_2) > u) \\
& - P(F_X(X_2) > u | F_Y(Y_2) > u, F_Y(Y_1) > u) P(F_Y(Y_2) > u | F_Y(Y_1) > u) P(F_Y(Y_1) > u) \\
& - P(F_Y(Y_2) > u | F_X(X_2) > u, F_X(X_1) > u) P(F_X(X_2) > u | F_X(X_1) > u) P(F_X(X_1) > u) \\
& + P(F_Y(Y_2) > u, F_X(X_2) > u | F_X(X_1) > u, F_Y(Y_1) > u) \cdot \\
& \qquad \cdot P(F_Y(Y_1) > u | F_X(X_1) > u) P(F_X(X_1) > u)
\end{aligned} \tag{8}$$

and that

$$\begin{aligned}
P(F_X(X_1) \le u < F_X(X_2)) &= P(F_X(X_2) > u) - P(F_X(X_2) > u, F_X(X_1) > u) \\
&= P(F_X(X_2) > u) - P(F_X(X_2) > u | F_X(X_1) > u) P(F_X(X_1) > u) \\
&\sim (1-u)\big[1 - P(F_X(X_2) > u | F_X(X_1) > u)\big]
\end{aligned} \tag{9}$$

as $u \uparrow 1$. Hence applying (6) we have (7). $\square$

Corollary 2.2 *Under the conditions of Proposition 2.1, if $\lambda^{(X)} \neq 1$ and $\lambda^{(Y)} \neq 1$, then*

$$(1-\lambda^{(X)})\mu_{Y|X} = (1-\lambda^{(Y)})\mu_{X|Y}.$$

Proposition 2.1 states a very interesting feature about $\mu_{Y|X}$ (respectively, $\mu_{X|Y}$ by Corollary 2.2), as this coefficient congregates both "temporal" and "spatial" dependence. We have temporal dependence measured by $\lambda^{(X)}$ and $\lambda^{(Y)}$ for time series $\{X_i\}$ and $\{Y_i\}$, and "spatial" dependence measured by $\lambda_{Y_{\{i\}}|X_{\{i\}}}$ for vectors (X_i, Y_i). Moreover it also includes the effect of temporal dependence in a given "location" into another by coefficients $\lambda_{X_{\{i\}}|Y_{\{i,i-1\}}}$ and $\lambda_{Y_{\{i\}}|X_{\{i,i-1\}}}$, as well as, the effect of "location" dependence in time by coefficient $\lambda_{X_{\{2\}},Y_{\{2\}}|X_{\{1\}},Y_{\{1\}}}$.

Remark 2.3 *Events $\bigcap_{\imath\in\mathcal{I}}\{F_X(X_\imath) > u\}$ in (6) can be replaced by $\min_{\imath\in\mathcal{I}}\{F_X(X_\imath) > u\}$. Hence the multivariate tail dependence coefficient $\lambda_{X_I,Y_J|X_L,Y_K}$ is actually a bivariate one, as the originally defined λ stated in (1), for random pairs $\big(\min_{i\in I,j\in J}\{F_X(X_i), F_Y(Y_j)\}, \min_{l\in L,k\in K}\{F_X(X_l), F_Y(Y_k)\}\big)$.*

Obviously, the dependence between consecutive random pairs (X_1, Y_1) and (X_2, Y_2) plays a role. Observe that, if they are (asymptotic) independent ($\lambda^{(Y)} = \lambda^{(Y)} = 0$) then $\mu_{X|Y} = \mu_{Y|X} = \lambda \equiv \lambda_{Y_2|X_2} \equiv \lambda_{Y_1|X_1}$, which makes sense because dependence only exists within each random pair.

By a similar procedure of Ledford and Tawn [18, 19] (1996, 1997), we consider a formulation stating the convergence rate of $P(Y_1 \le x < Y_2, X_1 \le x < X_2)$ to 0, as $x \to x_F$, in order to graduate the "strength" of dependence within asymptotic upcrossings independence. More precisely,

$$P(X_1 \le x < X_2, Y_1 \le x < Y_2) \sim P(X_1 \le x < X_2)^{1/\nu_{Y|X}} L^\star_{\nu_{Y|X}}(1/P(X_1 \le x < X_2)), \tag{10}$$

as $x \to x_F$, where $L^\star_{\nu_{Y|X}}$ is a slowly varying function at ∞, and the same conclusions for exceedances concerning the Ledford and Tawn coefficient $\eta_{X,Y}$, are derived. The coefficient $\nu_{Y|X}$ describes the type of limiting dependence between upcrossings of X_i's and Y_i's, and $L^\star_{\nu_{Y|X}}(x)$ its relative strength given a particular value of $\nu_{Y|X}$. Expressing equation (10) as

$$P(Y_1 \le x < Y_2 | X_1 \le x < X_2) \sim P(X_1 \le x < X_2)^{1/\nu_{Y|X}-1} L^\star_{\nu_{Y|X}}(1/P(X_1 \le x < X_2)),$$

we can also see how $\mu_{Y|X}$ changes with $\nu_{Y|X}$. When $\nu_{Y|X} = 1$ and $L^\star_{\nu_{Y|X}}(x) \not\to 0$ as $x \to \infty$ we have asymptotic dependence of the upcrossings (total dependence if $L^\star_{\nu_{Y|X}}(x) = 1$), and asymptotic independence otherwise. The cases $\nu_{Y|X} > 1/2$ and $\nu_{Y|X} < 1/2$ correspond to, respectively, positive and negative dependence, and $\nu_{Y|X} = 1/2$ an (almost) independence (perfect if $L^\star_{\nu_{Y|X}}(x) = 1$). Similarly, we can also generalize to the case of X and Y non-identically distributed, by considering

$$\begin{aligned} &P(F_X(X_1) \le 1-t < F_X(X_2), F_Y(Y_1) \le 1-t < F_Y(Y_2)) \\ &\qquad\qquad \sim P(F_X(X_1) \le 1-t < F_X(X_2))^{1/\nu_{Y|X}} L_{\nu_{Y|X}}(t), \end{aligned} \tag{11}$$

as $t \downarrow 0$, where function $L_{\nu_{Y|X}}(t)$ is slowly varying at 0.

Remark 2.4 *Observe that, under conditions of Corollary 2.2, in the case of upcrossings-tail dependence, $\mu_{Y|X}$ and $\mu_{X|Y}$ might differ (see Example 4.2) and within asymptotic upcrossings extremal independence, $\mu_{Y|X} = \mu_{X|Y} = 0$. If we assume the condition (2) of Ledford and Tawn for the random pairs (X_1, X_2) and (Y_1, Y_2), the strength of dependence measured by coefficient $\nu_{Y|X}$ and respective slow varying function, stated in (11), will only change on this latter one. It is easy to see if we rewritten expression in (11) as*

$$P(F_X(X_1) \le 1-t < F_X(X_2), F_Y(Y_1) \le 1-t < F_Y(Y_2)) \sim t^{1/\nu}\mathcal{L}_\nu(t), \text{ as } t \downarrow 0,$$

where in the last step we have applied (9) and (3), with slowly varying function $\mathcal{L}_\nu(t) = (1 - t^{1/\eta_{X_1,X_2}-1}L_{\eta_{X_1,X_2}}(t))^{1/\nu_{Y|X}} L_{\nu_{Y|X}}(t)$ or $\mathcal{L}_\nu(t) = (1 - t^{1/\eta_{Y_1,Y_2}-1}L_{\eta_{Y_1,Y_2}}(t))^{1/\nu_{X|Y}} L_{\nu_{X|Y}}(t)$ and $\nu = \nu_{Y|X} = \nu_{X|Y}$. Therefore, the conclusion about upcrossings tail dependence or independence between $\{F_X(X_1) \le u < F_X(X_2)\}$ and $\{F_Y(Y_1) \le u < F_Y(Y_2)\}$ do not change, only the strength of *dependence within the dependence case or independence case might differ. Therefore, from now on we consider (11) with $\nu_{Y|X}$ replaced by ν, i.e.,*

$$P(F_X(X_1) \leq 1-t < F_X(X_2), F_Y(Y_1) \leq 1-t < F_Y(Y_2)) \sim P(F_X(X_1) \leq 1-t < F_X(X_2))^{1/\nu} L_\nu(t), \tag{12}$$

with $L_\nu(t) \equiv L_{\nu_{Y|X}}(t)$.

Now we formulate a sufficient condition for (12) throughout the η's coefficients. By definition (3), $1/\eta$ is the regularly varying index of r.v. $\min(X,Y)$. Hence, the following extension for sets $I, J \subset \{1,...,n\}$,

$$P\Big(\min_{i\in I, j\in J}(F_X(X_i), F_Y(Y_j)) > 1-t\Big) \underset{t\downarrow 0}{\sim} t^{1/\eta_{X_I,Y_J}} L_{X_I,Y_J}(t), \tag{13}$$

where $L_{X_I,Y_J}(t)$ is a slowly varying function at 0, leads us to the coefficient η_{X_I,Y_J}.

Proposition 2.5 *Assume that (13) holds for any $I, J \subset \{1,2\}$. Let*

$$\eta = \max\{\eta_{X,Y}, \eta_{X_{\{2\}},Y_{\{1,2\}}}, \eta_{X_{\{1,2\}},Y_{\{1,2\}}}, \eta_{X_{\{1,2\}},Y_{\{1,2\}}}\},$$

where $\eta_{X,Y}$ stands for $\eta_{X_{\{2\}},Y_{\{2\}}} = \eta_{X_{\{1\}},Y_{\{1\}}}$, and L_η is the corresponding slowly varying function in (13).

(i) If $\eta = \eta_{X,Y}$, then (12) holds with $\nu = \eta_{X,Y}$, provided the left-hand side of (12) is non null;

(ii) If $\eta \neq \eta_{X,Y}$, then the left-hand side of (12) is null.

Proof. First observe that if $I' \subset I$ and $J' \subset J$ then

$$t^{1/\eta_{X_{I'},Y_{J'}}} L_{X_{I'},Y_{J'}}(t) \geq t^{1/\eta_{X_I,Y_J}} L_{X_I,Y_J}(t).$$

On the other hand, from (8), (12) and (13), we have

$$\begin{aligned} & P(F_X(X_1) \leq 1-t < F_X(X_2), F_Y(Y_1) \leq 1-t < F_Y(Y_2)) & (14)\\ \sim\ & t^{1/\eta} L_\eta(t)\Big[t^{1/\eta_{X,Y}-1/\eta} L_{X,Y}(t) L_\eta^{-1}(t) - t^{1/\eta_{X_{\{2\}},Y_{\{1,2\}}}-1/\eta} L_{X_{\{2\}},Y_{\{1,2\}}}(t) L_\eta(t)^{-1} \\ & -t^{1/\eta_{X_{\{1,2\}},Y_{\{2\}}}-1/\eta} L_{X_{\{1,2\}},Y_{\{2\}}}(t) L_\eta(t)^{-1} + t^{1/\eta_{X_{\{1,2\}},Y_{\{1,2\}}}-1/\eta} L_{X_{\{1,2\}},Y_{\{1,2\}}}(t) L_\eta(t)^{-1}\Big] \\ =\ & t^{1/\eta} L_\eta(t)\big[a_1(t) - a_2(t) - a_3(t) + a_4(t)\big] & (15) \end{aligned}$$

where $a_i(t)$ $(i = 1,...,4)$ denotes the absolute value of the ith term in the product of the right-hand side and satisfy the following properties:

(1) $a_1(t) \geq \max\{a_2(t), a_3(t)\} \geq \min\{a_2(t), a_3(t)\} \geq a_4(t) > 0$;

(2) $a_1(t) = 1$ or $a_1(t) \to 0$, as $t \to 0$;

(3) $a_1(t) - a_2(t) - a_3(t) + a_4(t) \geq 0$.

Now we look at all the possibilities for η.

If $\eta = \eta_{X,Y}$, then (15) becomes

$$t^{1/\eta} L_\eta(t)\big[1 - a_2(t) - a_3(t) + a_4(t)\big] \sim t^{1/\eta} L_\eta(t), \text{ as } t \to 0,$$

provided $1 - a_2(t) - a_3(t) + a_4(t)$ is non null.

If $\eta = \eta_{X_{\{2\}}, Y_{\{1,2\}}}$, then (15) is equal to

$$t^{1/\eta} L_\eta(t) \big[a_1(t) - 1 - a_3(t) + a_4(t) \big].$$

By conditions (1) and (2) above, we have $a_1(t) = 1$, and by (1) and (3) we have $a_3(t) \geq a_4(t)$ and $-a_3(t) + a_4(t) \geq 0$. Therefore, $a_3(t) = a_4(t)$, and hence probability in (14) is null.

If $\eta = \eta_{X_{\{1,2\}}, Y_{\{2\}}}$, then (15) is

$$t^{1/\eta} L_\eta(t) \big[a_1(t) - a_2(t) - 1 + a_4(t) \big],$$

and conditions (1) and (2) lead to $a_1(t) = 1$, and by (1) and (3) we have $a_2(t) = a_4(t)$. Hence probability in (14) is null.

If $\eta = \eta_{X_{\{1,2\}}, Y_{\{1,2\}}}$, then (15) is equal to

$$t^{1/\eta} L_\eta(t) \big[a_1(t) - a_2(t) - a_3(t) + 1 \big],$$

where, by (1) and (2), $a_1(t) = 1 = a_2(t) = a_3(t)$. Therefore, we also have (14) null. $\square$

Observe that, in a similar manner, we extend the upcrossings tail dependence coefficient $\mu_{Y|X}$, defined in (5), to the *multivariate upcrossings tail dependence coefficient*. More precisely, for sets $I, J, L, K \subset \{1, ..., n\}$,

$$\begin{aligned} \mu_{X_I, Y_J | X_L, Y_K} = \lim_{u \uparrow 1} P\Big(\bigcap_{i \in I, j \in J} & \{F_X(X_i) \leq u < F_X(X_{i+1}), F_Y(Y_j) \leq u < F_Y(Y_{j+1})\} \Big| \\ & \bigcap_{l \in L, k \in K} \{F_X(X_l) \leq u < F_X(X_{l+1}), F_Y(Y_k) \leq u < F_Y(Y_{k+1})\} \Big) \end{aligned} \tag{16}$$

Dependence also occurs when a single process is studied in terms of its temporal evolution.

More precisely, for a stationary process $\{X_i\}$, we can state the above mentioned tail dependence measures for random pairs (X_1, X_{1+m}), i.e. observations separated in time by a lag m ($m \in \mathbb{N}$). Hence, and considering marginal uniform normalization, we have the lag-m *tail dependence coefficient*,

$$\lambda_m = \lim_{u \uparrow 1} P(F(X_{1+m}) > u | F(X_1) > u), \tag{17}$$

as well as the lag-m *Ledford and Tawn coefficient*, η_m, such that

$$P(X_1 > F^{-1}(1-t), X_{1+m} > F^{-1}(1-t)) \sim t^{1/\eta_m} L_m(t), \text{ as } t \downarrow 0 \tag{18}$$

or, equivalently,

$$P(X_{1+m} > F^{-1}(1-t) | X_1 > F^{-1}(1-t)) \sim t^{1/\eta_m - 1} L_m(t), \text{ as } t \downarrow 0 \tag{19}$$

where F is the marginal d.f. of process $\{X_i\}$ and $L_m(t)$ is a slowly varying function at 0.

Similarly, we state the lag-m *upcrossings tail dependence coefficient*

$$\mu_m = \lim_{u \uparrow 1} P(F(X_{2+m}) \leq u < F(X_{3+m}) | F(X_1) \leq u < F(X_2)), \tag{20}$$

and also, as $t \downarrow 0$,

$$\begin{aligned} & P(X_1 \leq F^{-1}(1-t) < X_2, X_{2+m} \leq F^{-1}(1-t) < X_{3+m}) \\ & \qquad \sim P(X_1 \leq F^{-1}(1-t) < X_2)^{1/\nu_m} L_{\nu_m}(t), \end{aligned} \tag{21}$$

or, equivalently,

$$P(X_{2+m} \leq F^{-1}(1-t) < X_{3+m} | X_1 \leq F^{-1}(1-t) < X_2) \\ \sim P(X_1 \leq F^{-1}(1-t) < X_2)^{1/\eta_m - 1} L_{\nu_m}(t), \tag{22}$$

with function $L_{\nu_m}(t)$ slowly varying at 0.

Corollary 2.6 *For the lag-m upcrossings tail dependence coefficient in (20), we have*

$$\mu_m \sim \left(1-\lambda_1\right)^{-1}\Big[\lambda_{m+1} - \lambda_m \lambda_{\{3+m|2,2+m\}} - \lambda_1 \lambda_{\{3+m|1,2\}} + \lambda_1 \lambda_{\{2+m,3+m|1,2\}}\Big], \tag{23}$$

where we take $\lambda_{X_{I,J}|X_{L,K}} = \lambda_{I,J|L,K}$ since there is no ambiguity, provided the existence of these limits and $\lambda_1 \neq 1$.

As stated in Proposition 2.5, the coefficient ν_m relates with Ledford and Tawn coefficients throughout:

$$P(X_1 \leq F^{-1}(1-t) < X_2)^{1/\nu_m} L_{\nu_m}(t) \sim t^{1/\eta_{m+1}} L_{m+1}(t) - t^{1/\eta_{\{2,2+m,3+m\}}} L_{\{2,2+m,3+m\}}(t) \\ - t^{1/\eta_{\{1,2,3+m\}}} L_{\{1,2,3+m\}}(t) + t^{1/\eta_{\{1,2,2+m,3+m\}}} L_{\{1,2,2+m,3+m\}}(t). \tag{24}$$

Remark 2.7 *In applying the multivariate tail dependence coefficient (6) to consecutive r.v.'s of a sequence $\{X_i\}$, we are actually computing a bivariate tail dependence coefficient (see Remark 2.3) of a random pair of levels persisting in a fixed period of time, $\big(\min\{X_1, ..., X_r\}, \min\{X_{r+m}, ..., X_s\}\big)$ that will be studied in Section 4.2.*

3 Extremal Index and Upcrossings Index

For a stationary sequence $\{X_i\}$, some local dependence conditions concerning extremal events have been considered leading to short-range dependence measures, e.g., the extremal index θ (Leadbetter *et al.* [16], 1983) and the upcrossings index η (Ferreira [11], 2006), related with the presence of clustering of, respectively, exceedances and upcrossings of high levels u_n.

Definition 3.1 *Condition $\Delta(u_n)$ will be said to hold for $\{X_i\}$ if $\alpha_{n,l_n} \underset{n\to\infty}{\to} 0$ for some sequence $l_n = o(n)$, where*

$$\alpha(n,l) = \sup_{1 \leq k \leq n-l} \{|P(A \cap B) - P(A)P(B)| : A \in \mathcal{B}_1^k(u_n), B \in \mathcal{B}_{k+l}^n(u_n)\},$$

and $\mathcal{B}_i^j(u_n)$ denotes the σ-field generated by $\{X_i, ..., X_j\}$.

Condition $\mathrm{D}(u_n)$ will hold if we are under the same assumptions of condition $\Delta(u_n)$ above, but restricted to the events $\{X_s \leq u_n\}$, $i \leq s \leq j$.

The local dependence condition $\mathrm{D}'(u_n)$ considered in Leadbetter *et al.* [16] bounds the probability of more than one exceedance of u_n, on a time-interval of $r_n = [n/k_n]$ integers with $k_n \to \infty$, as $n \to \infty$.

Definition 3.2 *Condition $D'(u_n)$ will be said to hold for $\{X_i\}$ if for some sequence $\{k_n\}$ such that $k_n \underset{n\to\infty}{\to} \infty$, we have*

$$\limsup_{n\to\infty} n \sum_{j=2}^{r_n} P\big(X_1 > u_n, X_j > u_n\big) = 0.$$

Under condition $\mathrm{D}'(u_n)$, the exceedances of levels u_n tend to come out isolated, similar to an i.i.d. behavior, leading to unit extremal index. If condition $\mathrm{D}'(u_n)$ doesn't hold, then the exceedances of u_n tend to cluster. For such sequences, Leadbetter e Nandagopalan [17] stated another local dependence condition, $\mathrm{D}''(u_n)$, weaker than $\mathrm{D}'(u_n)$ (under $\mathrm{D}''(u_n)$ all values $0 \leq \theta \leq 1$ are possible), that inhibits rapid oscillations near high levels and hence restricts the local occurrence of upcrossings $\{X_j \leq u_n < X_{j+1}\}$.

Definition 3.3 *Condition $D''(u_n)$ will be said to hold for $\{X_i\}$ if condition $D(u_n)$ also holds and $(k_n)_n$ is such that*

$$k_n \underset{n\to\infty}{\longrightarrow} \infty \ , \ k_n \alpha_{n,l_n} \underset{n\to\infty}{\longrightarrow} 0 \ , \ k_n l_n/n \underset{n\to\infty}{\longrightarrow} 0 \ , \tag{25}$$

$k_n(1-F(u_n)) \underset{n\to\infty}{\longrightarrow} 0$ *and*

$$\limsup_{n\to\infty} n \sum_{j=2}^{r_n-1} P\big(X_1 > u_n, X_j \leq u_n < X_{j+1}\big) = 0.$$

Condition $\mathrm{D}''(u_n)$ can be slightly weakened by replacing "$X_1 > u_n$" by "$X_1 \leq u_n < X_2$" as we can see in the proof of Proposition 4.3.5 of Leadbetter and Nandagopalan [17].

For stationary normal sequences, if the covariances between X_i and X_j, $\rho_{|i-j|}$, satisfy the Berman's condition $\sum_{n=0}^{\infty} \rho_n^2 < \infty$, then $\mathrm{D}(u_n)$ and $\mathrm{D}'(u_n)$ hold for appropriate sequences $\{u_n\}$. We shall present in the next result a sufficient condition for $\mathrm{D}'(u_n)$ and $\mathrm{D}''(u_n)$ throughout the above dependence coefficients $\{\lambda_n\}$ and $\{\mu_n\}$.

Proposition 3.1 *Let $\{X_i\}$ be a stationary sequence.*

1. *If $nP(X_1 > u_n) \to \tau \geq 0$ then $D'(u_n)$ holds if and only if $\sum_{j=2}^{r_n} \lambda_{j-1}(u_n) \underset{n\to\infty}{\to} 0$ for any sequence $\{r_n = [n/k_n]\}$ with $\{k_n\}$ satisfying (25), where $\lambda_j(u_n) = P(X_{1+j} > u_n | X_1 > u_n)$, $j \geq 1$.*

2. *If $nP(X_1 \leq u_n < X_2) \to \varsigma \geq 0$ then $D''(u_n)$ holds if and only if $\sum_{j=2}^{r_n-1} \mu_{j-1}(u_n) \underset{n\to\infty}{\to} 0$ for any sequence $\{r_n = [n/k_n]\}$ with $\{k_n\}$ satisfying (25), where $\mu_j(u_n) = P(X_{2+j} \leq u_n < X_{3+j} | X_1 \leq 1 < u_n < X_2)$, $j \geq 1$.*

Proof. Observe that condition $\mathrm{D}'(u_n)$ is given by

$$\limsup_{n\to\infty} \ nP(X_1 > u_n) \sum_{j=2}^{r_n} \lambda_{j-1}(u_n) = 0,$$

and condition $\mathrm{D}''(u_n)$ becomes

$$\limsup_{n\to\infty} \ nP(X_1 \leq u_n < X_2) \sum_{j=2}^{r_n-1} \mu_{j-1}(u_n) = 0. \quad \square$$

We remark that if condition $\mathrm{D}'(u_n)$ holds for $u_n \equiv u_n^{(\tau)}$ satisfying $nP(X_1 > u_n^{(\tau)}) \to \tau$, then we find $\lambda_i = 0$, $i \geq 1$, provided the existence of these coefficients. Analogously, from the statement in *2.*, if $\mathrm{D}''(u_n)$ holds for $u_n \equiv u_n^{(\varsigma)}$ satisfying $nP(X_1 \leq u_n^{(\varsigma)} < X_2) \to \varsigma$, then if the coefficients μ_i $(i \geq 1)$ exist they must be null.

Consider notation $M_{i,j} = \max\{X_i, ..., X_j\}$ for $i \leq j$ and $M_{i,j} = -\infty$ for $i > j$.

Definition 3.4 *Condition $D^{(k)}(u_n)$ holds for $\{X_i\}$ when for some k_n as in (25),*

$$nP\big(X_1 > u_n \geq M_{2,k}, M_{k+1,r_n} > u_n\big) \underset{n\to\infty}{\to} 0.$$

The family of conditions $\mathrm{D}^{(k)}(u_n)$, for $k \geq 1$, considered in Chernick *et al.* [6] (1991) are sufficient to derive

$$\theta = \lim_{n\to\infty} P(M_{2,k} \leq u_n^{(\tau)} | X_1 > u_n^{(\tau)})$$

when the limit exists, where levels $u_n^{(\tau)}$ satisfy $nP(X_1 > u_n^{(\tau)}) \to \tau$, as $n \to \infty$. Under $\mathrm{D}'(u_n(\tau)) \equiv \mathrm{D}^{(1)}(u_n^{(\tau)})$ we have $\theta = 1$, and $\mathrm{D}^{(2)}(u_n^{(\tau)})$ leads to $\varsigma = \theta\tau$, where $\varsigma = \lim_{n\to\infty} nP(X_1 \leq u_n < X_2)$.

We now relate θ with the multivariate tail dependence coefficients.

Proposition 3.2 *If the stationary sequence $\{X_i\}$ satisfies $D^{(k)}(u_n^{(\tau)})$, then the extremal index is given by*

$$\theta \quad = \quad 1 - \sum_{2\leq i\leq k} \lambda_{\{i|1\}} + \sum_{2\leq i<j\leq k} \lambda_{\{i,j|1\}} + ... + (-1)^{k+1}\lambda_{\{2,...,k|1\}}, \tag{26}$$

where $\lambda_{\{i|1\}} \equiv \lambda_{i-1}$ given in (17), provided these limits exist.

Proof. Just observe that, as $n \to \infty$,

$$\begin{aligned}\theta \quad \sim \quad & 1 - \sum_{2\leq i\leq k} P(X_i > u_n^{(\tau)} | X_1 > u_n^{(\tau)}) + \sum_{2\leq i<j\leq k} P(X_i > u_n^{(\tau)}, X_j > u_n^{(\tau)} | X_1 > u_n^{(\tau)}) \\ & + ... + (-1)^{k+1} P(X_2 > u_n^{(\tau)}, ..., X_k > u_n^{(\tau)} | X_1 > u_n^{(\tau)}). \quad \square\end{aligned}$$

According to the remark after Proposition 3.1, we will find $\theta = 1$ under $D'(u_n^{(\tau)})$, since we have $0 \leq \lambda_{i_1,...,i_p|1} \leq \lambda_{i_1|1} \equiv \lambda_{i_1 - 1} = 0$, for any integers $1 < i_1 < ... < i_p$.

Replacing exceedances with upcrossings in the condition $\mathrm{D}^{(k)}(u_n)$ a generalization of condition $\mathrm{D}''(u_n)$ takes place. This new family of local conditions, slightly stronger than $\mathrm{D}^{(k)}(u_n)$, is defined below (cf. Ferreira [11]).

Consider notation $\widetilde{N}_n(B) = \sum_{i=1}^{n} \mathbf{1}_{\{X_i \leq u_n < X_{i+1}\}} \delta_{i/n}(B)$, $B \subset [0,1]$, and $\widetilde{N}_n[i/n, j/n] \equiv \widetilde{N}_{i,j}$

Definition 3.5 *For any $k \geq 2$, $\{X_i\}$ satisfies condition $\widetilde{D}^{(k)}(u_n)$ if condition $\Delta(u_n)$ holds and*

$$nP\big(X_1 \leq u_n < X_2, \widetilde{N}_{3,k} = 0, \widetilde{N}_{k+1,r_n} > 0\big) \underset{n\to\infty}{\to} 0,$$

for some sequence $r_n = [n/k_n]$ with $\{k_n\}$ satisfying (25).

We now define the upcrossings index, ϑ, which, as already mentioned, can be viewed as a measure of clustering of upcrossings of high levels u_n by the r.v.'s in $\{X_i\}$.

Definition 3.6 *If for each $\varsigma > 0$ there exits $\{\widetilde{u}_n^{(\varsigma)}\}$ such that $nP(X_1 \leq u_n^{(\varsigma)} < X_2) \to \varsigma$ and $P(\widetilde{N}_n(\widetilde{u}_n^{(\varsigma)}) = 0) \to \exp(-\vartheta\varsigma)$, for some constant $0 \leq \vartheta \leq 1$, then we say that the sequence $\{X_i\}$ has upcrossings index ϑ.*

Hence, under conditions $\Delta(u_n)$ and $\widetilde{D}^{(k)}(u_n^{(\varsigma})$ for some $k \geq 2$ and for each $\varsigma > 0$, then the upcrossings index of $\{X_i\}$ exists and is equal to ϑ if and only if

$$P(\widetilde{N}_{3,k}(\widetilde{u}_n^{(\varsigma)}) = 0 | X_1 \leq \widetilde{u}_n^{(\varsigma)} < X_2) \underset{n\to\infty}{\to} \vartheta,$$

for each $\varsigma > 0$ (Corollary 3.1 in Ferreira [11]). We also have the following relation between the upcrossings index and the extremal index:

$$\theta = \frac{\varsigma}{\tau}\vartheta \tag{27}$$

A relation between μ and the multivariate tail upcrossings coefficients defined in (16) can also be stated.

Proposition 3.3 *If the stationary sequence $\{X_i\}$ satisfies $\widetilde{D}^{(k)}(u_n^{(\varsigma)})$, we have,*

$$\vartheta \;=\; 1-\sum_{3\leq i\leq k}\mu_{\{i|1\}}+\sum_{3\leq i<j\leq k}\mu_{\{i,j|1\}}+...+(-1)^{k+1}\mu_{\{2,...,k|1\}}, \qquad (28)$$

where $\mu_{\{i|1\}}\equiv\mu_{i-2}$ given in (20), provided the existence of these limits.

Proof. Straightforward by considering,

$$\begin{aligned}\vartheta \;\sim\; & 1-\sum_{3\leq i\leq k}P(X_i\leq u_n^{(\tau)}<X_{i+1}|X_1\leq u_n^{(\tau)}<X_2)\\ & +\textstyle\sum_{3\leq i<j\leq k}P(X_i\leq u_n^{(\tau)}<X_{i+1},X_j\leq u_n^{(\tau)}<X_{j+1}|X_1\leq u_n^{(\tau)}<X_2)\\ & +...+(-1)^kP(X_3\leq u_n^{(\tau)}<X_4,...,X_k\leq u_n^{(\tau)}<X_{k+1}|X_1\leq u_n^{(\tau)}<X_2),\end{aligned}$$

as $n\to\infty$. $\square$

Under condition $\widetilde{D}^{(2)}(u_n^{(\varsigma)})$ we will find $\vartheta=1$ as a consequence of Proposition 3.1.

4 Examples

In this section, we illustrate the statements above with some examples.

Example 4.1 *Let $\{Y_n\}_{n\geq -2}$ be an i.i.d. sequence of standard uniform distributed r.v.'s. Consider $\{X_n\}_{n\geq 1}$ such that $X_n=\max(Y_n,Y_{n-2},Y_{n-3})$. This sequence has $\theta=1/3$, $\vartheta=1/2$ and satisfies conditions $\Delta(u_n)$ (it is 4-dependent), $D^{(3)}(u_n)$ and $\widetilde{D}^{(3)}(u_n)$ for levels u_n such that $nP(X_1>u_n)\underset{n\to\infty}{\to}\tau>0$ (Ferreira [11]).*

We compute the tail dependence coefficients, λ_m and η_m, given in (17) and (18), respectively. Observe that,

$$P(X_1>u,X_{1+m}>u)=1-P(X_1\leq u)-P(X_{1+m}\leq u)+P(X_1\leq u,X_{1+m}\leq u)$$

$$=1-P(Y_1\leq u,Y_{-1}\leq u,Y_{-2}\leq u)-P(Y_{1+m}\leq u,Y_{m-1}\leq u,Y_{m-2}\leq u)+$$

$$P(Y_1\leq u,Y_{-1}\leq u,Y_{-2}\leq u,Y_{1+m}\leq u,Y_{m-1}\leq u,Y_{m-2}\leq u)$$

$$=1-2u^3+u^5\mathbf{1}_{\{m\leq 3\}}+u^6\mathbf{1}_{\{m>3\}},$$

and hence,

$$\begin{aligned}\frac{P(X_1>u,X_{1+m}>u)}{P(X_1>u)} &= \tfrac{1-2u^3+u^5}{1-u^3}\mathbf{1}_{\{m\leq 3\}}+(1-u^3)\mathbf{1}_{\{m>3\}}\\ &= \tfrac{1-2u^3+u^5}{1-u^3}\mathbf{1}_{\{m\leq 3\}}+P(X_1>u)\mathbf{1}_{\{m>3\}},\end{aligned}$$

leading to $\lambda_m=(1/3)\mathbf{1}_{\{m\leq 3\}}+0\mathbf{1}_{\{m>3\}}$ (observe that $\theta=1-\lambda_1-\lambda_2$ and $D'(u_n)$ does not hold). Taking $u=1-t$, we obtain, for $m\leq 3$,

$$\frac{P(X_1>1-t,X_{1+m}>1-t)}{P(X_1>1-t)}\sim 1,\ t\downarrow 0.$$

Therefore, by (19) $\eta_m=1\cdot\mathbf{1}_{\{m\leq 3\}}+(1/2)\cdot\mathbf{1}_{\{m>3\}}$ and $L_m(t)=1$.

Now we compute the tail upcrossings coefficients, μ_m and ν_m, given in (20) and (21), respectively. Observe that,

$$P(X_1 \le u < X_2) = P(X_1 \le u) - P(X_1 \le u, X_2 \le u)$$

$$= P(Y_1 \le u, Y_{-1} \le u, Y_{-2} \le u) - P(Y_1 \le u, Y_{-1} \le u, Y_{-2} \le u, Y_2 \le u, Y_0 \le u, Y_{-1} \le u)$$

$$= u^3 - u^5 = u^3(1-u^2)$$

and that,

$$P(X_1 \le u < X_2, X_{2+m} \le u < X_{3+m})$$

$$= P(X_1 \le u, X_{2+m} \le u) - P(X_1 \le u, X_{2+m} \le u, X_2 \le u)$$

$$-P(X_1 \le u, X_{2+m} \le u, X_{3+m} \le u) + P(X_1 \le u, X_{2+m} \le u, X_2 \le u, X_{3+m} \le u)$$

$$= P(Y_1 \le u, Y_{-1} \le u, Y_{-2} \le u, Y_{2+m} \le u, Y_m \le u, Y_{m-1} \le u)$$

$$-P(Y_1 \le u, Y_{-1} \le u, Y_{-2} \le u, Y_{2+m} \le u, Y_m \le u, Y_{m-1} \le u, Y_2 \le u, Y_0 \le u, Y_{-1} \le u)$$

$$-P(Y_1 \le u, Y_{-1} \le u, Y_{-2} \le u, Y_{2+m} \le u, Y_m \le u, Y_{m-1} \le u, Y_{3+m} \le u, Y_{1+m} \le u, Y_{m-1} \le u)$$

$$+P(Y_1 \le u, Y_{-1} \le u, Y_{-2} \le u, Y_{2+m} \le u, Y_m \le u, Y_{m-1} \le u, Y_2 \le u, Y_0 \le u, Y_{-1} \le u$$

$$, Y_{3+m} \le u, Y_{1+m} \le u, Y_{m-1} \le u)$$

$$= u^6(1-u^2)^2$$

provided $m > 3$. If $m = 1$, we have,

$$P(X_1 \le u < X_2, X_3 \le u < X_4) = u^5(1-u)$$

if $m = 2$, then

$$P(X_1 \le u < X_2, X_4 \le u < X_5) = u^5(1-u-u^2+u^3)$$

and $m = 3$,

$$P(X_1 \le u < X_2, X_5 \le u < X_6) = u^6(1-u-u^2+u^3)$$

Hence, by (20), we obtain $\mu_1 = 1/2$ and $\mu_m = 0$ for $m > 1$ (observe now that $\vartheta = 1 - \mu_1$ and $D''(u_n)$ does not hold too). Replacing u by $1-t$ in the above expressions, we have successively, as $t \downarrow 0$,

$$\frac{P(X_1 \le 1-t < X_2, X_{2+m} \le 1-t < X_{3+m})}{P(X_1 \le 1-t < X_2)} \sim (1-t)^3(1-(1-t)^2) \sim 2t$$

provided $m > 3$, whereas for $m = 1$,

$$\frac{P(X_1 \le 1-t < X_2, X_3 \le 1-t < X_4)}{P(X_1 \le 1-t < X_2)} \sim \tfrac{1}{2-t} - t$$

for $m = 2$,

$$\frac{P(X_1 \le 1-t < X_2, X_4 \le 1-t < X_5)}{P(X_1 \le 1-t < X_2)} \sim t(1-t)^2 \sim t$$

and $m = 3$,

$$\frac{P(X_1 \le 1-t < X_2, X_5 \le 1-t < X_6)}{P(X_1 \le 1-t < X_2)} \sim t(1-t)^3 \sim t$$

Therefore, from (22) we derive $\nu_1 = 1$ *and* $L_{\nu_1}(t) = \frac{1}{2-t} - t$ *corresponding to tail upcrossings dependence, and for* $m > 1$, $\nu_m = 1/2$ *and* $L_{\nu_m}(t) = 1$, *i.e., an almost total independence.*

Example 4.2 *Let* $\{Y_n\}_{n\geq -2}$ *be an i.i.d. sequence of standard uniform distributed r.v.'s. Consider* $\{(X_{n,1}, X_{n,2}\}_{n\geq 1}$ *such that* $X_{n,1} = \max(Y_n, Y_{n-2}, Y_{n-3})$ *and* $X_{n,2} = Y_{n+1}$, $n \geq 1$. *We have,*

$$P(X_{1,2} \leq u < X_{2,2}) = P(Y_2 \leq u < Y_3) = u(1-u),$$

as well as,

$$\begin{aligned} P(X_{1,1} \leq u < X_{2,1}) &= P(Y_1 \leq u, Y_{-1} \leq u, Y_{-2} \leq u) - P(Y_1 \leq u, Y_{-1} \leq u, Y_{-2} \leq u, Y_2 \leq u, Y_0 \leq u) \\ &= u^3 - u^5 = u^3(1-u^2), \end{aligned}$$

and also,

$$\begin{aligned} & P(X_{1,2} \leq u < X_{2,2}, X_{1,1} \leq u < X_{2,1}) \\ = \;& P(Y_2 \leq u < Y_3, \max(Y_1, Y_{-1}, Y_{-2}) \leq u < \max(Y_2, Y_0, Y_{-1})) \\ = \;& P(Y_2 \leq u < Y_3, Y_1 \leq u, Y_{-1} \leq u, Y_{-2} \leq u, Y_0 \leq u) \\ = \;& u^4(1-u)^2 \end{aligned} \tag{29}$$

Hence,

$$\mu = \lim_{u\uparrow 1} P(X_{1,2} \leq u < X_{2,2} | X_{1,1} \leq u < X_{2,1}) = 0 = \lim_{u\uparrow 1} P(X_{1,1} \leq u < X_{2,1} | X_{1,2} \leq u < X_{2,2}).$$

Observe now that,

$$\frac{P(X_{1,2} \leq 1-t < X_{2,2}, X_{1,1} \leq 1-t < X_{2,1})}{P(X_{1,2} \leq 1-t < X_{2,2})} \sim P(X_{1,2} \leq 1-t < X_{2,2})(1-t)^2, \text{ as } t \downarrow 0,$$

leading to $\nu = 1/2$ *with* $L_\nu(t) = (1-t)^2$, *and that*

$$\frac{P(X_{1,2} \leq 1-t < X_{2,2}, X_{1,1} \leq 1-t < X_{2,1})}{P(X_{1,1} \leq 1-t < X_{2,1})} \sim \frac{t(1-t)}{2-t} = t\left(1 - \frac{1}{2-t}\right), \text{ as } t \downarrow 0,$$

hence $\nu = 1/2$ *with* $L_\nu(t) = 1 - \frac{1}{2-t}$.

Now consider sequence $\{(X_{n,1}, X_{n,2}\}_{n\geq 1}$ *such that* $X_{n,1} = \max(Y_n, Y_{n-2}, Y_{n-3})$ *and* $X_{n,2} = Y_n$, $n \geq 1$. *Only the joint probability in (29) changes, becoming*

$$\begin{aligned} P(X_{1,2} \leq u < X_{2,2}, X_{1,1} \leq u < X_{2,1}) &= P(Y_2 \leq u < Y_3, \max(Y_1, Y_{-1}, Y_{-2}) \leq u < \max(Y_2, Y_0, Y_{-1})) \\ &= P(Y_2 \leq u < Y_3, Y_1 \leq u, Y_{-1} \leq u, Y_{-2} \leq u) \\ &= u^3(1-u). \end{aligned}$$

Therefore we have upcrossings-tail dependence, since,

$$\lim_{u\uparrow 1} P(X_{1,2} \leq u < X_{2,2} | X_{1,1} \leq u < X_{2,1}) = 1/2 \quad \text{and} \quad \lim_{u\uparrow 1} P(X_{1,1} \leq u < X_{2,1} | X_{1,2} \leq u < X_{2,2}) = 1,$$

the last one corresponding to perfect dependence.

4.1 M4 processes

Smith and Weissman (1996) extend Deheuvels' definition to the so called multivariate maxima of moving maxima (henceforth M4) process:

$$Y_{i,d} = \max_l \max_k a_{l,k,d} Z_{l,i-k},\ d = 1, ..., D, -\infty < i < \infty,$$

for nonnegative constants $\{a_{l,k,d}, l \geq 1, -\infty < k < \infty\}$ satisfying $\sum_{l=1}^{\infty} \sum_{k=-\infty}^{\infty} a_{l,k,d} = 1$ for $d = 1, ..., D$, and $\{Z_{l,k}, l \geq 1, -\infty < k < \infty\}$ being an array of independent unit Fréchet random variables which have distribution form $F(x) = \exp(-1/z)$, $z > 0$. These are very flexible for temporally dependent multivariate extreme value models. The tail dependence concerning exceedances, i.e., tail dependence coefficients

$$\lambda_{dd'_r} = \lim_{x\to\infty} P(Y_{1+r,d'} > x | Y_{1,d})$$

and analogous extended versions, $\eta_{dd'_r}$, of Ledford and Tawn coefficient, η, have been derived in Heffernan *et al.* [15] (2007). More precisely,

$$P(Y_{1,d} < x, Y_{1+r,d'} < x) = \exp\Big\{ -\sum_{l=1}^{\infty}\sum_{k=-\infty}^{\infty} \max(a_{l,k,d}, a_{l,k+r,d'}) x^{-1} \Big\}$$

and, as $x \to \infty$,

$$\exp\Big\{\sum_{l=1}^{\infty}\sum_{k=-\infty}^{\infty} \max(a_{l,k,d}, a_{l,k+r,d'}) x^{-1}\Big\} \sim 1 - \sum_{l=1}^{\infty}\sum_{k=-\infty}^{\infty} \max(a_{l,k,d}, a_{l,k+r,d'}) x^{-1}$$

and

$$\lambda_{dd'_r}^{(M4)} = 2 - \textstyle\sum_{l=1}^{\infty}\sum_{k=-\infty}^{\infty} \max(a_{l,k,d}, a_{l,k+r,d'}) \quad \text{and} \quad \eta_{dd'_r}^{(M4)} = 1.$$

Hence, for sufficiently large x,

$$\begin{aligned}
&P(Y_{r+1,d'} \leq x < Y_{r+2,d'}, Y_{1,d} \leq x < Y_{2,d})\\
=&P(Y_{r+1,d'} \leq x, Y_{1,d} \leq x) - P(Y_{r+1,d'} \leq x, Y_{1,d} \leq x, Y_{2,d} \leq x)\\
&-P(Y_{r+1,d'} \leq x, Y_{1,d} \leq x, Y_{r+2,d'} \leq x) + P(Y_{r+1,d'} \leq x, Y_{1,d} \leq x, Y_{2,d} \leq x, Y_{r+2,d'} \leq x)\\
=&\exp\Big\{ -\sum_{l=1}^{\infty}\sum_{k=-\infty}^{\infty} \max(a_{l,k,d}, a_{l,k+r,d'}) x^{-1}\Big\}\\
&-\exp\Big\{ -\textstyle\sum_{l=1}^{\infty}\sum_{k=-\infty}^{\infty} \max(a_{l,k,d}, a_{l,k+r,d'}, a_{l,k+1,d}) x^{-1}\Big\}\\
&-\exp\Big\{ -\sum_{l=1}^{\infty}\sum_{k=-\infty}^{\infty} \max(a_{l,k,d}, a_{l,k+r,d'}, a_{l,k+r+1,d'}) x^{-1}\Big\}\\
&+\exp\Big\{ -\sum_{l=1}^{\infty}\sum_{k=-\infty}^{\infty} \max(a_{l,k,d}, a_{l,k+r,d'}, a_{l,k+1,d}, a_{l,k+r+1,d'}) x^{-1}\Big\}\\
\sim&\sum_{l=1}^{\infty}\sum_{k=-\infty}^{\infty}\max(a_{l,k,d}, a_{l,k+r,d'}, a_{l,k+1,d}) x^{-1} + \sum_{l=1}^{\infty}\sum_{k=-\infty}^{\infty}\max(a_{l,k,d}, a_{l,k+r,d'}, a_{l,k+r+1,d'}) x^{-1}\\
&-\sum_{l=1}^{\infty}\sum_{k=-\infty}^{\infty}\max(a_{l,k,d}, a_{l,k+r,d'}) x^{-1} - \sum_{l=1}^{\infty}\sum_{k=-\infty}^{\infty}\max(a_{l,k,d}, a_{l,k+r,d'}, a_{l,k+1,d}, a_{l,k+r+1,d'}) x^{-1}\\
=&A\, x^{-1},
\end{aligned} \tag{30}$$

and also

$$\begin{aligned}
& P(Y_{1,d} \leq x < Y_{2,d}) = P(Y_{1,d} \leq x) - P(Y_{1,d} \leq x, Y_{2,d} \leq x) \\
\sim\ & \textstyle\sum_{l=1}^{\infty} \sum_{k=-\infty}^{\infty} \max(a_{l,k,d}, a_{l,k+1,d}) x^{-1} - \displaystyle\sum_{l=1}^{\infty} \sum_{k=-\infty}^{\infty} a_{l,k,d} x^{-1} \\
=\ & \textstyle\sum_{l=1}^{\infty} \sum_{k=-\infty}^{\infty} \max(a_{l,k,d}, a_{l,k+1,d}) x^{-1} - x^{-1} \\
=\ & B\, x^{-1},
\end{aligned} \tag{31}$$

which is non null if $a_{l,k,d}$ is non decreasing as a function of k (otherwise upcrossing events, $\{Y_{i,d} \leq x < Y_{i+1,d}\}$, would be impossible). Therefore, under this assumption, by (30) and (31), we obtain,

$$\mu_{dd'_r}^{(M4)} = P(Y_{r+1,d'} \leq x < Y_{r+2,d'} | Y_{1,d} \leq x < Y_{2,d}) \sim \tfrac{A}{B}$$

corresponding to upcrossings-tail dependence. Hence, $\nu_{dd'_r}^{(M4)} = 1$ with slowly varying function $L_{\nu_{dd'_r}^{(M4)}} \equiv A/B$.

Since all variables in model M4 are asymptotically dependent, Heffernan *et al.* [15] (2007) propose an extension in order to include also asymptotical independence. More precisely, they present

$$Y_{i,d} = \max\big(U_{i,d}^{1/\alpha}, \max_l \max_k a_{l,k,d} Z_{l,i-k}\big),\ d = 1, ..., D, -\infty < i < \infty, \tag{32}$$

where $\alpha > 0$ and $\{U_{i,d}, -\infty < i < \infty, d = 1, ..., D\}$ are an array of positive independent r.v.'s and independent of $Z_{l,i}$. As before we consider unit Fréchet marginals.

Observe that we now have,

$$P(Y_{i,d} < x) = \exp\{-x^{-\alpha} - x^{-1}\}$$

as well as,

$$P(Y_{1,d} < x, Y_{1+r,d'} < x) = \exp\Big\{ - 2x^{-\alpha} - \sum_{l=1}^{\infty} \sum_{k=-\infty}^{\infty} \max(a_{l,k,d}, a_{l,k+r,d'}) x^{-1} \Big\}.$$

In this case, we have,

$$\lambda_{dd'_r}^{(EM4)} = \begin{cases} 0 & , \alpha < 1 \\ \lambda_{dd'_r}^{(M4)} & , \alpha \geq 1 \end{cases} \quad \text{and} \quad \eta_{dd'_r}^{(EM4)} = \begin{cases} \max(1/2, \alpha) & , \alpha < 1 \\ 1 & , \alpha \geq 1 \end{cases} \tag{33}$$

See Heffernan *et al.* [15] (2007) for details.

Similarly to (30) and (31) we derive successively,

$$\begin{aligned}
& P(Y_{r+1,d'} \le x < Y_{r+2,d'}, Y_{1,d} \le x < Y_{2,d}) \\
= \;& P(Y_{r+1,d'} \le x, Y_{1,d} \le x) - P(Y_{r+1,d'} \le x, Y_{1,d} \le x, Y_{2,d} \le x) \\
& - P(Y_{r+1,d'} \le x, Y_{1,d} \le x, Y_{r+2,d'} \le x) + P(Y_{r+1,d'} \le x, Y_{1,d} \le x, Y_{2,d} \le x, Y_{r+2,d'} \le x) \\
= \;& \exp\Big\{ - 2x^{-\alpha} - \sum_{l=1}^{\infty} \sum_{k=-\infty}^{\infty} \max(a_{l,k,d}, a_{l,k+r,d'}) x^{-1} \Big\} \\
& - \exp\Big\{ - 3x^{-\alpha} - \textstyle\sum_{l=1}^{\infty} \sum_{k=-\infty}^{\infty} \max(a_{l,k,d}, a_{l,k+r,d'}, a_{l,k+1,d}) x^{-1} \Big\} \\
& - \exp\Big\{ - 3x^{-\alpha} - \sum_{l=1}^{\infty} \sum_{k=-\infty}^{\infty} \max(a_{l,k,d}, a_{l,k+r,d'}, a_{l,k+r+1,d'}) x^{-1} \Big\} \\
& + \exp\Big\{ - 4x^{-\alpha} - \sum_{l=1}^{\infty} \sum_{k=-\infty}^{\infty} \max(a_{l,k,d}, a_{l,k+r,d'}, a_{l,k+1,d}, a_{l,k+r+1,d'}) x^{-1} \Big\} \\
\sim \;& A\, x^{-1} + 3x^{-2\alpha},
\end{aligned}$$

and

$$\begin{aligned}
& P(Y_{1,d} \le x < Y_{2,d}) = P(Y_{1,d} \le x) - P(Y_{1,d} \le x, Y_{2,d} \le x) \\
\sim \;& x^{-\alpha} + \textstyle\sum_{l=1}^{\infty} \sum_{k=-\infty}^{\infty} \max(a_{l,k,d}, a_{l,k+1,d}) x^{-1} - \displaystyle\sum_{l=1}^{\infty} \sum_{k=-\infty}^{\infty} a_{l,k,d} x^{-1} \\
= \;& x^{-\alpha} + B\, x^{-1},
\end{aligned}$$

Therefore, denoting $\mu_{dd'_r}^{(EM4)}$ for extended M4 process in (32), we have

$$\mu_{dd'_r}^{(EM4)} = \begin{cases} 0 & \text{, if } \alpha < 1 \\ \mu_{dd'_r}^{(M4)} & \text{, if } \alpha \ge 1 \end{cases}$$

When $\alpha < 1$, as $x \to \infty$,

$$\frac{P(Y_{r+1,d'} \le x < Y_{r+2,d'}, Y_{1,d} \le x < Y_{2,d})}{P(Y_{1,d} \le x < Y_{2,d})^{1/\nu_{dd'_r}^{(EM4)}}} \sim \frac{Ax^{-1} + 3x^{-2\alpha}}{[x^{-\alpha} + Bx^{-1}]^{1/\nu_{dd'_r}^{(EM4)}}},$$

which implies, $\nu_{dd'_r}^{(EM4)} = \max(1/2, \alpha)$ with slowly varying function $L_{\nu_{dd'_r}^{(EM4)}}(x) = 3\mathbf{1}_{\{\alpha \le 1/2\}} + A\mathbf{1}_{\{\alpha > 1/2\}}$. If $\alpha \ge 1$, then $\nu_{dd'_r}^{(EM4)} = \nu_{dd'_r}^{(M4)} = 1$ with slowly varying function $L_{\nu_{dd'_r}^{(EM4)}} \equiv L_{\nu_{dd'_r}^{(M4)}}$. Observe that coefficient $\nu_{dd'_r}^{(EM4)}$ coincides with $\eta_{dd'_r}^{(EM4)}$ in (33).

4.2 Levels that persist for a fixed period of time

The main objective of an extreme value analysis is to estimate the probability of events that are more extreme than any that have already been observed. By way of example, suppose that a sea-wall projection requires a coastal defense from all sea-levels, for the next 100 years. Extremal models are a precious tool that enables extrapolations of this type. However, an adverse situation may also be the permanency of high values in time. Draisma [7] broaches this problem with regard to successive high tide water levels registered on some places of Holland's coast which may damage the sand dunes and hence give rise to devastating floods. More formally, given a time series of water levels, $\{X_1, ..., X_n\}$, he presents a new sequence $\{Y_i\}$, such that,

$$Y_i = \min(X_i, ..., X_{i+s}), \tag{34}$$

where s is some fixed positive integer, that is, $\{Y_i\}$ is a sequence where each observation y_i is a value that persist for $s+1$ successive periods of time. We will look at the extremal behavior of $\{Y_i\}$ by considering first that $\{X_i\}$ is an i.i.d. sequence and then considering two particular stationary cases of $\{X_i\}$: $pARMAX$ and $ARMAX$. The sequence $\{Y_i\}$ is obviously stationary, hence it exists a common marginal d.f., which we will denote by F_Y. In the following we will use notation $a_t = F_Y^{-1}(1-t)$.

4.2.1 $\{X_i\}$ is i.i.d.

Let $\{X_i\}$ be an i.i.d. sequence. We have that $\{Y_i\}$ satisfying (34) is $(s+1)$-dependent and satisfies condition $\mathrm{D}'(u_n)$ (Leadbetter *et al.* [16], 1983). Assuming the regularly varying condition (38) and given the independence of $\{X_i\}$,

$$1 - F_Y(x) = (1 - F(x))^{s+1} = x^{-\frac{1}{\gamma/(s+1)}} \left(L_F(x)\right)^{s+1}$$

and hence, $\gamma_Y = \gamma/(s+1)$. Observe also that

$$F_Y^{-1}(1-t) = F^{-1}\big(1 - t^{1/(s+1)}\big).$$

Considering the random pair (Y_1, Y_{1+m}) composed by two r.v.'s with a lag-distance m, we have

$$P\Big(Y_1 > F_Y^{-1}\big(1-t\big), Y_{1+m} > F_Y^{-1}\big(1-t\big)\Big) = \begin{cases} t^{1+m/(s+1)} & , m \leq s \\ t^2 & , m > s\,. \end{cases}$$

and hence, $\eta_m = (s+1)/(s+m+1)$ for $m \leq s$ and $\eta_m = 1/2$ for $m > s$, with $L_m(t) = 1$ for all $m \in \mathbb{N}$ (for details see Ferreira and Canto e Castro [10], 2008). It is straightforward that $\lambda_m^{(Y)} = 0$, for all $m \in \mathbb{N}$, and hence, by (23), $\mu_m^{(Y)} = 0$, which agrees with the fact that condition $\mathrm{D}'(u_n)$ holds and $\theta = 1$ (Ferreira and Canto e Castro [10], 2008). From (28) we also have $\vartheta = 1$.

Now we focus on the calculation of coefficient $\nu_m^{(Y)}$. Note that,

$$P(Y_{2+m} \leq x < Y_{3+m}, Y_1 \leq x < Y_2) = 0, \text{ if } m \leq s. \tag{35}$$

For $m > s$, given the independence and stationarity of sequence $\{X_i\}$, we have

$$P(Y_{2+m} \leq x < Y_{3+m}, Y_2 \leq x < Y_1) = P(Y_1 \leq x < Y_2)^2,$$

and hence, $\nu_m^{(Y)} = 1/2$ and $L_{\nu_m^{(Y)}}(t) = 1$.

4.2.2 $\{X_i\}$ is stationary: $pARMAX$ and $ARMAX$

As already mentioned, the motivation for studying the sequence of levels that persist for a fixed period of time emerges from its potential applicability to natural phenomenon data. Whenever the independence seems an unrealistic assumption, we must consider dependent models. Max-autoregressive processes have revealed very useful in what respects the extremal analysis of time series. We consider for sequence $\{X_i\}$ the processes, $pARMAX$ and $ARMAX$, defined below in (36) and (37), respectively, given their suitably for extreme values modeling, easily derived finite-dimensional d.f.'s and quite different tail behavior concerning measures based on exceedances of high values. See Ferreira and Canto e Castro [10] for details.

Consider $\{Z_i\}$ a sequence of i.i.d. copies of r.v. Z with positive support and marginal d.f. F_Z. A sequence $\{X_i\}$ is said to be a $pARMAX$ process if,

$$X_i = X_{i-1}^c \vee Z_i \ , 0 < c < 1, \ \ i = 0, \pm 1, \pm 2, ... \tag{36}$$

and is said to be an $ARMAX$ process if,

$$X_i = cX_{i-1} \vee Z_i \ , 0 < c < 1, \ \ i = 0, \pm 1, \pm 2, ... \tag{37}$$

with X_i independent of Z_j, for all integer $i < j$. For the sake of stationarity in the $pARMAX$ case, the innovations $\{Z_i\}$ have support in $[1, \infty[$.

We start by analyzing the processes themselves (some auxiliary calculations are in Appendix), then we study sequence $\{Y_i\}$ of levels $ARMAX$ and $ARMAX$ persisting in time. We shall always consider, both with Pareto-type marginal d.f. F,

$$1 - F(x) = x^{-1/\gamma} L_F(x), \tag{38}$$

where L_F is a slow varying function at $+\infty$ and γ (the tail index) is positive, which is the most interesting case. Let $\{X_i\}$ be a $pARMAX$ process satisfying (36). Based on relations (A.4)-(A.8), we have that, as $t \downarrow 0$,

$$P(X_1 > F^{-1}(1-t), X_{1+m} > F^{-1}(1-t)) \sim t^2 \mathbf{1}_{\{c^m \leq 1/2\}} + t^{1/c^m} \mathbf{1}_{\{c^m > 1/2\}}$$

and hence, by (18), we obtain $\eta_m = \max(1/2, c^m)$ and $L_m(t) = 1.\mathbf{1}_{\{c^m \leq 1/2\}} + \mathcal{L}_m(t).\mathbf{1}_{\{c^m > 1/2\}}$ with $\mathcal{L}_m(t)$ defined in (A.7) (see Ferreira and Canto e Castro [10] for details). We have $\lambda_m = 0$, for all $m \in \mathbb{N}$ and hence, by (23), we also have $\mu_m = 0$. Observe that in $pARMAX$ processes the local dependence condition $\mathrm{D}'(u_n)$ holds and the extremal index is unit ($\theta = 1 - \lambda_1$). By relation (28) we have also an unit upcrossings index (i.e., $\vartheta = 1 - \mu_1 = 1$).

For a process $\{X_i\}$ satisfying $ARMAX$ recursion (37), we have that, as $t \downarrow 0$,

$$P(X_1 > F^{-1}(1-t), X_{1+m} > F^{-1}(1-t)) \sim tc^{m/\gamma}$$

which leads to, $\eta_m = 1$ and $L_m(t) = c^{m/\gamma}$, for all $m \in \mathbb{N}$ (see Ferreira and Canto e Castro [10] for details), and hence we have $\lambda_m = c^{m/\gamma}$. In the $ARMAX$ processes the local dependence condition $\mathrm{D}'(u_n)$ does not hold and $\theta = 1 - c^{1/\gamma}$ (Alpuim [1] 1989), which is in agreement with, respectively, Propositions 3.1 and 3.2 and remarks therein.

Analogously, and replacing t by $1-u$, we can obtain the following probabilities, as $u \uparrow 1$, in order to derive μ_m in (23):

$$\begin{aligned} \lambda_{\{3+m|2+m,2\}} &\sim P(F(X_{3+m}) > u | F(X_{2+m}) > u, F(X_2) > u) \\ &\sim \frac{1-3u+\frac{u^2}{1-(1-u)c^{1/\gamma}}+\frac{u^2}{1-(1-u)c^{(m+1)/\gamma}}+\frac{u^2}{1-(1-u)c^{m/\gamma}}-\frac{u^3}{(1-(1-u)c^{m/\gamma})(1-(1-u)c^{1/\gamma})}}{1-2u+\frac{u^2}{1-(1-u)c^{m/\gamma}}} \\ &\sim \frac{c^{(m+1)/\gamma}(1-u)}{c^{m/\gamma}(1-u)} \sim c^{1/\gamma}, \end{aligned} \tag{39}$$

$$\begin{aligned} \lambda_{\{3+m|2,1\}} &\sim P(F(X_{3+m}) > u | F(X_2) > u, F(X_1) > u) \\ &\sim \frac{1-3u+\frac{u^2}{1-(1-u)c^{1/\gamma}}+\frac{u^2}{1-(1-u)c^{(m+1)/\gamma}}+\frac{u^2}{1-(1-u)c^{(m+2)/\gamma}}-\frac{u^3}{(1-(1-u)c^{(m+1)/\gamma})(1-(1-u)c^{1/\gamma})}}{1-2u+\frac{u^2}{1-(1-u)c^{1/\gamma}}} \\ &\sim \frac{c^{(m+2)/\gamma}(1-u)}{c^{1/\gamma}(1-u)} \sim c^{(m+1)/\gamma}, \end{aligned} \tag{40}$$

and

$$\lambda_{\{3+m,2+m|2,1\}} \sim \quad P(F(X_{3+m})>u, F(X_{2+m})>u|F(X_2)>u, F(X_1)>u)$$
$$\sim \Big[1-4u+\tfrac{2u^2}{1-(1-u)c^{1/\gamma}}+\tfrac{2u^2}{1-(1-u)c^{(m+1)/\gamma}}+\tfrac{u^2}{1-(1-u)c^{(m+2)/\gamma}}+\tfrac{u^2}{1-(1-u)c^{m/\gamma}}$$
$$-\tfrac{2u^3(1-(1-u)c^{1/\gamma})^{-1}}{(1-(1-u)c^{(m+1)/\gamma})}-\tfrac{2u^3(1-(1-u)c^{1/\gamma})^{-1}}{(1-(1-u)c^{m/\gamma})}+\tfrac{u^4(1-(1-u)c^{1/\gamma})^{-2}}{(1-(1-u)c^{m/\gamma})}\Big]\Big/ \tag{41}$$
$$\Big[1-2u+\tfrac{u^2}{1-(1-u)c^{1/\gamma}}\Big]$$
$$\sim \quad \tfrac{c^{(m+2)/\gamma}(1-u)}{c^{1/\gamma}(1-u)} \sim c^{(m+1)/\gamma},$$

Hence, by (23), we also obtain $\mu_m = 0$ in the $ARMAX$ process. The local dependence condition $D''(u_n)$ holds (Canto e Castro [3], 1992) and by (28) we obtain upcrossings index $\vartheta = 1$.

Now we compute coefficient ν_m in (21). Consider first the $pARMAX$ process and $a_t = F^{-1}(1-t)$. Based on the $pARMAX$ relations (A.4)-(A.8), after some calculations we derive, as $t \downarrow 0$,

$$P(X_{3+m}>a_t, X_2>a_t) \sim t^2 + t^{1/c^{m+1}}\mathcal{L}_{m+1}(t),$$

$$P(X_{3+m}>a_t, X_{2+m}>a_t, X_2>a_t) \sim t^3 + t^{1/c^{m+1}}\mathcal{L}_{m+1}(t),$$

$$P(X_{3+m}>a_t, X_2>a_t, X_1>a_t) \sim t^3 + t^{1/c^{m+2}}\mathcal{L}_{m+2}(t)$$

and

$$P(X_{3+m}>a_t, X_{2+m}>a_t, X_2>a_t, X_1>a_t) \sim t^4 + t^{2/c}\mathcal{L}_1(t)^2 + t^{1/c^{m+2}}\mathcal{L}_{m+2}(t)$$

Therefore, we have that,

$$P(X_1 \le F^{-1}(1-t) < X_2, X_{2+m} \le F^{-1}(1-t) < X_{3+m}) \sim t^2 - 2t^3 + t^4 + t^{2/c}\mathcal{L}_1(t)^2 \sim t^2$$

and also $P(X_1 \le F^{-1}(1-t) < X_2) \sim t$. Hence, by (21), we have $\nu_m = 1/2$ and $L_{\nu_m}(t) \sim 1$ for all $m \in \mathbb{N}$ which corresponds to (almost) total independence. (See Proposition 2.5).

For $ARMAX$ process, if we apply (39)-(41) in (24), we have,

$$t^{1/\nu_m}L_{\nu_m}(t) \sim tc^{(m+1)/\gamma} - tc^{(m+1)/\gamma} - tc^{(m+2)/\gamma} + tc^{(m+2)/\gamma},$$

leading us to a null limit. Going further on the rate of the approximation and based on relations (A.9)-(A.10), after some calculations, we obtain

$$P(X_1 \le F^{-1}(1-t) < X_2, X_{2+m} \le F^{-1}(1-t) < X_{3+m}) \sim (1-c^{1/\gamma})^2(1-c^{m/\gamma})t^2, \text{ as } t \downarrow 0,$$

as well as, $P(X_1 \le F^{-1}(1-t) < X_2) \sim t(1-c^{1/\gamma})$. Hence, according to (21), we have $\nu_m = 1/2$ and $L_{\nu_m}(t) = (1-c^{1/\gamma})(1-c^{m/\gamma})$.

Now we turn to the sequence $\{Y_i\}$. Deriving results in a dependence context for $\{Y_i\}$ involves more calculations and so, in the sequel, we restrict ourselves to the case $s=1$ in (34), though we presume that similar results will be valid for any finite s.

We treat first the case where $\{X_i\}$ is a $pARMAX$ process as in (36). Based on relations (A.4)-(A.8) and after some calculations, we have, for $m>1$,

$$P(Y_{1+m}>a_t, Y_1>a_t) \sim \quad t^2\mathbf{1}_{\{c^m \le 1/2\}} + t^{1/c^m}\mathcal{L}^{(Y)}_{m+1}(t)\mathbf{1}_{\{c^m>1/2\}} \tag{42}$$

where slow varying function $\mathcal{L}_j^{(Y)}(t)$ is given in (A.7) (details can be seen in Ferreira and Canto e Castro [10], 2008). Hence it is straightforward that $\lambda_m^{(Y)} = 0$ (agrees with the fact that $\mathrm{D}'(u_n)$ holds and $\theta = 1$). The case $m = 1$ is similar. Therefore, by (23), we obtain $\mu_m^{(Y)} = 0$ for all $m \in \mathbb{N}$ (hence $\vartheta = 1$). Note that, from (42), we have $\eta_m^{(Y)} = \max(1/2, c^m)$ and $L_m^{(Y)}(t) \sim \mathbf{1}_{\{c^m \le 1/2\}} + \mathcal{L}_{m+1}^{(Y)}(t)\mathbf{1}_{\{c^m > 1/2\}}$.

In order to compute $\nu_m^{(Y)}$, observe that,

$$
\begin{aligned}
& P(Y_{2+m} \le x < Y_{3+m}, Y_1 \le x < Y_2) \\
= \; & P(Y_{3+m} > x, Y_2 > x) - P(Y_{3+m} > x, Y_{2+m} > x, Y_2 > x) \\
& - P(Y_{3+m} > x, Y_1 > x, Y_2 > x) + P(Y_{3+m} > x, Y_{2+m} > x, Y_2 > x, Y_1 > x) \\
= \; & P(X_1 < x, X_{2+m} < x) - P(X_1 < x, X_2 < x, X_{2+m} < x) \\
& - P(X_1 < x, X_3 < x, X_{2+m} < x) - P(X_1 < x, X_{2+m} < x, X_{3+m} < x) \\
& - P(X_1 < x, X_{2+m} < x, X_{4+m} < x) + P(X_1 < x, X_2 < x, X_3 < x, X_{2+m} < x) \\
& + P(X_1 < x, X_2 < x, X_{2+m} < x, X_{3+m} < x) + P(X_1 < x, X_2 < x, X_{2+m} < x, X_{4+m} < x) \\
& + P(X_1 < x, X_3 < x, X_{2+m} < x, X_{3+m} < x) + P(X_1 < x, X_3 < x, X_{2+m} < x, X_{4+m} < x) \\
& + P(X_1 < x, X_{2+m} < x, X_{3+m} < x, X_{4+m} < x) \\
& - P(X_1 < x, X_2 < x, X_3 < x, X_{2+m} < x, X_{3+m} < x) \\
& - P(X_1 < x, X_2 < x, X_3 < x, X_{2+m} < x, X_{4+m} < x) \\
& - P(X_1 < x, X_2 < x, X_{2+m} < x, X_{3+m} < x, X_{4+m} < x) \\
& - P(X_1 < x, X_3 < x, X_{2+m} < x, X_{3+m} < x, X_{4+m} < x) \\
& + P(X_1 < x, X_2 < x, X_3 < x, X_{2+m} < x, X_{3+m} < x, X_{4+m} < x).
\end{aligned}
$$

As already noticed in (35), the probability above is null if $m = 1$. By (A.4) and (A.5) and recalling notation $a_t = F_Y^{-1}(1 - t)$, we have

$$
\begin{aligned}
& P(Y_{2+m} \le a_t < Y_{3+m}, Y_1 \le a_t < Y_2) \\
= \; & \frac{F^2(a_t)}{F\left(a_t^{1/c^{1+m}}\right)} - \frac{F^3(a_t)}{F\left(a_t^{1/c}\right) F\left(a_t^{1/c^m}\right)} - \frac{F^3(a_t)}{F\left(a_t^{1/c^2}\right) F\left(a_t^{1/c^{m+1}}\right)} - \frac{F^3(a_t)}{F\left(a_t^{1/c}\right) F\left(a_t^{1/c^{m+1}}\right)} \\
& - \frac{F^3(a_t)}{F\left(a_t^{1/c^2}\right) F\left(a_t^{1/c^{m+1}}\right)} + \frac{F^4(a_t)}{F^2\left(a_t^{1/c}\right) F\left(a_t^{1/c^{m-1}}\right)} + \frac{F^4(a_t)}{F^2\left(a_t^{1/c}\right) F\left(a_t^{1/c^m}\right)} \\
& + \frac{F^4(a_t)}{F\left(a_t^{1/c}\right) F\left(a_t^{1/c^2}\right) F\left(a_t^{1/c^m}\right)} + \frac{F^4(a_t)}{F\left(a_t^{1/c}\right) F\left(a_t^{1/c^2}\right) F\left(a_t^{1/c^{m-1}}\right)} \\
& + \frac{F^4(a_t)}{F^2\left(a_t^{1/c^2}\right) F\left(a_t^{1/c^{m-1}}\right)} + \frac{F^4(a_t)}{F^2\left(a_t^{1/c}\right) F\left(a_t^{1/c^{m+1}}\right)} - \frac{F^5(a_t)}{F^3\left(a_t^{1/c}\right) F\left(a_t^{1/c^{m-1}}\right)} \\
& - \frac{F^5(a_t)}{F^2\left(a_t^{1/c}\right) F\left(a_t^{1/c^2}\right) F\left(a_t^{1/c^{m-1}}\right)} - \frac{F^5(a_t)}{F^3\left(a_t^{1/c}\right) F\left(a_t^{1/c^m}\right)} \\
& - \frac{F^5(a_t)}{F^2\left(a_t^{1/c}\right) F\left(a_t^{1/c^2}\right) F\left(a_t^{1/c^{m-1}}\right)} + \frac{F^6(a_t)}{F^4\left(a_t^{1/c}\right) F\left(a_t^{1/c^{m-1}}\right)}
\end{aligned}
\tag{43}
$$

Since, as $t \downarrow 0$, we have

$$1 - F\left(a_t^{1/c^j}\right) \sim \begin{cases} t^{1/(2c^j)}\left(\mathcal{L}_j^{(Y)}(t)\right)^{1/2} & , c \leq 1/2 \\ t^{1/c^{j-1}}\mathcal{L}_{j-1}^{(Y)}(t)\big) & , c > 1/2\,, \end{cases}$$

with slow varying function $\mathcal{L}_j^{(Y)}(t)$ given in (A.7) (see Ferreira and Canto e Castro [10], 2008), if we apply (A.6)-(A.7), after some calculations we obtain, as $t \downarrow 0$,

$$P(Y_{2+m} \leq a_t < Y_{3+m}, Y_1 \leq a_t < Y_2) \sim t^2$$

and also $P(Y_1 \leq a_t < Y_2) \sim t$. Therefore, by definition in (21), we have for $pARMAX$ case, $\nu_m^{(Y)} = 1/2$ and $L_{\nu_m^{(Y)}}(t) \sim 1$.

Regarding the $ARMAX$ process in (37), we have

$$P(Y_{1+m} > a_t, Y_1 > a_t) \sim tc^{m/\gamma},$$

as $t \downarrow 0$ (see details in Ferreira and Canto e Castro [10], 2008), and hence we have, $\lambda_m^{(Y)} = c^{m/\gamma}$ (in agreement with the fact that $\mathrm{D}'(u_n)$ does not hold and $\theta = 1 - c^{1/\gamma}$), $\eta_m^{(Y)} = 1$ and $L_m^{(Y)}(t) \sim c^{m/\gamma}$ for all $m \in \mathbb{N}$. A similar reasoning to that in (43) leads to

$$\begin{aligned} & P(Y_{2+m} \leq a_t < Y_{3+m}, Y_1 \leq a_t < Y_2) \\ = \; & \frac{F^2(a_t)}{F(a_t/c^{1+m})} - \frac{F^3(a_t)}{F(a_t/c)F(a_t/c^m)} - \frac{F^3(a_t)}{F(a_t/c^2)F(a_t/c^{m+1})} - \frac{F^3(a_t)}{F(a_t/c)F(a_t/c^{m+1})} \\ & - \frac{F^3(a_t)}{F(a_t/c^2)F(a_t/c^{m+1})} + \frac{F^4(a_t)}{F^2(a_t/c)F(a_t/c^{m-1})} + \frac{F^4(a_t)}{F^2(a_t/c)F(a_t/c^m)} + \frac{F^4(a_t)}{F(a_t/c)F(a_t/c^2)F(a_t/c^m)} \\ & + \frac{F^4(a_t)}{F(a_t/c)F(a_t/c^2)F(a_t/c^{m-1})} + \frac{F^4(a_t)}{F^2(a_t/c^2)F(a_t/c^{m-1})} + \frac{F^4(a_t)}{F^2(a_t/c)F(a_t/c^{m+1})} - \frac{F^5(a_t)}{F^3(a_t/c)F(a_t/c^{m-1})} \\ & - \frac{F^5(a_t)}{F^2(a_t/c)F(a_t/c^2)F(a_t/c^{m-1})} - \frac{F^5(a_t)}{F^3(a_t/c)F(a_t/c^m)} - \frac{F^5(a_t)}{F^2(a_t/c)F(a_t/c^2)F(a_t/c^{m-1})} + \frac{F^6(a_t)}{F^4(a_t/c)F(a_t/c^{m-1})} \end{aligned}$$

Since we have, as $t \downarrow 0$,

$$1 - F(a_t/c^j) \sim 1 - F_Y(c^{j-1}a_t) \sim c^{(j-1)/\gamma}t$$

in the $ARMAX$ case (Ferreira and Canto e Castro [10], 2008), after some calculations we derive

$$P(Y_{2+m} \leq a_t < Y_{3+m}, Y_1 \leq a_t < Y_2) \sim (1 - c^{1/\gamma})^2(1 - c^{(m-1)/\gamma})t^2,$$

as well as,

$$P(Y_1 \leq a_t < Y_2) \sim (1 - c^{1/\gamma})t.$$

Therefore, according to (20) we have $\mu_m^{(Y)} = 0$ (agrees with the fact that $\mathrm{D}''(u_n)$ holds; we have then $\vartheta = 1$), and by (21), $\nu_m^{(Y)} = 1/2$ and $L_{\nu_m^{(Y)}}(t) \sim (1 - c^{(m-1)/\gamma})$.

5 Inference: some notes

The estimation of coefficient μ can be made through the multivariate tail dependence coefficients $\lambda_{X_I,Y_J|X_L,Y_K}$ given the relation stated in (7). Observe that they can be defined via the notion of copula, introduced by Sklar [24] (1959). A copula C is a cumulative distribution function whose margins are uniformly distributed on $[0,1]$, i.e., $C(u_1,...,u_d) = P(F_1(X_1) \leq u_1,...,F_d(X_d) \leq u_d)$, where $F_1,...,F_d$ are the continuous marginal d.f.'s of random vector $(X_1,...,X_d)$ and $(u_1,...,u_d) \in$

$[0,1]^d$. The copula is unique as long as the marginal d.f.'s are continuous, a requisite that we assume. For instance, we have

$$\lambda = \lim_{u\uparrow 1}\ 2 - \frac{1-C(u,u)}{1-u} = \lim_{u\uparrow 1}\ 2 - \frac{\log C(u,u)}{\log u}.$$

Parametric estimation methods, based on either a specific distribution or family of distributions or a specific copula or family of copulas, as well as nonparametric estimation procedures are already known in literature. For a survey see Frahm *et al.* [13] (2005) and Schmidt and Stadtmüller [23] (2006). In the sequel we shall focus on nonparametric estimation.

Under conditions somewhat wide in this context, Schmidt and Stadtmüller [23] (2006) prove strong consistency and asymptotic normality of the general nonparametric estimator

$$\widehat{\lambda}_{Z_I|Z_L} = \sum_{j=1}^{n} \mathbf{1}_{\{R_{nl}^{(j)}>n-k,\ \forall l\in I\cup L\}} \Big/ \sum_{j=1}^{n} \mathbf{1}_{\{R_{nl}^{(j)}>n-k,\ \forall l\in L\}}$$

for the general coefficient $\lambda_{Z_I|Z_L}$, considering random vector $Z=(Z_1,...,Z_d)$, where $I\cup L\subset\{1,...,d\}$ $(I\cap L=\emptyset)$, $R_{nl}^{(j)}$ denotes de rank of component $Z_l^{(j)}$, $k=k(n)\to\infty$ and $k/n\to 0$ as $n\to\infty$. In our case, the most interesting situation is to consider the non-independent sequence, (X_1,X_2,Y_1,Y_2), (X_3,X_4,Y_3,Y_4),...,$(X_{n-1},X_n,Y_{n-1},Y_n)$. If we assume a regularity condition for the joint tail of (X_1,X_2,Y_1,Y_2) and a uniform bound on the probability that both X_1 and X_2, or Y_1 and Y_2, belong to an extremal interval, similar to conditions (C2) and (C3) in Drees [9] (2003), we still derive asymptotic normality with eventually modified variance (see Proposition 2.1 and Theorem 2.1 in Drees [9] 2003, and Theorem 6/10 in Schmidt and Stadtmüller [23] 2006). Therefore, by plugging in the respective tail dependence coefficient estimators in expression (7), we derive estimator,

$$\widehat{\mu} = (1-\widehat{\lambda}_{X_{\{2\}}|X_{\{1\}}})^{-1}\Big[\widehat{\lambda}_{Y|X} - \widehat{\lambda}_{Y_{\{2\}}|Y_{\{1\}}}\widehat{\lambda}_{X_{\{2\}}|Y_{\{1,2\}}} - \widehat{\lambda}_{X_{\{2\}}|X_{\{1\}}}\widehat{\lambda}_{Y_{\{2\}}|X_{\{1,2\}}} + \\ + \widehat{\lambda}_{Y|X}\widehat{\lambda}_{X_{\{2\}},Y_{\{2\}}|X_{\{1\}},Y_{\{1\}}}\Big],$$

which is also strong consistent (straightforward from Theorem 11 Schmidt and Stadtmüller [23] 2006) and asymptotic normal. One important practical problem arises in the optimal choice of the parameter k which relates to the usual variance-bias problem. An algorithm to choose the optimal threshold k can be seen in Schmidt and Stadtmüller [23] (2006).

Other estimators arise from the relation between μ and the upcrossings index ϑ which in turn relates with the extremal index θ by $\vartheta = \frac{\tau}{\varsigma}\theta$ in (27). More precisely, under conditions $\Delta(u_n)$ and $\widetilde{D}^{(3)}(u_n)$, we have

$$\mu_1 = 1-\vartheta.$$

with μ_1 given in (20). Estimation of ϑ can be done through the extremal index θ, modified by consistent estimates of the mean number of exceedances (τ) and the mean number of upcrossings (ς) of high levels. There are several estimators of the extremal index in literature. For a survey see Ancona-Navarrete and Tawn [2].

Now we consider coefficient ν introduced in (12). Given the relations stated with other parameters well-known and studied in the literature, we can also derive quite straightforward estimators for ν. More precisely, under conditions of Proposition 2.5, an estimator suggesting it-self is,

$$\widehat{\nu} = \max(\widehat{\eta}, \widehat{\eta}_{X_{\{2\}},Y_{\{1,2\}}}, \widehat{\eta}_{X_{\{1,2\}},Y_{\{2\}}}, \widehat{\eta}_{X_{\{1,2\}},Y_{\{1,2\}}}),$$

Observe that coefficient η_{X_I,Y_J} in (13) corresponds to the tail index of r.v. $\min(X_I,Y_J)$ for which many estimators with good properties have been established (hill, pickands, maximum-likelihood, moments, power weighted moments, are the most known). Other estimators have also been proposed. For a survey see Coles *et al.* [5] (1999), Peng [21] (1999) and Draisma *et al.* [8] (2004).

In a future research, we intend to derive other estimators and respective asymptotic properties, as well as analyze and compare them with the above mentioned estimators through simulation.

A Appendix: $ARMAX$ and $pARMAX$ processes

We derive some useful properties about processes $pARMAX$ in (36) and $ARMAX$ in (37), both with Pareto-type marginal d.f. F given in (38). We denote left-end-point, x_*, and right-end-point, $x_F = +\infty$. Formulation (38) means also that $1-F$ is a regularly varying function at ∞ of order $-1/\gamma$. Equivalently, we consider a regularly varying tail quantile function of order $-\gamma$,

$$F^{-1}(1-t) = t^{-\gamma} L_{F^{-1}}(t), \tag{A.1}$$

with function $L_{F^{-1}}$ slowly varying at 0. Since,

$$F(F^{-1}(1-t)) \sim F\big(t^{-\gamma} L_{F^{-1}}(t)\big) = 1 - t\big[L_{F^{-1}}(t)\big]^{-1/\gamma} L_F\big(t^{-\gamma} L_{F^{-1}}(t)\big),$$

we have the following relation between L_F and $L_{F^{-1}}$:

$$\big[L_{F^{-1}}(t)\big]^{-1/\gamma} L_F\big(t^{-\gamma} L_{F^{-1}}(t)\big) \sim 1,\ t \downarrow 0. \tag{A.2}$$

$$\tag{A.3}$$

The stationarity equation of $pARMAX$ in (36) is given by

$$F(x) = F(x^{1/c}) F_Z(x)\,.$$

whilst for $ARMAX$ in (37) it is given by

$$F(x) = F(x/c) F_Z(x)\,,$$

Using the latest, we derive the respective *m-step* transition probability functions (t.p.f.) from x to $]-\infty, y]$: for $pARMAX$ process we have,

$$Q^m(x,]-\infty, y]) := P\big(X_{n+m} \le y | X_n = x\big) = \tfrac{F(y)}{F(y^{1/c^m})} \mathbf{1}_{\left\{x \le y^{1/c^m}\right\}}.$$

and for $ARMAX$ process it is given by,

$$Q^m(x,]-\infty, y]) := P\big(X_{n+m} \le y | X_n = x\big) = \tfrac{F(y)}{F(y/c^m)} \mathbf{1}_{\left\{x \le y/c^m\right\}},$$

where $\mathbf{1}_{\{.\}}$ denotes the indicator function.

In the following we derive multivariate d.f.'s within each process.

- for $pARMAX$ recursion in (36), we have

$$P(X_i \le y, X_j \le y) \quad = \quad \int_{x_*}^{y} Q^{j-i}(x,]-\infty, y]) F(dx) = \frac{F^2(y)}{F\big(y^{1/c^{j-i}}\big)} \tag{A.4}$$

Moreover, for the multivariate case,

$$\begin{aligned} & P(X_{i_1} \le y, .., X_{i_k} \le y) \\ = & \int_{x_*}^{F^{-1}(u)} \cdots \int_{x_*}^{F^{-1}(u)} Q^{i_k - i_{k-1}}\big(x_{i_{k-1}},]-\infty, y]\big) \prod_{j=2}^{k-1} Q^{i_{k-j} - i_{k-j+1}}(x_{i_{k-j}}, dx_{i_{k-j+1}}) F(dx_{i_1}) \qquad \text{(A.5)} \\ = & \frac{F^k(y)}{\prod_{j=2}^{k} F\big(y^{1/c^{i_j - i_{j-1}}}\big)} \end{aligned}$$

Observe now that,

$$F(F^{-1}(1-t)^{1/c^j}) = F\big(t^{-\gamma/c^j}(L_{F^{-1}}(t))^{1/c^j}\big) = 1 - t^{1/c^j}\mathcal{L}_j(t), \tag{A.6}$$

where

$$\mathcal{L}_j(t) = (L_{F^{-1}}(t))^{-1/(\gamma c^j)} L_F\big(t^{-\gamma/c^j}(L_{F^{-1}}(t))^{-1/c^j}\big) \tag{A.7}$$

By (A.1)-(A.2), we have that

$$\mathcal{L}_j(t) \text{ is slow varying, as } t \downarrow 0. \tag{A.8}$$

- for $ARMAX$ recursion in (37), in a similar manner we derive,

$$P(X_i \leq y, X_j \leq y) = \frac{F^2(y)}{F\big(y/c^{j-i}\big)} \tag{A.9}$$

In the multivariate case we have

$$P(X_{i_1} \leq y, .., X_{i_k} \leq y) = \frac{F^k(y)}{\prod_{j=2}^{k} F\big(y/c^{i_j - i_{j-1}}\big)}$$

and

$$F(F^{-1}(1-t)/c^j) = F\big(t^{-\gamma}/c^j L_{F^{-1}}(t)\big) = 1 - tc^{j/\gamma}\mathfrak{L}_j(t).$$

where, by (A.1)-(A.2), we have

$$\mathfrak{L}_j(t) = (L_{F^{-1}}(t))^{-1/\gamma} L_F\big(t^{-\gamma}(L_{F^{-1}}(t))/c^j\big) \sim 1, \text{ as } t \downarrow 0. \tag{A.10}$$

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