General stuttering $Beta(p, q)$ Cantor-like random sets

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Abstract. *Random damage — deletion of $(X_{1:2}, X_{2:2})$ from $[0, 1]$ — followed by random repair, meaning the union of $(Y_{1:2}, Y_{2:2})$ to the damaged set, where X and Y are independent random variables with support $[0, 1]$, are used at each step of the iterative construction of Cantor-like random sets. The Hausdorff dimension is computed under various randomness patterns that represent various degrees of redundancy, namely using Beta and extension of beta random variables to model either damage or repair, or both damage and repair.*

Keywords. *Random Cantor sets, Hausdorff dimension, beta distributions, order statistics.*

1 Introduction

Redundancy is an important asset in the survival of complex biological systems, and in the reliability and quality control of industrial systems. For instance, in the recovery of information in damaged hardware, information can be extracted both from the damaged and the undamaged tracks, and in the next step redundancy is cleared up — an operation that mirrors the mathematical operation of set union, in the sense that it discards repetitions. Random repair models brought in substantial progress in Probability Theory, cf. v.g. the chapters "Doubling with Repair", and "Mathematical Theory of Reliability Growth" in [4]. In here, we use random repair at each step of the iterative procedure used to construct a Cantor-like set, allowing for redundancy in the sense that repair can operate upon non-damaged zones.

A huge class of deterministic and of random Cantor-like sets can be constructed using several iterative procedures (Falconer, [3]; Pesin and Weiss, [5]). In the most common constructions, some part of the set from the former iteration is removed at each step. Pestana and Aleixo [7] introduced a stuttering procedure, in which at each step deletion (damage) is followed by partial random reconstruction (repair), working out in detail the Hausdorff dimension of the limiting fractal for several combinations of deterministic, uniform random and $Beta(2, 2)$ random deletion/reconstruction. Here, random deletion and random repair mean the removal of a random segment and the union of a random segment, respectively, independent of each other, determined by the order statistics of a random sample of size 2 taken from specified parent populations. Formally, a stuttering Cantor-like random set is defined as follows:

- Let $F_0 = [0,1]$.
- Construction of F_1:
  1. (Damaging stage) Generate two random points X_1 and X_2, where the parent random variable X has support $S = [0,1]$ and delete from $[0,1]$ the set $(X_{1:2}, X_{2:2})$.
  2. (Repair stage) Generate two random points Y_1 and Y_2 independent from (X_1, X_2), where the parent random variable Y has support $S = [0,1]$.

  $$F_1 = [F_0 - (X_{1:2}, X_{2:2})] \cup (Y_{1:2}, Y_{2:2}).$$

  A simple example, using uniform damage and repair:
  If $X \stackrel{\text{d}}{=} Y \frown Uniform(0,1)$, as $(X_1, X_2, , Y_1, Y_2) \stackrel{\text{d}}{=} (U_1, U_2, U_3, U_4)$ where the U_k, $k = 1, \ldots, 4$ are independent replica of the standard uniform random variable, and there are $4! = 24$ possible reorderings to consider when dealing with order statistics, it is easily established that random set F_1 is a mixture of $\widetilde{N}_1 = \begin{cases} 1 & 2 & 3 \\ \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \end{cases}$ random variables (some of which are degenerate): $F_1 = \bigcup_{i=1}^{N_1} S_{i,1}$ or, more explicitly,

  $$F_1 = \begin{cases} [0,1] & \text{if } Y_{1:2} < X_{1:2} \text{ and } Y_{2:2} > X_{2:2} & (\text{with probability } \frac{1}{6}) & \textbf{(a)} \\ [0, X_{1:2}] \cup [X_{2:2}, 1] & \text{if } Y_{2:2} < X_{1:2} \text{ or } Y_{1:2} > X_{2:2} & (\text{with probability } \frac{1}{3}) & \textbf{(b)} \\ [0, Y_{2:2}] \cup [X_{2:2}, 1] & \text{if } Y_{1:2} < X_{1:2} < Y_{2:2} < X_{2:2} & (\text{with probability } \frac{1}{6}) & \textbf{(c)} \\ [0, X_{1:2}] \cup [Y_{1:2}, 1] & \text{if } X_{1:2} < Y_{1:2} < X_{2:2} < Y_{2:2} & (\text{with probability } \frac{1}{6}) & \textbf{(d)} \\ [0, X_{1:2}] \cup [Y_{1:2}, Y_{2:2}] \cup [X_{2:2}, 1] & \text{if } X_{1:2} < Y_{1:2} < Y_{2:2} < X_{2:2} & (\text{with probability } \frac{1}{6}) & \textbf{(e)} \end{cases}$$

- F_k results from applying the above "stuttering" procedure to each segment $S_{i,k-1}$ of F_{k-1}, $k = 2, 3, \ldots$: in each segment $S_{i,k-1}$ in F_{k-1}
  1. (Damaging stage) Generate two random points $X_{1;\,i,k-1}, X_{2;\,i,k-1}$, and delete the middle random segment $(X_{1:2;\,i,k-1}, X_{2:2:\,i,k-1})$ from the corresponding segment $S_{i,k-1}$ in F_{k-1};
  2. (Repair stage) Generate two random points $Y_{1;\,i,k-1}$, $Y_{2;\,i,k-1}$, independent from $X_{1;\,i,k-1}$, $X_{2;\,i,k-1}$, and perform the union of the "damaged set" obtained in the previous stage with $(Y_{1:2;\,i,k-1}, Y_{2:2;\,i,k-1})$.

  $F_k = \bigcup_i \{[S_{i,k-1} - (X_{1:2;\,i,k-1}, X_{2:2:\,i,k-1})] \cup (Y_{1:2;\,i,k-1}, Y_{2:2;\,i,k-1})\}$, is what results from this "stuttering random repair". So, F_k is the union of $\widetilde{N}_k = \sum_{j=1}^{\widetilde{N}_{k-1}} \widetilde{N}_{1,j}$ of a random number of independent replicas of the random variable $\widetilde{N}_1$. In other words, F_k is the union of a random number of mixtures of random variables F_{k_i} "similar" (up to scaling) to F_1 as exhibited above, $F_k = \bigcup_{i=1}^{\widetilde{N}_k} S_{i,k}$.

- $\mathcal{F} = \bigcap_{k=1}^{\infty} F_k$ is thus the stuttering Cantor-like random set obtained by uniform damage counteracted by uniform repair.

2 Random repair and Hausdorff dimension

Observe also that in terms of the independent replica $U_k \frown Beta(1,1)$, $k = 1,\ldots,4$, the expressions (**b**), (**c**), (**d**) and (**e**) are

$$(\mathbf{b}) = \{[0, U_{3:4}] \cup [U_{4:4}, 1]\} \cup \{[0, U_{1:4}] \cup [U_{2:4}, 1]\},$$

$$(\mathbf{c}) = [0, U_{3:4}] \cup [U_{4:4}, 1],$$

$$(\mathbf{d}) = [0, U_{1:4}] \cup [U_{2:4}, 1],$$

$$(\mathbf{e}) = [0, U_{1:4}] \cup [U_{2:4}, U_{3:4}] \cup [U_{4:4}, 1],$$

a useful representation when the goal is to compute expectations, since $U_{3:4} \stackrel{\mathrm{d}}{=} 1 - U_{2:4} \frown Beta(3,2)$ and $1 - U_{4:4} \stackrel{\mathrm{d}}{=} U_{1:4} \stackrel{\mathrm{d}}{=} U_{3:4} - U_{2:4} \frown Beta(1,4)$. Hence, in view of the self-similarity in distribution in the successive stages, the Hausdorff dimension s of the fractal $\mathcal{F}$ is the exponent s such that $\sum_{i=1}^{\tilde{N}_1} \mathrm{E}[S_{i,1}^s]$ expands to the original size 1 of F_0, i.e., the solution $s \approx 0.669783$ of

$$\frac{1}{6} + \frac{8}{(s+3)(s+4)} + \frac{28}{(s+1)(s+2)(s+3)(s+4)} = 1.$$

In Pestana *et al.* [6] it has been shown that the Hausdorff dimension of a Cantor-like random fractal obtained by the iterative removal of a middle random set (in the precise sense that its endpoints were the order statistics of two uniform points in the segments of the previous iteration) — hence with no repair stage — is approximately 0.561553. Hence a counteracting random repair raises by almost 20% the Hausdorff dimension of the limiting fractal.

Pestana *et al.* [6] and Rocha *et al.* [8] have also computed the Hausdorff dimension of Cantor-like random sets generated using order statistics of $Beta(p,q)$ distributions. In fact, the family of beta distributions has a diversity of forms appropriate to model very diverse patterns of randomness in $(0,1)$ — symmetric if $p = q$, U-shaped if $p, q \in (0,1)$, uniform if $p = q = 1$, J-shaped if one of the parameters is 1 and de other greater than 1, unimodal if $p, q > 1$. It is therefore interesting to investigate the value of the Hausdorff dimension for different kinds of stuttering Cantor-like random sets, constructed by several combinations of damage and repair methodologies. In Table 1 we present partial results for some damage/repair combinations of the parameters p and q when damage is deterministic in the sense that it is determined by the expectations of order statistics.

In Table 1 we use deterministic damage, i.e. at each step the segments removed have as extreme points the expected values of the minimum and the maximum of a random sample of size 2 of a truncated $Beta(p,q)$ on $(a_{i,k-1}, b_{i,k-1})$, where the support $(a_{i,k-1}, b_{i,k-1})$ is in each case the segment $S_{i,k-1}$ obtained in the former step. The increase of the Hausdorff dimension over the dimension of the corresponding damage/non repair fractals, as shown in Table 2, is striking.

Results are of course much less impressive when the random repair chosen doesn't mach the damage infliged, for instance when damage is $Beta(1,2)$ and the repair is $Beta(2,1)$, an expected consequence of the intersection of the expected random middle intervals determined by the order statistics used in the damage and in the repair phases being much smaller.

In Table 3 we use random damage and random repair, using the minimum and the maximum of random samples of size 2 from $Beta(p_1, q_1)$ in the damage phase and from $Beta(p_2, q_2)$ in the repair phase. Observe that the Hausdorff dimension decreases under randomness in the damage stage. Once again, compare with the corresponding Hausdorff dimension without repair stage, shown in Table 4.

Computations for small integer values p and q are cumbersome but feasible (in Table 3 we register the "exact" values for the cases $p = q = 1$ and $p = q = 2$ both in the damage and in the repair phases), but this is indeed a situation where Monte Carlo methods easily provide satisfactory

Table 1: Hausdorff dimension of some stuttering Cantor-like random sets, damage using the expected minimum and maximum of two $Beta(p_1, q_1)$ order statistics, and $Beta(p_2, q_2)$ minimum and maximum random repair.

Expected Extremes Damage	**Random Repair**									
	$(\frac{1}{2},\frac{1}{2})$	(1,1)	(1,2)	(1,3)	(2,1)	(2,2)	(2,3)	(3,1)	(3,2)	(3,3)
$(\frac{1}{2},\frac{1}{2})$	0.6772	0.6774	0.6772	0.6751	0.6772	0.6775	0.6764	0.6750	0.677	0.6770
(1,1)	0.7141	0.7406	0.7216	0.6897	0.7221	0.7639	0.7529	0.6890	0.7521	0.7737
(1,2)	0.7313	0.7463	0.7720	0.7620	0.7039	0.7511	0.7813	0.6824	0.7125	0.7473
(1,3)	0.7404	0.7448	0.7805	0.7898	0.7081	0.7372	0.7677	0.6943	0.7080	0.7280
(2,1)	0.7313	0.7465	0.7042	0.6823	0.7709	0.7511	0.7122	0.7636	0.7806	0.7475
(2,2)	0.7582	0.7791	0.7664	0.7415	0.7654	0.7986	0.7898	0.7419	0.7895	0.8082
(2,3)	0.7710	0.7871	0.7953	0.7793	0.7600	0.8003	0.8144	0.7405	0.7739	0.8049
(3,1)	0.7401	0.7450	0.7077	0.6943	0.7808	0.7380	0.7083	0.7901	0.7674	0.7282
(3,2)	0.7707	0.7878	0.7609	0.7406	0.7948	0.7996	0.7734	0.7796	0.8148	0.8050
(3,3)	0.7851	0.8030	0.7914	0.7725	0.7909	0.8199	0.8123	0.7725	0.8121	0.8286

Table 2: Hausdorff dimension of Cantor-like sets, damage using the expected minimum and maximum of two $Beta(p, q)$ order statistics (no repair).

$B(\frac{1}{2},\frac{1}{2})$	$B(1,1)$	$B(1,2)$	$B(1,3)$	$B(2,1)$	$B(2,2)$	$B(2,3)$	$B(3,1)$	$B(3,2)$	$B(3,3)$
0.5715	0.6309	0.6663	0.6880	0.6663	0.6999	0.7203	0.6880	0.7203	0.7397

Table 3: Hausdorff dimension of some stuttering Cantor-like random sets, random damage and random repair using the minimum and maximum of $Beta(p, q)$ random variables.

Random Damage	**Random Repair**									
	$(\frac{1}{2},\frac{1}{2})$	(1,1)	(1,2)	(1.3)	(2,1)	(2,2)	(2,3)	(3,1)	(3,2)	(3,3)
$(\frac{1}{2},\frac{1}{2})$	0.5637	0.5795	0.5661	0.5430	0.5646	0.5768	0.5706	0.5447	0.5734	0.5684
(1,1)	0.6560	0.6698*	0.6574	0.6379	0.6601	0.6723	0.6644	0.6371	0.6684	0.6717
(1,2)	0.6931	0.7029	0.7149	0.7082	0.6739	0.7009	0.7138	0.6540	0.6848	0.7030
(1,3)	0.7121	0.7118	0.7381	0.7391	0.6881	0.7124	0.7269	0.6648	0.6880	0.7121
(2,1)	0.6916	0.7015	0.6741	0.6564	0.7148	0.7054	0.6821	0.7056	0.7143	0.7031
(2,2)	0.7309	0.7453	0.7355	0.7191	0.7335	0.7516*	0.7461	0.7190	0.7426	0.7519
(2,3)	0.7537	0.7634	0.7666	0.7539	0.7443	0.7685	0.7719	0.7258	0.7531	0.7662
(3,1)	0.7082	0.7144	0.6864	0.6676	0.7396	0.7089	0.6934	0.7398	0.7285	0.7079
(3,2)	0.7498	0.7627	0.7455	0.7303	0.7640	0.7644	0.7510	0.7545	0.7699	0.7668
(3,3)	0.7691	0.7853	0.7722	0.7620	0.7717	0.7902	0.7838	0.7560	0.7853	0.7895

(* exact value)

approximations. For the computation of the Hausdorff dimension s for the various combinations of random and repair shown in tables 1 and 2, approximations with error less than 10^{-4} have been

Table 4: Hausdorff dimension of Cantor-like sets, damage using the minimum and maximum of two $Beta(p,q)$ order statistics (no repair).

$B(\frac{1}{2},\frac{1}{2})$	$B(1,1)$	$B(1,2)$	$B(1,3)$	$B(2,1)$	$B(2,2)$	$B(2,3)$	$B(3,1)$	$B(3,2)$	$B(3,3)$
0.4320	0.5616	0.6189	0.6497	0.6189	0.6693	0.6965	0.6497	0.6965	0.7214

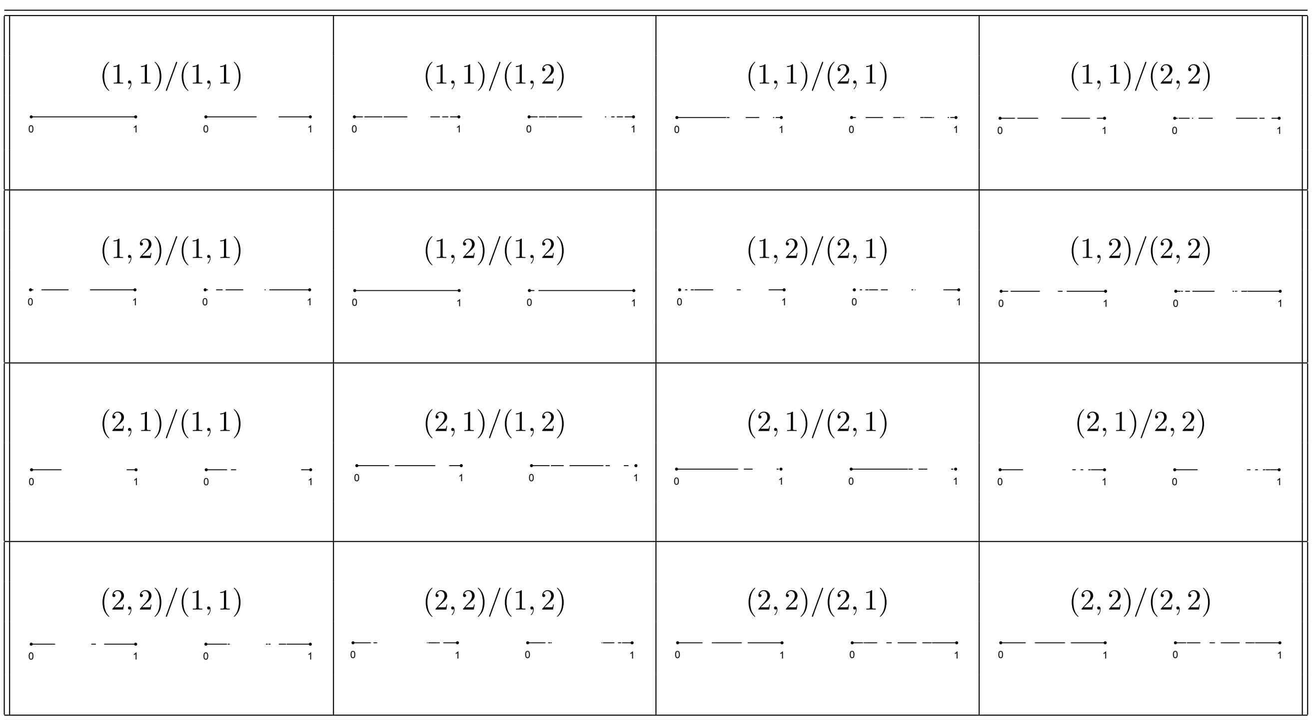
obtained using 5 000 runs for the computation of s.

3 Beta and BetaBoop stuttering damage/repair F_k sets

In section 2 we computed the Hausdorff dimension of stuttering Cantor-like sets to have some clue on the asymptotic effect of different combinations of damage/repair. For practical purposes, the observation of what happens after a rather small number of steps of damage/repair is crucial.

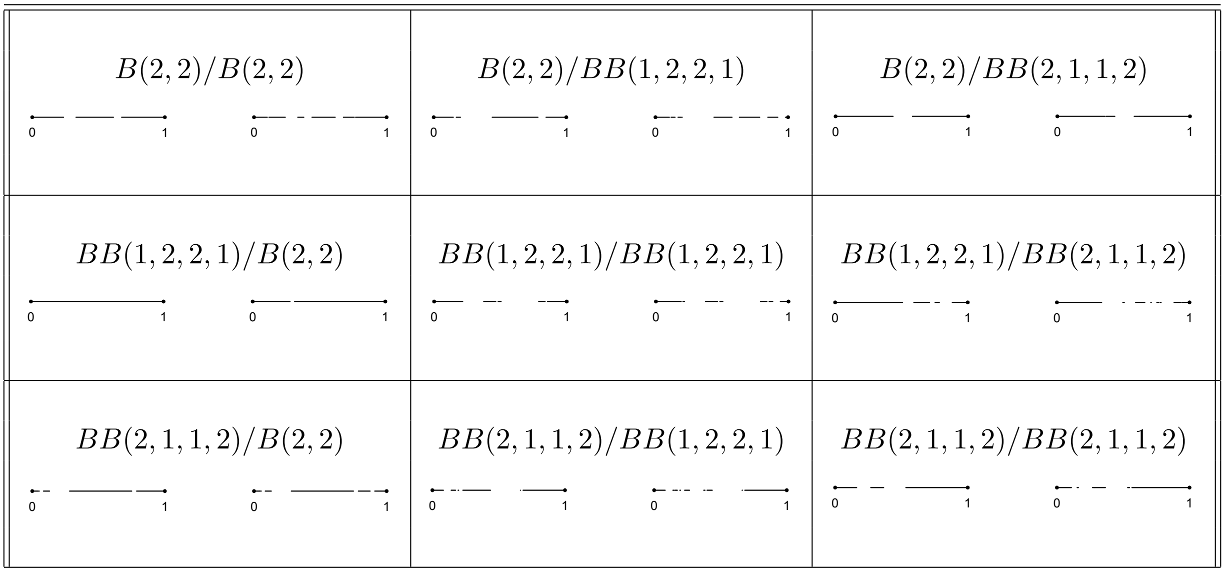
Therefore, we developed a program in R to exhibit F_k, for a chosen value of k. In Fig. 1 we exhibit F_2 and F_3, for various combinations of $Beta(p_d,q_d)$ and $Beta(p_r,q_r)$, for steps $k \in \{2,3\}$.

Figure 1: Plots of F_2 and F_3 for $Beta(p_d,q_d)$ random damage and $Beta(p_r,q_r)$ random repair. The parameters of damage/repair are indicated as $(p_d,q_d)/(p_r,q_r)$. In each cell the left figure is the result after 2 steps, the right figure the result after 3 steps.



Brilhante *et al.* [2] extended the $Beta(p,q)$ family by observing that $1-x$ is the linear truncation of $-\ln x$, and Brilhante *et al.* [1] considered a general $BetaBoop(p,q,P,Q)$ family of random variables with probability density function $f_{p,q,P,Q}(x) = c\,x^{p-1}(1-x)^{q-1}(-\ln(1-x))^{P-1}(-\ln x)^{Q-1}\mathrm{I}_{(0,1)}(x)$, $p,q,P,Q > 0$ (which, for $P = Q = 1$ reduces to the $Beta(p,q)$ family, and for $q = P = 1$ is the $Betinha(p,Q)$ studied in Brilhante *et al.* [2]). In Fig. 2 we exhibit F_2 and F_3, for combinations of $Beta(2,2)$ and $BetaBoop(2,1,1,2)$ or $BetaBoop(1,2,2,1)$, for damage/repair.

Figure 2: Plots of F_2 and F_3 for combinations of $Beta(2,2)$, $BetaBoop(1,2,2,1)$ and $BetaBoop(2,1,1,2)$ random damage and random repair. In each cell the left figure is the result after 2 steps, the right figure the result after 3 steps. Beta parameters are indicated as $B(p,q)$ and BetaBoop parameters as $BB(p,q,P,Q)$, for damage/repair considered in each cell.



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