A Mean-of-order-p Class of Value-at-Risk Estimators

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October 11, 2013

Abstract

The main objective of *statistics of univariate extremes* lies in the estimation of quantities related to extreme events. In many areas of application, like *finance*, *insurance* and *statistical quality control*, a typical requirement is to estimate a *high quantile*, i.e. the *Value at Risk* at a level q (VaR_q), high enough, so that the chance of an exceedance of that value is equal to q, small. In this paper we deal with the semi-parametric estimation of VaR_q, for heavy tails, introducing a new class of VaR-estimators based on a class of *mean-of-order-p* (MOP) *extreme value index* (EVI)-estimators, recently introduced in the literature. Interestingly, the MOP EVI-estimators can have a mean square error smaller than that of the classical EVI-estimators, even for small values of k. They are thus a nice basis to build alternative VaR-estimators not only around optimal levels, but for other levels too. The new VaR-estimators are compared with the classical ones, not only asymptotically, but also for finite samples, through Monte-Carlo techniques.

AMS 2000 subject classification. Primary 62G32; Secondary 65C05.

Keywords and phrases. Heavy tails, high quantiles, Monte-Carlo simulation, semi-parametric estimation, statistics of extremes, value at risk

1 Introduction and scope of the paper

A relevant situation in risk management is the risk of a big loss that occurs rarely or even very rarely. Such a risk is generally expressed as the *Value at Risk* (VaR), i.e. the size of the loss that occurred with a fixed small probability, q. We are thus dealing with a (high) *quantile*,

$$\chi_{1-q} \equiv \mathrm{VaR}_q := F^{\leftarrow}(1-q),$$

of an unknown cumulative distribution function (CDF) F, with $F^{\leftarrow}(y) = \inf\{x : F(x) \geq y\}$, the generalized inverse function of F. As usual, let us denote $U(t)$ the generalized inverse function of $1/(1-F)$. Then, for small q, we want to estimate the parameter

$$\mathrm{VaR}_q = U\left(1/q\right), \quad q = q_n \rightarrow 0, \quad nq_n \leq 1,$$

i.e. we want to extrapolate beyond the sample. Since we are dealing with a small probability q, we may confine ourselves to modeling the tail. Moreover, since in real applications one often encounters heavy tails, we shall assume that the CDF underlying the data satisfies

$$1 - F(x) \ \sim \ c\, x^{-1/\gamma}, \quad \text{as } x \rightarrow \infty, \tag{1.1}$$

for some positive constant c. Equivalently, and for some $C > 0$,

$$U(t) \ \sim \ C\, t^{\gamma}, \quad \text{as } t \rightarrow \infty, \tag{1.2}$$

where the notation $a(y) \sim b(y)$ means that $a(y)/b(y) \rightarrow 1$, as $y \rightarrow \infty$. The parameter γ in either (1.1) or (1.2) is the *extreme value index* (EVI), the primary parameter of extreme (and large) events.

Generally (Gnedenko, 1943), if we consider a random sample $(X_1, \ldots, X_n)$ from F and we can find attraction coefficients (a_n, b_n), with $a_n > 0$ and $b_n \in \mathbb{R}$, such that the sequence of suitably normalized maxima, $\{(X_{n:n} - b_n)/a_n\}_{n\geq 1}$, converges to a non-degenerate random variable (RV), such a RV is compulsory of the type of a general *extreme value* (EV) CDF,

$$\mathrm{EV}_\gamma(x) = \begin{cases} \exp(-(1+\gamma x)^{-1/\gamma}, 1+\gamma x > 0, & \text{if} \quad \gamma \neq 0, \\ \exp(-\exp(-x)), \ x > 0, & \text{if} \quad \gamma = 0. \end{cases} \tag{1.3}$$

We then say that F is in the max-domain of attraction of EV_γ, and use the notation $F \in \mathcal{D}_{\mathcal{M}}(\mathrm{EV}_\gamma)$. If (1.1) holds, or equivalently (1.2) holds, the aforementioned result also holds, but with $\gamma > 0$.

Weissman (1978) proposed the following semi-parametric VaR-estimator:

$$Q_{\widehat{\gamma}}^{(q)}(k) := X_{n-k:n} \left(\frac{k}{nq}\right)^{\widehat{\gamma}}, \tag{1.4}$$

where $X_{n-k:n}$ is the $(k+1)$-th top order statistic (o.s.), $\widehat{\gamma}$ any consistent estimator for γ and Q stands for quantile. Further details on semi-parametric estimation of extremely high quantiles for any real EVI can be found in de Haan and Rootzén (1993) and Ferreira *et al.* (2003). For heavy right-tails, Gomes and Figueiredo (2003), Matthys and Beirlant (2003) Mathys *et al.* (2004), Gomes and Pestana (2007) and Caeiro and Gomes (2008, 2009), among others, dealt with reduced bias VaR-estimation, a topic out of the scope of this paper.

The estimator in (1.4) is an *asymptotic* estimator, in the sense that it provides useful estimates when the sample size n is high. Also, and as usual in semi-parametric estimation of parameters of extreme events, we need to work with an *intermediate* sequence of integers,

$$k = k_n \to \infty, \quad k \in [1, n), \qquad k = o(n) \qquad \text{as } n \to \infty. \tag{1.5}$$

For heavy tails, the classical EVI-estimator, usually the one which is used in (1.4), for a semi-parametric quantile estimation, is the Hill estimator $\widehat{\gamma} = \widehat{\gamma}(k) =: \mathrm{H}(k)$ (Hill, 1975), with the functional expression,

$$\mathrm{H}(k) := \frac{1}{k}\sum_{i=1}^{k} V_{ik}, \quad V_{ik} = \ln \frac{X_{n-i+1:n}}{X_{n-k:n}}, \ 1 \leq i \leq k. \tag{1.6}$$

If we insert in (1.4) the Hill estimator $\mathrm{H}(k)$, we get the so-called Weissman-Hill quantile or VaR_q-estimator, with the obvious notation, $Q_{\mathrm{H}}^{(q)}(k)$. Since $Q_{\mathrm{H}}^{(q)}(k)$ is skewed (see Gomes and Pestana, 2007), it is advisable to work with the lnVaR estimator

$$\ln Q_{\widehat{\gamma}}^{(q)}(k) = \ln X_{n-k:n} + \widehat{\gamma}(k)\ \ln\left(\frac{k}{nq}\right), \tag{1.7}$$

for any consistent EVI-estimator, $\widehat{\gamma}(k)$. Again, if we plug $\mathrm{H}(k)$ into (1.7), we get the so-called Weissman-Hill lnVaR estimator, with the obvious notation, $\ln Q_{\mathrm{H}}^{(q)}(k)$.

In order to be able to study the asymptotic behavior of $\ln Q_{\mathrm{H}}^{(q)}(k)$, as well as of alternative lnVaR_q-estimators, it is useful to impose a second-order expansion on the tail function $1-F$ or on the function U. Here we shall assume that we are working in Hall-Welsh class of models (Hall and Welsh, 1985), where, as $t \to \infty$ and with $C,\ \gamma > 0$, $\rho < 0$ and β non-zero,

$$U(t) = Ct^{\gamma}\big(1 + A(t)/\rho + o\left(t^{\rho}\right)\big), \qquad A(t) = \gamma\ \beta\ t^{\rho}. \tag{1.8}$$

The class in (1.8) is a wide class of models, that contains most of the heavy-tailed parents useful in applications, like the *Fréchet*, the *Generalized Pareto* and the *Student*-t_ν, with ν degrees of freedom. Indeed, (1.8) implies either (1.1) or (1.2).

Note next that we can write

$$\mathrm{H}(k) = \sum_{i=1}^{k} \ln\left(\frac{X_{n-i+1:n}}{X_{n-k:n}}\right)^{1/k} = \ln\left(\prod_{i=1}^{k}\frac{X_{n-i+1:n}}{X_{n-k:n}}\right)^{1/k}, \quad 1 \le i \le k < n.$$

The Hill estimator is thus the logarithm of the *geometric mean* (or *mean-of-order*-0) of $U_{ik} := X_{n-i+1:n}/X_{n-k:n},\ 1 \le i \le k < n$. More generally, Brilhante *et al.* (2013) considered as basic statistics the *mean-of-order-p* (MOP) of U_{ik}, $1 \le i \le k$, $p \in \mathbb{R}_0^+$, i.e., the class of statistics

$$\mathrm{A}_p(k) = \begin{cases} \left(\frac{1}{k}\sum\limits_{i=1}^{k} U_{ik}^{p}\right)^{1/p} & \text{if} \quad p > 0 \\ \\ \left(\prod\limits_{i=1}^{k} U_{ik}\right)^{1/k} & \text{if} \quad p = 0, \end{cases}$$

and the following class of EVI-estimators:

$$\mathrm{H}_p(k) \equiv \mathrm{MOP}(k) \equiv \widehat{\gamma}^{\mathrm{H}_p}(k) := \begin{cases} \big(1 - \mathrm{A}_p^{-p}(k)\big)/p, & \text{if} \quad 0 < p < 1/\gamma, \\ \\ \ln \mathrm{A}_0(k) = \mathrm{H}(k), & \text{if} \quad p = 0, \end{cases} \tag{1.9}$$

with $\mathrm{H}_0(k) \equiv \mathrm{H}(k)$, given in (1.6). This class of MOP EVI-estimators, studied in Brilhante *et al.* (2013), depends now on this *tuning* parameter $p \geq 0$, and was shown to be highly flexible.

The aim of this paper is to find the asymptotic and finite sample properties of alternative estimators for $\ln\mathrm{VaR}_q$, replacing, in (1.7), $\ln Q_{\widehat{\gamma}}^{(q)}(k)$ by the new $\ln\mathrm{VaR}_q$ estimators $\ln Q_{\mathrm{H}_p}^{(q)}(k)$ based on the MOP EVI-estimator, $\mathrm{H}_p(k)$, in (1.9). If we adequately choose p, the new estimators have an asymptotic mean square error (MSE) smaller than the Weissman-Hill lnVaR-estimators for all k. Consequently, they are alternatives to the previous estimators not only around optimal levels but for all k. The outline of the paper is as follows. In Section 2, we briefly discuss general first and second-order frameworks under a heavy-tailed set-up. The classes of EVI and VaR-estimators under study are discussed in Section 3. In Section 4 we deal with asymptotic properties of the EVI and lnVaR-estimators under consideration. Section 5 is devoted to a Monte-Carlo simulation, that enables the derivation of the distributional properties of the new classes of MOP lnVaR-estimators, comparatively to the Weissman-Hill lnVaR-estimators. Finally, in Section 6, we provide some general remarks on the topic.

2 A brief review of general first and second-order conditions for heavy right tails

In the area of *statistics of extremes* and whenever working with large values, i.e. with the right tail of the model F underlying the data, a model F is usually said to be *heavy-tailed* whenever the right tail-function,

$$\overline{F} := 1 - F$$

is a regularly varying function with a negative index of regular variation equal to $-1/\gamma$, $\gamma > 0$. We then use the notation $\overline{F} \in \mathcal{R}_{-1/\gamma}$. Note that a regularly varying function with an index of regular variation equal to $a \in \mathbb{R}$, i.e. an element of $\mathcal{R}_a$, is a positive measurable function $g(\cdot)$ such that for all $x > 0$, $g(tx)/g(t) \to x^a$, as $t \to \infty$ (see Bingham *et al.*, 1987, for details on regular variation). Heavy-tailed models are thus such that $\overline{F}(x) = x^{-1/\gamma} L(x)$, $\gamma > 0$, with $L \in \mathcal{R}_0$, a *regularly varying* function with an *index of regular variation* equal to zero, also called a *slowly varying* function at infinity. Equivalently, with $F^{\leftarrow}(x) := \inf\{y : F(y) \geq x\}$, the *reciprocal quantile function* $U(t) := F^{\leftarrow}(1 - 1/t),\ t \geq 1$, is of regular variation with index γ (de Haan, 1984), i.e. $U \in \mathcal{R}_\gamma$. If either (1.1) or (1.2) hold the slowly varying functionn $L(\cdot)$ behaves as a constant.

We thus have the validity of any of the equivalent and general first-order conditions,

$$F \in \mathcal{D}_{\mathcal{M}}^+ := \mathcal{D}_{\mathcal{M}} \left(EV_\gamma\right)_{\gamma > 0} \quad \Longleftrightarrow \quad \overline{F} \in RV_{-1/\gamma} \quad \Longleftrightarrow \quad U \in RV_\gamma. \tag{2.1}$$

The second-order parameter ρ (≤ 0) measures the rate of convergence in the general first-order conditions, in (2.1), and can be defined as the non-positive parameter in the limiting relation,

$$\lim_{t \to \infty} \frac{\ln U(tx) - \ln U(t) - \gamma \ln x}{A(t)} = \begin{cases} \frac{x^\rho - 1}{\rho}, & \text{if} \quad \rho < 0, \\ \ln x, & \text{if} \quad \rho = 0, \end{cases}$$

$x > 0$, and where $|A|$ must be of regular variation with an index ρ (Geluk and de Haan, 1987). This condition has been widely accepted as an appropriate condition to specify the right tail of a Pareto-type distribution in a semi-parametric way and enables easily the derivation of the non-degenerate bias of EVI and VaR-estimators, under a semi-parametric framework. Further developments of the topic can be found in de Haan and Ferreira (2006). If we consider only negative values of ρ, we are then in the class of models in (1.8), or equivalently either (1.1) or (1.2) holds.

3 EVI and VaR-estimators under heavy-tailed frameworks: asymptotic behavior

Let $\mathcal{N}(\mu, \sigma^2)$ stand for a normal RV with mean value μ and variance σ^2. In Section 3.1 we deal with the asymptotic behavior of the EVI-estimators under consideration, already known. A parallel study is performed in Section 3.2, for the VaR-estimators.

3.1 The EVI-estimators

It follows from the results of de Haan and Peng (1998) that in Hall-Welsh class of models in (1.8), and for intermediate k-values, i.e. if (1.5) holds,

$$\sqrt{k}\,(\mathrm{H}(k)-\gamma) \ \stackrel{d}{=}\ \mathcal{N}\left(0,\gamma^2\right)+\sqrt{k}\left(\frac{\gamma\ \beta\ (n/k)^\rho}{1-\rho}\right)(1+o_p(1)), \tag{3.1}$$

where the bias $\gamma\ \beta\ \sqrt{k}\ (n/k)^\rho/(1-\rho)$ can be very large, moderate or small, i.e. go to infinity, constant or zero, as $n \to \infty$.

Just as proved in Brilhante *et al.* (2013), the result in (3.1) can be generalized. Under the same conditions as above, for $0 \leq p < 1/(2\gamma)$, and with $\mathrm{H}_p(k)$ given in (1.9),

$$\sqrt{k}\,(\mathrm{H}_p(k)-\gamma) \ \stackrel{d}{=}\ \mathcal{N}\left(0,\frac{\gamma^2(1-p\gamma)^2}{1-2p\gamma}\right)+\sqrt{k}\left(\frac{\gamma\ \beta\ (n/k)^\rho(1-p\gamma)}{1-\rho-p\gamma}\right)(1+o_p(1)). \tag{3.2}$$

3.2 Extreme quantile or VaR-estimators

Under condition (1.8), the asymptotic behavior of $\ln Q_{_\mathrm{H}}(k)$ is well-known (Weissman, 1978):

$$\frac{\sqrt{k}}{\ln(k/(nq))}\left(\ln Q_{_\mathrm{H}}^{(q)}(k)-\ln \mathrm{VaR}_q\right) \ \stackrel{d}{=} \mathcal{N}\Big(\frac{\lambda}{1-\rho},\gamma^2\Big),$$

provided that the sequence $k = k_n$ satisfies the condition $\lim\limits_{n\to\infty} \sqrt{k}A(n/k) = \lambda \in \mathbb{R}$, finite, with $A(\cdot)$ the function in (1.8).

Regarding VaR-estimation, we shall here consider, as possible alternatives to the classical Weissman-Hill lnVaR-estimator, $\ln Q_{_\mathrm{H}}^{(q)}(k)$, the class of estimators

$$\ln Q_{_{\mathrm{H}_p}}^{(q)}(k) := \ln X_{n-k+1:n} + \mathrm{H}_p(k)\ \ln\left(\frac{k}{nq}\right), \tag{3.3}$$

with H_p given in (1.9).

As previously mentioned, for intermediate k, i.e., whenever (1.5) holds, we are here dealing with semi-parametric lnVaR_q estimators, of the type of $\ln Q_{\widehat{\gamma}}^{(q)}$ in (1.7), where $\widehat{\gamma} \equiv \widehat{\gamma}(k)$ can be any semi-parametric estimator of the tail index γ, and it is here considered to be the MOP EVI-estimator in (1.9). We may state the following:

Theorem 1. *In* Hall-Welsh *class of models in* (1.8)*, for intermediate k, i.e. k-values such that* (1.5) *holds, whenever*

$$\ln(n\ p_n) = o\left(\sqrt{k}\right), \tag{3.4}$$

$\sqrt{k}\ A(n/k) \to \lambda$, finite, possibly non-null, and for any $p < 1/(2\gamma)$

$$\frac{\sqrt{k}}{\ln(k/(nq))}\left(\ln Q^{(q)}_{\mathrm{H}_p}(k) - \ln \mathrm{VaR}_q\right) \underset{n\to\infty}{\overset{d}{\longrightarrow}} \mathcal{N}\left(0, \frac{\gamma^2(1-p\gamma)^2}{1-2p\gamma}\right), \tag{3.5}$$

with H_p *any of the estimators in* (1.9)*.*

Proof. We may write

$$\ln X_{n-k+1:n} \ \stackrel{d}{=}\ \ln U(n/k) + \frac{\gamma\ B_k}{\sqrt{k}} + o_p(A(n/k)),$$

with B_k asymptotically standard normal. Since

$$\ln \mathrm{VaR}_q = \ln U\left(\frac{1}{q}\right) = \ln U\left(\frac{n}{k} \times \frac{k}{nq}\right),$$

we have, with $A(t)$ the function in (1.8),

$$\begin{aligned}
\ln Q^{(q)}_{\widehat{\gamma}}(k) - \ln \mathrm{VaR}_q \ &\stackrel{d}{=}\ -\left(\ln U\left(\frac{n}{k} \times \frac{k}{nq}\right) - \ln U\left(\frac{n}{k}\right)\right) + \frac{\gamma\ B_k}{\sqrt{k}} \\
&\qquad + \widehat{\gamma}(k)\ \ln\left(\frac{k}{nq}\right) + o_p(A(n/k)) \\
&\stackrel{d}{=} (\widehat{\gamma}(k) - \gamma)\ln\left(\frac{k}{nq}\right) + \frac{\gamma\ B_k}{\sqrt{k}} - \frac{(k/(nq))^\rho - 1}{\rho}\ A(n/k)(1+o(1)) + o_p(A(n/k)).
\end{aligned}$$

Consequently, since $(k/(nq))^\rho = o(1)$,

$$\ln Q^{(q)}_{\widehat{\gamma}}(k) - \ln \mathrm{VaR}_q \ \stackrel{d}{=}\ (\widehat{\gamma}(k) - \gamma)\ln\left(\frac{k}{nq}\right) + \frac{\gamma\ B_k}{\sqrt{k}} + \frac{A(n/k)}{\rho} + o_p(A(n/k)).$$

The dominant term is thus of the order of $\left\{\ln(k/(nq))/\sqrt{k}\right\}$, that must converge towards zero, and this is true due to condition (3.4). The results in (3.5) follow thus from (3.2).

∎

Apart from the MOP lnVaR-estimator, in (3.3), we have further considered in the lnVaR-estimator in (1.7), the replacement of the estimator $\widehat{\gamma}(k)$ by one of the most simple corrected-bias Hill estimator, the one in Caeiro *et al.* (2005). Such a class is defined as

$$\mathrm{CH}(k) \equiv \mathrm{CH}(k;\hat{\beta},\hat{\rho}) := \mathrm{H}(k)\Big(1-\hat{\beta}(n/k)^{\hat{\rho}}/(1-\hat{\rho})\Big). \tag{3.6}$$

The estimators in (3.6) can be second-order minimum-variance reduced-bias (MVRB) EVI-estimators, for adequate levels k and an adequate external estimation of the vector of second-order parameters, (β,ρ), introduced in (1.8), i.e. the use of $\mathrm{CH}(k)$ can enable us to eliminate the dominant component of bias of the Hill estimator, $\mathrm{H}(k)$, keeping its asymptotic variance. Indeed, from the results in Caeiro *et al.* (2005), we know that it is possible to adequately estimate the second-order parameters β and ρ, so that we get

$$\sqrt{k}\,(\mathrm{CH}(k)-\gamma) \ \stackrel{d}{=}\ \mathcal{N}\left(0,\gamma^2\right)+o_p\big(\sqrt{k}(n/k)^{\rho}\big),$$

i.e. $\mathrm{CH}(k)$ overpasses $\mathrm{H}(k)$ for all k. Overviews on reduced-bias estimation can be found in Chapter 6 of Reiss and Thomas (2007), Gomes *et al.* (2008a) and Beirlant *et al.* (2012).

For the estimation of the vector of second-order parameters (β,ρ), we propose an algorithm of the type of the ones presented in Gomes and Pestana (2007), where it is used the β-estimator in Gomes and Martins (2002) and the simplest ρ-estimator in Fraga Alves *et al.* (2003), both computed at a level $k_1=\lfloor n^{0.999}\rfloor$, with the notation $\lfloor x\rfloor$ standing for the integer part of x. More recent estimators of β can be found in Gomes *at al.* (2010) and Caeiro and Gomes (2012a,b). For alternative estimation of ρ, see Goegebeur *at al.* (2008, 2010) and Ciuperca and Mercadier (2010).

4 Simulated behaviour of the lnVaR estimators

We have implemented large-scale multi-sample Monte-Carlo simulation experiments of size 5000×20, essentially for the new classes of lnVaR-estimators, $\ln Q^{(p)}_{\mathrm{H}_p}(k)$, in (3.3), with H_p given in (1.9), for a few values of p, for sample sizes $n=100, 200, 500, 1000, 2000$ and 5000, and $\gamma=0.1, 0.25, 0.5$ and 1, from the following models:

1. Fréchet(γ) model, with CDF $F(x) = \exp(-x^{-1/\gamma}),\ x \geq 0\ (\rho = -1)$;
2. Extreme value model, with CDF $F(x) = \mathrm{EV}_\gamma(x)$, in (1.3) ($\rho = -\gamma$);
3. Burr(γ, ρ) model, with CDF $F(x) = 1-(1+x^{-\rho/\gamma})^{1/\rho}$, $x \geq 0$, for the aforementioned values of γ and for $\rho = -0.25, -0.5$ and -1;
4. Generalized Pareto model, with CDF $F(x)$ =$\mathrm{GP}_\gamma(x) = 1 + \ln \mathrm{EV}_\gamma(x) = 1 - \big(1+\gamma x\big)^{-1/\gamma}$, $x \geq 0$ ($\rho = -\gamma$).

We have further considered

5. Student-t_ν underlying parents, with $\nu = 4$ ($\gamma = 1/\nu = 0.25;\ \rho = -2/\nu = -0.5$), with probability density function
$$f(x;\nu) = \frac{\gamma((\nu+1)/2)}{\sqrt{\pi\nu}\Gamma(\nu/2)}\big(1+x^2/\nu\big)^{-(\nu+1)/2}, \quad t \in \mathbb{R}.$$

For details on multi-sample simulation, see Gomes and Oliveira (2001).

4.1 Mean values and MSE patterns

For each value of n and for each of the aforementioned models, we have first simulated the mean values (E) and root MSEs (RMSEs) of the lnVaR-estimators under consideration, as functions of the number of top order statistics k involved in the estimation, and on the basis of the first run of size 5000. As an illustration, we present Figures 1 and 2, respectively associated with $\mathrm{GP}_{0.25}$ and $\mathrm{EV}_{0.25}$ parents. In these figures, we show, for $n = 1000$, $q = 1/n$, and on the basis of the first $N = 5000$ runs, the simulated patterns of mean value, $E[\cdot]$, and root mean squared error, $RMSE[\cdot]$, of a few RV's $\ln Q^{(p)}_{\widehat{\gamma}}(k) - \ln \chi_q$, based on the statistics $\ln Q^{(p)}_{\widehat{\gamma}}(k)$ in (1.7), with $\widehat{\gamma}$ replaced by both H_p, in (1.9), for some values of p, and CH in (3.6). We shall use the obvious notations $\ln Q_p$ and $\ln Q_{\mathrm{CH}}$. Apart from $p = 0$, associated with the Weissman-Hill lnVaR-estimator, we have considered $p = p_j = j/(10\gamma)$, $j = 1, 2, 4$, all within the framework of Theorem 1, as well as $j = 7$ for which we can no longer guarantee the asymptotic normality of the new lnVaR-estimators.

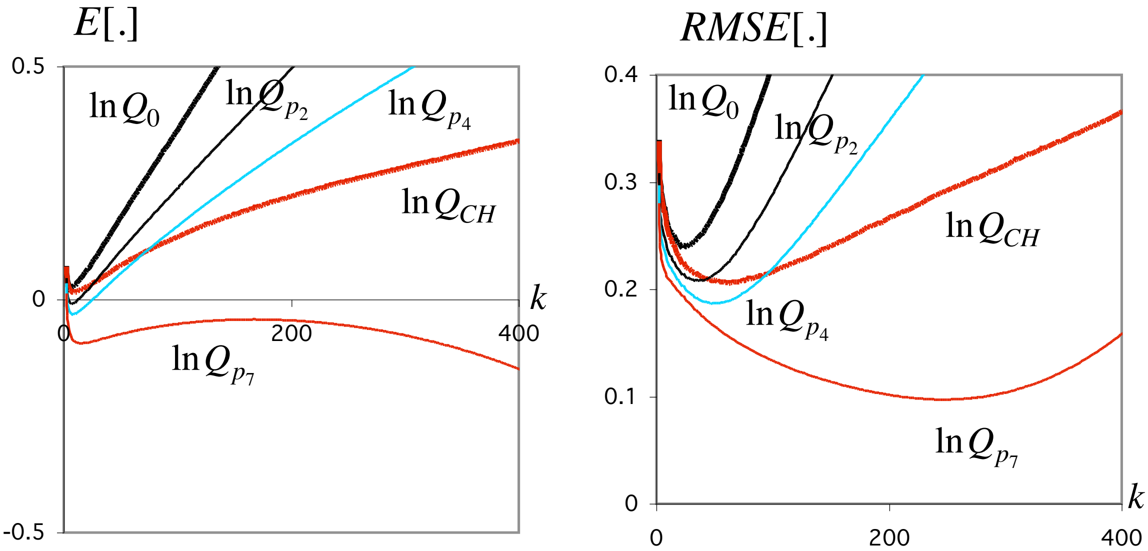

Figure 1: Underlying GP parent with $\gamma = 0.25$ $(\rho = -0.25)$

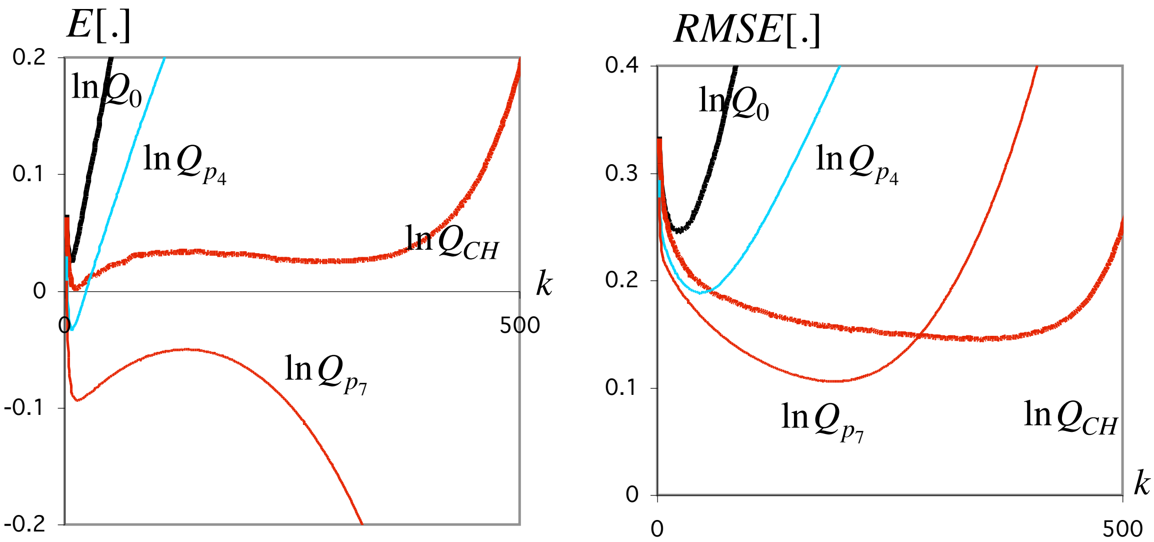

Figure 2: Underlying EV parent with $\gamma = 0.25$ $(\rho = -0.25)$

We further present in Figures 3 and 4, similar results but for two other models, a Student t_4 $(\rho = -0.5)$ and a Fréchet(1) $(\rho = -1)$, where the patterns are slightly different, from the ones obtained before for $\rho = -0.25$.

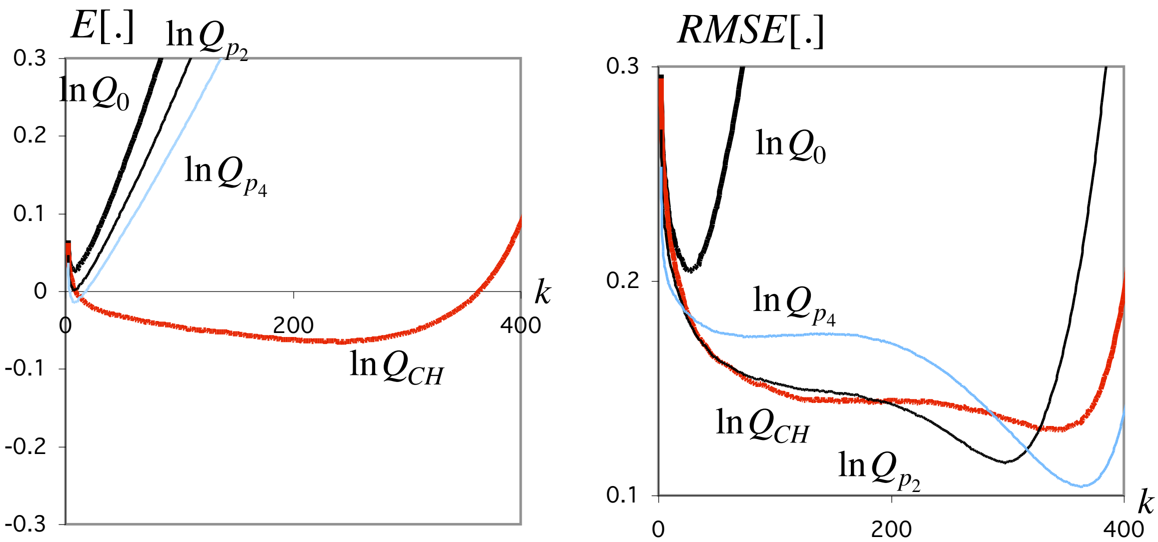

Figure 3: Underlying *Student* t_ν parent, with $\nu = 4$ ($\gamma = 1/\nu = 0.25, \rho = -2/\nu = -0.5$)

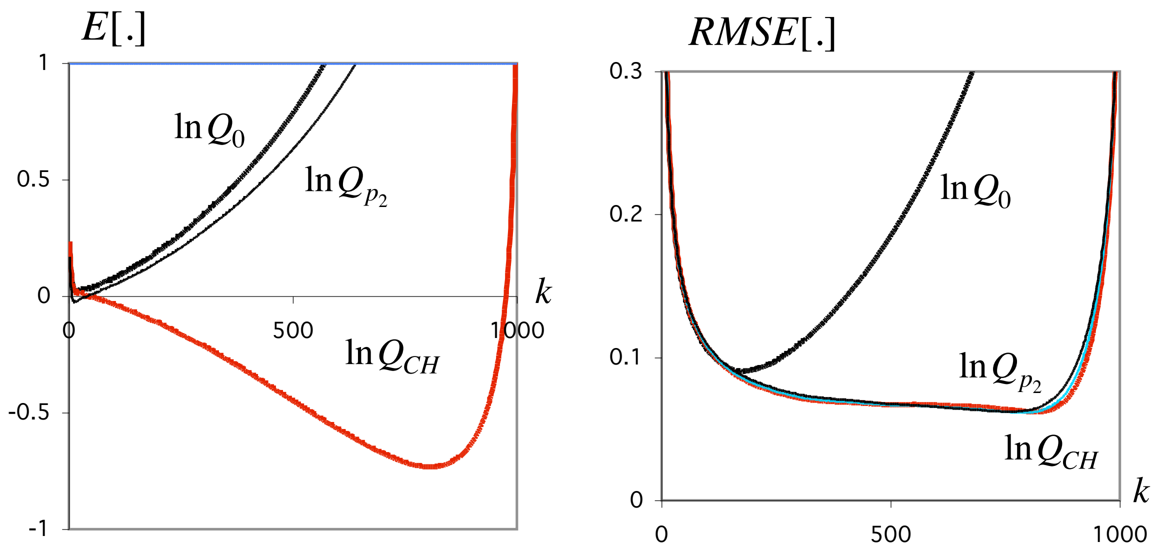

Figure 4: Underlying *Fréchet* parent, with $\gamma = 1$ ($\rho = -1$)

A few comments:

- The obtained results lead us to strongly advise the use of the log-quantile estimator $\ln Q_p$ for an adequate value of p, provided by a bootstrap algorithm of the type of the ones devised for an EVI-estimation in Gomes *et al.* (2012, 2013).

- For values of $|\rho| \leq 0.25$ the use of $\ln\mathrm{Var}Q_p$, with $p = p_7$, enables always a reduction in RMSE. Moreover, the bias is also reduced comparatively with the bias of the Weissman-Hill lnVaR-estimator with the obtention of estimates closer to the target value VaR_q, for $q = 1/n$. Note however that such a value of p is out of the scope of Theorem 1.
- Such a reduction is particularly high for values of ρ close to zero, even when we work with models again out of the scope of Theorem 1, like the the log-gamma and the log-Pareto. This is surely due to the high bias of the Weissman-Hill lnVaR-estimators for this type of models with $\rho = 0$.

4.1.1 RMSEs and relative efficiency indicators at optimal levels

We have computed the Weissman-Hill lnVaR-estimator $\ln Q_{\mathrm{H}}^{(q)}(k) \equiv \ln Q_{\mathrm{H}_0}^{(q)}(k)$, with $\ln Q_{\widehat{\gamma}}^{(q)}(k)$ defined in (1.7), at the simulated value of $k_{0|\mathrm{H}_0}^{(q)} := \arg\min_k \mathrm{RMSE}\big(\ln Q_{\mathrm{H}}^{(q)}(k)\big)$, the simulated optimal k in the sense of minimum RMSE, not relevant in practice, but providing an indication of the best possible performance of the Weissman-Hill lnVaR-estimator. Such an estimator is denoted by $\ln Q_{00}$. We have also computed $\ln Q_{p0}$, for a few values of p, and the simulated indicators,

$$\mathrm{REFF}_{p|0} := \frac{\mathrm{RMSE}\,(\ln Q_{00})}{\mathrm{RMSE}\,(\ln Q_{p0})}. \tag{4.1}$$

A similar REFF-indicator, $\mathrm{REFF}_{\mathrm{CH}|0}$ has also been computed for the lnVaR-estimator based on CH EVI-estimators, in (3.6).

Remark 1. *This indicator has been conceived so that an indicator higher than one means a better performance than the one of the* Weissman-Hill lnVaR-*estimator. Consequently, the higher these indicators are, the better the associated* lnVaR-*estimators perform, comparatively to* $\ln Q_{00}$.

As an illustration of the results obtained for the different lnVaR-estimators under consideration, we present Tables 1 and 2. In the first row, we provide the RMSE of $\ln Q_{00}$, denoted RMSE_0, so that we can easily recover the RMSE of all other estimators. The following rows provide the REFF-indicators of the lnVaR-estimators based on CH and on H_p. An underlined and **bold** mark is used for the highest REFF indicator. Among the MOP lnVaR-estimators within the scope of Theorem 1, we place the highest value in *italic* and underlined, whenever such a value is not the highest one among all estimators under consideration.

Table 1: Simulated RMSE of $\ln Q_{00}$, $q = 1/n$ (first row) and REFF-indicators of $\ln Q_{\mathrm{CH}|0}$ and $\ln Q_{p_j|0}$, for $p_j = j/(10\gamma)$, $j = 1, 2, 4$ and 7, for GP and EV parents, with $\gamma = 0.25$ ($\rho = -0.25$), together with 95% CIs

n	100	200	500	1000	2000	5000
GP parent, $\gamma = 0.25$						
RMSE_0	0.345 ± 0.0351	0.307 ± 0.0356	0.268 ± 0.0297	0.245 ± 0.0259	0.224 ± 0.0216	0.200 ± 0.0201
CH	1.243 ± 0.0086	1.492 ± 0.0092	1.256 ± 0.0074	1.174 ± 0.0035	1.131 ± 0.0035	1.093 ± 0.0029
p_1	1.116 ± 0.0026	1.098 ± 0.0018	1.082 ± 0.0018	1.072 ± 0.0019	1.063 ± 0.0015	1.053 ± 0.0011
p_2	1.299 ± 0.0055	1.239 ± 0.0047	1.187 ± 0.0030	1.160 ± 0.0039	1.136 ± 0.0031	1.11 ± 0.0021
p_4	$\underline{\mathbf{2.477}} \pm 0.0115$	$\underline{\mathbf{2.676}} \pm 0.0182$	$\underline{\mathbf{2.442}} \pm 0.0147$	$\underline{\mathit{2.361}} \pm 0.0158$	$\underline{\mathit{2.350}} \pm 0.0154$	$\underline{\mathit{2.430}} \pm 0.0163$
p_7	1.239 ± 0.0064	1.269 ± 0.0085	1.633 ± 0.0133	$\underline{\mathbf{2.516}} \pm 0.0166$	$\underline{\mathbf{4.438}} \pm 0.0153$	$\underline{\mathbf{7.738}} \pm 0.0571$
EV parent, $\gamma = 0.25$						
RMSE_0	0.353 ± 0.0358	0.311 ± 0.0364	0.270 ± 0.0276	0.246 ± 0.0257	0.224 ± 0.0216	0.200 ± 0.0209
CH	1.009 ± 0.0048	1.056 ± 0.0080	1.290 ± 0.0137	1.695 ± 0.0086	1.299 ± 0.0086	1.173 ± 0.0043
p_1	1.050 ± 0.0055	1.078 ± 0.0066	1.500 ± 0.0262	1.754 ± 0.0107	1.438 ± 0.0088	1.253 ± 0.0050
p_2	1.088 ± 0.0064	1.099 ± 0.0056	$\underline{\mathbf{1.751}} \pm 0.0180$	$\underline{\mathbf{1.886}} \pm 0.0115$	1.662 ± 0.0091	1.356 ± 0.0057
p_4	$\underline{\mathbf{1.126}} \pm 0.0081$	$\underline{\mathbf{1.146}} \pm 0.0052$	1.154 ± 0.0066	1.312 ± 0.0096	$\underline{\mathbf{2.849}} \pm 0.0230$	$\underline{\mathbf{4.772}} \pm 0.0244$
p_7	1.049 ± 0.0075	1.106 ± 0.0065	1.111 ± 0.0057	1.086 ± 0.0067	1.053 ± 0.0057	1.003 ± 0.0064

Remark 2. *We now provide a few comments related to the* REFF*-indicators:*

- *Note that the functionals T under play are functions of $\ln X$. Consequently, if X is a Fréchet(γ) RV, denoted F_γ, the uniform transformation enables us to write*

$$\exp\big(-\mathrm{F}_\gamma^{-1/\gamma}\big) \stackrel{d}{=} U \quad \Longleftrightarrow \quad \ln \mathrm{F}_\gamma \stackrel{d}{=} -\gamma \ln(-\ln(U)),$$

  i.e. $\ln \mathrm{F}_\gamma/\gamma$ *does not depend on* γ. *This also happens with a Burr(γ,ρ) model. For such a RV, now denoted* $\mathrm{B}_{\gamma,\rho}$, *we get*

$$\big(1 + \mathrm{B}_{\gamma,\rho}^{-\rho/\gamma}\big)^{1/\rho} \stackrel{d}{=} U \quad \Longleftrightarrow \quad \ln \mathrm{B}_{\gamma,\rho} \stackrel{d}{=} -\gamma \ln\big(U^\rho - 1\big)/\rho,$$

  i.e. again $\ln \mathrm{B}_{\gamma,\rho}/\gamma$ *does not depend on* γ.

- *Due to the above mentioned reasons, the* REFF*-indicators and* $\mathrm{RMSE}_0)/\gamma$ *do not depend on* γ *for* Fréchet *and* Burr(γ,ρ) *underlying parents.*

Table 2: Simulated RMSE of $\ln Q_{00}$, $q = 1/n$ (first row) and REFF-indicators of $\ln Q_{\mathrm{CH}|0}$ and $\ln Q_{p_j|0}$, for $p_j = j/(10\gamma)$, $j = 1, 2$, and 4, for Student t_4 ($\gamma = 0.25, \rho = -0.5$) and Fréchet parents with $\gamma = 0.25$ ($\rho = -1$), together with 95% CIs

n	100	200	500	1000	2000	5000
Student t_4 parent, $(\gamma, \rho) = (0.25, -0.5)$						
RMSE_0	0.300 ± 0.0291	0.264 ± 0.0264	0.228 ± 0.0232	0.207 ± 0.0192	0.188 ± 0.0180	0.164 ± 0.0132
CH	0.991 ± 0.0170	1.030 ± 0.0058	1.164 ± 0.0092	**1.609** ± 0.0115	**1.684** ± 0.00115	**1.468** ± 0.0083
p_1	1.093 ± 0.0033	1.079 ± 0.0021	1.064 ± 0.0024	1.054 ± 0.0016	1.047 ± 0.0013	1.039 ± 0.0014
p_2	1.219 ± 0.0041	1.181 ± 0.0041	1.137 ± 0.0045	1.116 ± 0.0028	1.097 ± 0.0025	1.077 ± 0.0021
p_4	**1.524** ± 0.0078	**1.348** ± 0.0044	**1.242** ± 0.0063	*1.195* ± 0.0049	*1.154* ± 0.0044	*1.113* ± 0.0034
Fréchet parent, $(\gamma, \rho) = (0.25, -1)$						
RMSE_0	0.683 ± 0.0757	0.600 ± 0.0642	0.501 ± 0.0435	0.432 ± 0.0368	0.372 ± 0.0308	0.303 ± 0.0307
CH	0.899 ± 0.0035	0.850 ± 0.0683	0.929 ± 0.0035	0.968 ± 0.0066	1.019 ± 0.0066	**1.144** ± 0.0058
p_1	1.033 ± 0.0010	1.028 ± 0.0017	1.023 ± 0.0011	1.021 ± 0.0011	1.019 ± 0.0012	1.018 ± 0.0009
p_2	**1.066** ± 0.0023	1.052 ± 0.0038	1.038 ± 0.0020	**1.033** ± 0.0020	**1.029** ± 0.0023	*1.026* ± 0.0022
p_4	1.101 ± 0.0045	**1.070** ± 0.0062	**1.042** ± 0.0039	1.028 ± 0.0033	1.017 ± 0.0042	1.011 ± 0.0034

- *Also, with the same notation,* $\ln \mathrm{B}_{\gamma,\rho}/\gamma = \ln \mathrm{B}_{1,\rho}$. *Moreover, with the obvious notation* GP_γ *for a* GP RV*, with* EVI γ*, we have* $\mathrm{GP}_\gamma = \ln \mathrm{B}_{\gamma,-\gamma}/\gamma$. *Consequently, the equivalent tables for* $\mathrm{Burr}(\gamma, \rho)$ RVs*, with* $\rho = -0.25$*, are trivially obtained from* Table 1, GP *parent. Particularly, the* REFF *indicators are the same.*

4.1.2 Mean values of the EVI-estimators at optimal levels

As an illustration of the bias of the new lnVaR-estimators, again at optimal levels, see Tables 3 and 4. We there present, for $n = 100,\ 200,\ 500,\ 1000,\ 2000$ and 5000, the simulated mean values at optimal levels of the lnVaR-estimators under study. Information on 95% confidence intervals (CIs), computed on the basis of the 20 replicates with 5000 runs each, is again provided. Among the estimators considered, the one providing the smallest squared bias is underlined, and written in **bold**.

Further note the following facts:

- Again for Fréchet and Burr underlying parents, and as already justified above, T/γ does not depend on γ, again with T denoting any of the aforementioned EVI-estimators.

Table 3: Simulated mean values, at optimal levels, of $T_{00} := \ln Q_{00} - \ln \chi_q$, $q = 1/n$ (first row) and REFF-indicators of $\ln Q_{\mathrm{CH}|0} - \ln \chi_q$ and $\ln Q_{p_j|0} - \chi_q$, for $p_j = j/(10\gamma)$, $j = 1, 2, 4$ and 7, for GP and EV parents, with $\gamma = 0.25$ ($\rho = -0.25$), together with 95% CIs

n	100	200	500	1000	2000	5000
GP parent, $\gamma = 0.25$						
H	0.077 ± 0.0058	0.078 ± 0.0040	0.081 ± 0.0056	0.084 ± 0.0051	0.083 ± 0.0043	0.083 ± 0.0037
CH	-0.093 ± 0.0041	$\underline{\mathbf{0.002}} \pm 0.0031$	0.090 ± 0.0038	0.089 ± 0.0044	0.086 ± 0.0035	0.085 ± 0.0027
p_1	0.063 ± 0.0051	0.070 ± 0.0055	0.073 ± 0.0040	0.074 ± 0.0023	0.077 ± 0.0027	0.078 ± 0.0021
p_2	0.063 ± 0.0051	0.060 ± 0.0035	0.064 ± 0.0029	0.067 ± 0.0029	0.071 ± 0.0028	0.073 ± 0.0022
p_4	$\underline{\mathbf{-0.009}} \pm 0.0021$	0.016 ± 0.0024	$\underline{\mathbf{0.011}} \pm 0.0013$	$\underline{\mathbf{0.006}} \pm 0.0008$	$\underline{\mathbf{0.006}} \pm 0.0022$	0.003 ± 0.0009
p_7	-0.153 ± 0.0046	-0.171 ± 0.0024	-0.120 ± 0.0011	-0.053 ± 0.0006	-0.009 ± 0.0004	-0.001 ± 0.0003
EV parent, $\gamma = 0.25$						
H	$\underline{\mathbf{0.075}} \pm 0.0060$	$\underline{\mathbf{0.081}} \pm 0.0056$	0.080 ± 0.0050	0.083 ± 0.0041	0.084 ± 0.0040	0.080 ± 0.0038
CH	-0.083 ± 0.0088	-0.086 ± 0.0058	-0.067 ± 0.0040	0.028 ± 0.0034	0.084 ± 0.0032	0.084 ± 0.0027
p_1	-0.083 ± 0.0045	-0.094 ± 0.0053	-0.031 ± 0.0056	0.020 ± 0.0028	0.080 ± 0.0022	0.076 ± 0.0025
p_2	-0.103 ± 0.0041	-0.094 ± 0.0051	$\underline{\mathbf{-0.003}} \pm 0.0031$	$\underline{\mathbf{0.012}} \pm 0.0021$	0.065 ± 0.0012	0.073 ± 0.0017
p_4	-0.156 ± 0.0034	-0.111 ± 0.0040	-0.120 ± 0.0027	-0.114 ± 0.0013	$\underline{\mathbf{-0.045}} \pm 0.0007$	$\underline{\mathbf{0.005}} \pm 0.0003$
p_7	-0.238 ± 0.0026	-0.177 ± 0.0015	-0.137 ± 0.0021	-0.124 ± 0.0039	-0.122 ± 0.0028	-0.131 ± 0.0037

Table 4: Simulated mean values, at optimal levels, of $T_{00} := \ln Q_{00} - \ln \chi_q$, $q = 1/n$ (first row) and REFF-indicators of $\ln Q_{\mathrm{CH}|0} - \ln \chi_q$ and $\ln Q_{p_j|0} - \chi_q$, for $p_j = j/(10\gamma)$, $j = 1, 2$ and 4, for Student t_4 ($\gamma = 0.25, \rho = -0.5$) and Fréchet parents with $\gamma = 0.25$ ($\rho = -1$), together with 95% CIs

n	100	200	500	1000	2000	5000
Student t_4 **parent,** $(\gamma, \rho) = (0.25, -0.5)$						
H	0.065 ± 0.0520	0.065 ± 0.0041	0.073 ± 0.0055	0.071 ± 0.0028	0.071 ± 0.0031	0.070 ± 0.0032
CH	-0.072 ± 0.0023	-0.079 ± 0.0060	-0.080 ± 0.0026	$\underline{\mathbf{-0.011}} \pm 0.0024$	$\underline{\mathbf{0.031}} \pm 0.0017$	$\underline{\mathbf{0.051}} \pm 0.0012$
p_1	$\underline{\mathbf{0.052}} \pm 0.0052$	0.062 ± 0.0051	0.064 ± 0.0035	0.066 ± 0.0030	0.068 ± 0.0029	0.067 ± 0.0018
p_2	0.053 ± 0.0062	$\underline{\mathbf{0.051}} \pm 0.0033$	0.053 ± 0.0044	0.062 ± 0.0025	0.064 ± 0.0022	0.063 ± 0.0020
p_4	0.074 ± 0.0044	0.053 ± 0.0036	$\underline{\mathbf{0.052}} \pm 0.0026$	0.055 ± 0.0027	0.057 ± 0.0018	0.058 ± 0.0020
Fréchet parent, $(\gamma, \rho) = (0.25, -1)$						
H	0.227 ± 0.0089	0.213 ± 0.0101	0.197 ± 0.0052	0.183 ± 0.0046	0.160 ± 0.0071	0.137 ± 0.0038
CH	-0.232 ± 0.0100	-0.198 ± 0.0092	$\underline{\mathbf{-0.162}} \pm 0.0039$	$\underline{\mathbf{-0.139}} \pm 0.0050$	$\underline{\mathbf{-0.116}} \pm 0.0039$	$\underline{\mathbf{-0.010}} \pm 0.0027$
p_1	0.222 ± 0.0105	0.209 ± 0.0072	0.188 ± 0.0058	0.178 ± 0.0051	0.158 ± 0.0054	0.135 ± 0.0039
p_2	0.201 ± 0.0075	0.209 ± 0.0085	0.186 ± 0.0060	0.169 ± 0.0039	0.157 ± 0.0027	0.132 ± 0.0034
p_4	$\underline{\mathbf{0.191}} \pm 0.0060$	$\underline{\mathbf{0.191}} \pm 0.0076$	0.180 ± 0.0045	0.169 ± 0.0038	0.155 ± 0.0041	0.132 ± 0.0028

- For $\gamma+\rho = 0$, the results associated with Burr(γ, ρ) parents are equal to the ones associated with GP(γ) parents.
- Regarding bias, the MOP lnVaR–estimators often outperform the MVRB EVI-estimators whenever $|\rho| < 0.5$.

5 Concluding remarks

- It is clear that Weissman-Hill lnVaR-estimation leads to a strong over-estimation of the EVI and the MOP provides a more adequate lnVaR-estimation, being even able to beat the MVRB lnVaR-estimators in a large variety of situations.
- The patterns of the estimators' sample paths are always of the same type, in the sense that for all k the lnVaR-estimator, $\ln Q^{(q)}_{H_p}$ decreases as p increases.

Acknowledgements. Research partially supported by National Funds through **FCT** – Fundação para a Ciência e a Tecnologia, project PEst-OE/MAT/UI0006/2011, and EXTREMA, PTDC/MAT/101736/2008.

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