INFERENCE ON THE LOCATION PARAMETER OF EXPONENTIAL POPULATIONS

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A token of friendship to Professor J. T. Mexia on his 70th birthday

Abstract: Studentization and analysis of variance are simple in Gaussian families because $\overline{X}$ and S^2 are independent random variables. We exploit the independence of the spacings in exponential populations with location λ and scale δ to develop simple ways of dealing with inference on the location parameter, namely by developing an analysis of scale in the homocedastic independent k-sample problem.

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1 Introduction

In what follows, $\boldsymbol{X} = (X_1, X_2, \ldots, X_n)$ denotes a random sample of size $n_X = n$ from the population X; $\overline{X} = \frac{1}{n}\sum_{k=1}^{n} X_k$ and $S_X^2 = \frac{1}{n-1}\sum_{k=1}^{n}(X_k - \overline{X})^2$. $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_k$, $\overline{X}_1, \ldots, \overline{X}_k$, $S^2_{X_1}, \ldots, S^2_{X_k}$ — which for simplicity we shall denote $S_1^2, \ldots, S_k^2$ — will have similar meaning. Also for simplicity, we shall write $n_{X_j} = n_j$, $j = 1, \ldots, k$, and $N = n_1 + \cdots + n_k$ the dimension of the combined sample. A pooled variance estimator $S_p^2 = \frac{(n_1-1)S_1^2 + \cdots + (n_k-1)S_k^2}{N-k}$ will also be used.

The exact sampling distribution of functions of low empirical moments of Gaussian populations $X_k \frown Gaussian(\mu_k, \sigma_k)$ is easily derived, and hence we have a wealth of results to estimate, or to test, the location parameter, or to compare location parameters of several Gaussian populations, even for small samples, at least assuming homocedasticity.

For the one-sample and the homocedastic independent two-sample cases, Student (1908) devised how to get rid of the nuisance scale parameter in σ in $\frac{\overline{X}-\mu}{\sigma\sqrt{\frac{1}{n}}}$ and in $\frac{\overline{X}_1-\overline{X}_2-(\mu_1-\mu_2)}{\sigma\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}}$, respectively, dividing those standard Gaussian random variables by appropriate functions of estimators of σ.

For the heterocedastic case, Smith (1936), Welch (1938), Satterthwaite (1946) and Aspin (1948) established that $\dfrac{\overline{X}_1-\overline{X}_2-(\mu_1-\mu_2)}{\sqrt{\frac{\sigma_1^2}{n_1}+\frac{\sigma_2^2}{n_2}}}$ has an approximate t_ν distribution, where the fractional number of degrees of freedom ν can be estimated by $\tilde{\nu}=\dfrac{(\frac{s_1^2}{n_1}+\frac{s_2^2}{n_2})^2}{\frac{s_1^4}{n_1^2(n_1-1)}+\frac{s_2^4}{n_2^2(n_2-1)}}$.

Aspin (1949) tabulated high quantiles of this 'fractional' t distribution. Alternatively, inference on $\mu_1-\mu_2$ can use random pairing of $X_{1,k}$ from $\boldsymbol{X}_1$ with Y_{2,ν_k} chosen using simple random sampling without replacement from $\boldsymbol{X}_2$, cf. Sheffé (1943, 1944), so that using the one-sample t-test for the 'random' differences $D_k=X_{1,k}-Y_{2,\nu_k}$, $k=1,\ldots,\min(n_1,n_2)$ is in order.

Following David and Nagaraja (2003) we shall speak of studentization, in general populations, whenever we divide a random variable $T=T(\Theta(\boldsymbol{X}),\theta,\xi)$, which is a function of an estimator $\Theta(\boldsymbol{X})$ of the parameter of interest θ, depending both on this parameter and on a nuisance parameter ξ, by some appropriate function $h(\Xi(\boldsymbol{X}))$ of an estimator $\tilde{\xi}=\Xi(\boldsymbol{X})$ of the nuisance parameter, so that the ratio no longer depends on the nuisance parameter ξ. Therefore, this ratio can be used as a fulcral variable, useful to construct interval estimators of θ; on the other hand, the above mentioned ratio is an adequate test statistic, under the null hypothesis $H_0:\ \theta=\theta_0$. We shall consider it *external studentization* when T and $h(\Xi)$ are independent, and *internal studentization* otherwise.

The homocedastic independent k-sample problem, in Gaussian population, has been solved by Fisher (1925). Instead of getting rid of the nuisance parameter, Fisher's ANalysis Of VAriance — in the sense of decomposing an estimator of the overall dispersion into components — uses the influence of the location parameter estimates on the scale parameter estimator. More precisely, as the variance σ^2 is the smaller second order moment, i.e., $\sigma^2<\mathbb{E}[(X-A)^2]$ for all $A\neq\mathbb{E}[X]$ (and, similarly, $s_{\boldsymbol{x}}^2=\frac{1}{n-1}\sum_{k=1}^n(x_k-\overline{x})^2<\frac{1}{n-1}\sum_{k=1}^n(x_k-a)^2$ for all $a\neq\overline{x}=\frac{1}{n}\sum_{k=1}^n x_k$), splitting the Total Sum of Squares $TSS=\sum_{i=1}^k\sum_{j=1}^{n_i}(X_{ij}-\overline{\overline{X}})^2$, where $\overline{\overline{X}}$ stands for the overall mean $\overline{\overline{X}}=\frac{1}{N}\sum_{i=1}^k\sum_{j=1}^{n_i}X_{ij}$ of the combined sample $\underset{\sim}{\boldsymbol{X}}=(\boldsymbol{X}_1,\ldots.\boldsymbol{X}_k)$, into the Between Sum of Squares $BSS=\sum_{i=1}^k n_i(\overline{X}_i-\overline{\overline{X}})^2$ and the Within Sum of Squares $WSS=\sum_{i=1}^k\sum_{j=1}^{n_i}(X_{ij}-\overline{X}_i)^2$, we obtain two *independent* estimators of the variance, namely the means squares $\frac{BSS}{k-1}$ and $S_p^2=\frac{WSS}{N-k}$. Moreover, while S_p^2 is an unbiased estimator of σ^2, $\frac{BSS}{k-1}$ is unbiased under the null hypothesis $\mu_1=\mu_2=\cdots=\mu_k=\mu$, but when the alternative is true $\frac{BSS}{k-1}$ is biased, and hence it must *overestimate* σ^2, and the F-ratio $\frac{\frac{BSS}{k-1}}{\frac{WSS}{N-k}} \underset{|H_0}{\frown} F_{k-1,N-k}$ should detect gross departures from the null hypothesis.

Once again, Welch–Satterthwaite pathbreaking techniques are useful in constructing approximate solutions without assuming homocedasticity (Satterthwaite, 1946; Welch, 1951; Oehlert, 2000).

Inference on location and scale in Gaussian populations is thus simple for two main reasons:

1. If $\boldsymbol{X}$ is a random sample from $X \frown Gaussian(\mu,\sigma)$, $\overline{X}$ and S^2 are independent random variables, and thus in the two-sample setting, $\overline{X}_1 - \overline{X}_2$ and S_p^2 are independent random variables. Thus deducing the exact distribution of the studentized variables $\frac{\overline{X}-\mu}{S\sqrt{\frac{1}{n}}}$ and $\frac{\overline{X}_1-\overline{X}_2-(\mu_1-\mu_2)}{S_p\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}}$ (Student, 1908) has been a simple task.

   Unfortunately, independence of $\overline{X}$ and S^2 is a characterization of the Gaussian populations, and studentization in other populations, e.g. uniform populations, is a hard task, even for samples of size 3 (Perlo, 1933).

2. Assume that we want to estimate the location parameter θ of an absolutely continuous population X using the maximum likelihood estimator $\hat{\theta}$. Under what conditions is $\hat{\theta} = \overline{X}$?

   Given a sample $\boldsymbol{x} = (x_1,\ldots,x_n)$, $\hat{\theta}$ must satisfy $\sum_{k=1}^{n} g'(\hat{\theta}-x_k) = 0$, where $g(\theta - x_k) = \ln[f(x_k \mid \theta)]$. The special case $x_1 = \cdots = x_{n-1} = 0$ and $x_n = nu$ implies that $(n-1)g'(u)+g'((1-n)u) = 0,\ n = 2,3,\ldots$; the case $n=2$ shows at once that g must be an odd function. Thus $g'(nu) = ng'(u),\ u \in \mathbb{R},\ n = 2,3,\ldots$, and henceforth g' is a linear function. From this, $g'(u) = Cu \implies g(u) = \frac{1}{2}\,C\,u^2 + b$, satisfying the condition $\int_{\mathbb{R}} f(x|\theta)\,\mathrm{d}u = 1$. Then

   $$f(x \mid \theta) = \sqrt{\frac{\alpha}{2\,\pi}}\mathrm{e}^{-\frac{1}{2}\,\alpha\,(x-\theta)^2}\mathbb{I}_{\mathbb{R}}(x), \quad \alpha > 0.$$

   This observation (Gauss, 1809) shows that the Gaussian family is unique, in the realm of absolutely continuous distributions, in having $\overline{X}$ as the maximum likelihood estimator of its location parameter $\theta = \mathbb{E}[X]$.

In the sequel we shall investigate how a characterization of the exponential populations — namely that $S_1 := X_{1:n}$ and the spacings $S_k := X_{k:n} - X_{k-1:n},\ k = 2,\ldots,n$ are independent — can be used to establish simple results in what concerns inference the location parameter λ of $Y = \delta X + \lambda \frown Exponential(\lambda,\delta)$, where $X \frown Exponential(1)$ is the standard exponential, with probability density function $f_X(x) = \mathrm{e}^{-x}\,\mathbb{I}_{(0,\infty)}(x)$.

2 Inference on the location of exponential random variables

2.1 The one-sample case

Let $\boldsymbol{y}$ be a random sample of size n from $Y \frown \textit{Exponential}(\lambda, \delta),\ \lambda \in \mathbb{R}, \delta > 0$, i.e., with distribution function $F_Y(y) = \left(1 - \exp(-\frac{y-\lambda}{\delta})\right) \mathbb{I}_{[\lambda,\infty)}(y)$.

The maximum likelihood estimators of the parameters are $\widehat{\lambda} = Y_{1:n}$ and $\widehat{\delta} = \overline{Y} - Y_{1:n}$. From those we can easily construct unbiased estimators of λ and of δ, $\widetilde{\lambda} = Y_{1:n} - \frac{\widetilde{\delta}}{n}$ and $\widetilde{\delta} = \frac{1}{n-1} \sum_{k=1}^{n} (Y_k - Y_{1:n})$. Moreover,

$$\widehat{\delta} = \overline{Y} - Y_{1:n} = \frac{1}{n} \sum_{k=2}^{n} (Y_{k:n} - Y_{1:n}) = \frac{1}{n} \sum_{k=2}^{n} (n+1-k) S_k,$$

where the spacings $S_k = Y_{k:n} - Y_{k-1:n} \frown \textit{Exponential}\left(0, \frac{\delta}{n+1-k}\right),\ k = 2, \ldots, n$ are independent and independent of $S_1 = Y_{1:n} - \lambda \frown \textit{Exponential}\left(0, \frac{\delta}{n}\right)$, cf. David and Nagaraja, (2003).

Thus the studentized random variable

$$\frac{Y_{1:n} - \lambda}{\widehat{\delta}} = \frac{\dfrac{Y_{1:n} - \lambda}{\delta}}{\dfrac{\widehat{\delta}}{\delta}}$$

is the quotient of the independent random variables $\dfrac{Y_{1:n} - \lambda}{\delta} \frown \textit{Exponential}(0, \frac{1}{n})$ and $\dfrac{\widehat{\delta}}{\delta} \frown \textit{Gamma}\left(n-1, 0, \frac{1}{n}\right)$.

Therefore, the probability density function of $\dfrac{Y_{1:n} - \lambda}{\widehat{\delta}}$ is

$$f_{\frac{Y_{1:n}-\lambda}{\widehat{\delta}}}(x) = \frac{n-1}{(1+x)^n} \mathbb{I}_{(0,\infty)}(x),$$

i.e.,

$$\frac{Y_{1:n} - \lambda}{\widehat{\delta}} \frown \textit{Pareto}(0, n-1)$$

can be used either to construct confidence intervals for the location parameter λ, or to test some specific hypothesis about where its value lies. Observe that the quantiles of this distribution are $x_{n,1-\alpha} = \alpha^{-\frac{1}{n-1}} - 1$.

2.2 The 'homocedastic' independent two samples case

Although Johnson *et al.* (1995, p. 193) and Brilhante and Kotz (2008) consider more general asymmetric Laplace random variables, herein we shall consider only the simpler situation of equal sample sizes, leading to the usual Laplace, obtained as $X_1 - X_2$, where X_1 and X_2 are iid exponential random variables.

Let $\boldsymbol{Y}_1$ be a random sample of size $\frac{n}{2}$ ($n > 2$ even) from $Y_1 \frown \mathit{Exponential}(\lambda_1, \delta)$, $\boldsymbol{Y}_2$ a random sample of size $\frac{n}{2}$ from $Y_2 \frown \mathit{Exponential}(\lambda_2, \delta)$ — hence we are assuming a 'homocedastic' situation $\delta_1 = \delta_2 = \delta$ —, and $\boldsymbol{Y}_1$ and $\boldsymbol{Y}_2$ independent. We shall denote $Y_{k:n}^{(1)}$ the k-th order statistics of $\boldsymbol{Y}_1$, $X_{j:n}^{(2)}$ the j-th order statistics of $\boldsymbol{Y}_2$.

Using arguments analogous to those stated in the one sample case, it is easy to establish that the studentized random variable

$$W = \frac{Y_{1:\frac{n}{2}}^{(1)} - Y_{1:\frac{n}{2}}^{(2)} - (\lambda_1 - \lambda_2)}{\widehat{\delta}} = \frac{\dfrac{Y_{1:\frac{n}{2}}^{(1)} - \lambda_1 - (Y_{1:\frac{n}{2}}^{(2)} - \lambda_2)}{\delta}}{\dfrac{\widehat{\delta}}{\delta}}$$

where the random variables

$$\frac{Y_{1:\frac{n}{2}}^{(1)} - Y_{1:\frac{n}{2}}^{(2)} - (\lambda_1 - \lambda_2)}{\delta} \frown \mathit{Laplace}\left(\frac{2}{n}\right)$$

and

$$\frac{\widehat{\delta}}{\delta} = \frac{\left(\overline{Y}_1 - Y_{1:\frac{n}{2}}^{(1)}\right) + \left(\overline{Y}_2 - Y_{1:\frac{n}{2}}^{(2)}\right)}{2} \frown \mathit{Gamma}\left(n-2, \frac{1}{n}\right)$$

are independent[1]. From this we readily obtain the probability density function of W,

$$f_W(x) = \frac{\frac{n}{2} - 1}{2\left(1 + \frac{|x|}{2}\right)^{n-1}} \mathbb{I}_{\mathbb{R}}(x).$$

and the upper quantiles

$$w_{n,1-\alpha} = 2\left((2\alpha)^{\frac{1}{2-n}} - 1\right), \ \alpha < \frac{1}{2}$$

(where n is the combined sample size).

[1] In the general case, with eventually unequal sample sizes n_1 and n_2, the adequate pooled estimator of the dispersion parameter δ is $\widehat{\delta} = \frac{n_1\left(\overline{Y}_1 - Y_{1:n_1}^{(1)}\right) + n_2\left(\overline{Y}_2 - X_{1:n_2}^{(2)}\right)}{n_1 + n_2}$.

2.3 The 'homocedastic' independent k samples case

Let

$$\boldsymbol{Y}_1 = (Y_{11}, \ldots, Y_{1n})$$
$$\vdots$$
$$\boldsymbol{Y}_k = (Y_{k1}, \ldots, Y_{kn})$$

be independent random samples from k populations from $Y_{ij} \frown Exponential(\lambda_i, \delta)$, $i = 1, \ldots, k;\ j = 1, \ldots, n$. Once again, herein we shall only deal with the simplified problem of a balanced design, with all sample sizes equal.

For simplicity, we denote $\overline{\lambda}$ the average $\frac{1}{k}\sum_{i=1}^{k}\lambda_i$, and we assume that the $\boldsymbol{Y}_i,\ i = 1, \ldots k$, have been ordered so that $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k$; henceforth $Y_1 \preceq Y_2 \preceq \cdots \preceq Y_k$, where $W \prec Z$ means that W is stochastically not greater than Z.

We shall use the notations $Y_{1:N} = \min_{i,j} Y_{ij},\ i = 1, \ldots, k,\ j = 1, \ldots, n,\ Y_{1:n}^{(i)} = \min_j Y_{ij},\ i = 1, \ldots, k$, and more generally $Y_{k:n}^{(i)}$ the $k-$th order statistic from the $i-$th random sample.

We now split the Total Sum of Spacings $TSSp$ into a Between Sum of Spacings $BSSp$ and a Within Sum of Spacings $WSSp$:

$$TSSp = \sum_{i=1}^{k}\sum_{j=1}^{n}(Y_{ij} - Y_{1:N}) = BSSp + WSSp =$$
$$= \sum_{i=1}^{k} n\left(Y_{1:n}^{(i)} - Y_{1:N}\right) + \sum_{i=1}^{k}\sum_{j=1}^{n}\left(Y_{ij} - Y_{1:n}^{(i)}\right)$$

The simple observation that

$$\begin{aligned} WSSp &= \textstyle\sum_{i=1}^{k}\sum_{j=2}^{n}\left(Y_{j:n}^{(i)} - Y_{1:n}^{(i)}\right) \\ &= \underbrace{\sum_{i=1}^{k}\underbrace{\sum_{j=2}^{n}(n+1-j)\left[Y_{j:n}^{(i)} - Y_{j-1:n}^{(i)}\right]}_{Gamma(n-1,\delta)}}_{Gamma(N-k,\delta)} \end{aligned}$$

shows that $\dfrac{WSSp}{N-k}$ is an unbiased estimator of δ, either under H_0 or under H_A.

Under H_0, $BSSp \frown Gamma(k-1, \delta)$, and therefore $\dfrac{BSSp}{k-1}$ is an unbiased estimator of δ. However, under H_A, $\dfrac{BSSp}{k-1}$ is a biased estimator of δ.

As a consequence, an ANOSp (**AN**alysis **O**f **Sp**acings) table similar to the one-way ANOVA table

Sum of Spacings	$d.f.$	δ estimator	$F-ratio$
$BSSp=\sum_{i=1}^{k} n\left[Y_{1:n}^{(i)}-Y_{1:N}\right]$	$k-1$	$\frac{BSSp}{k-1}$	$\frac{\frac{BSSp}{k-1}}{\frac{WSSp}{N-k}} \frown_{\mid H_0} F_{(2(k-1),2(N-k))}$
$WSSp=\sum_{i=1}^{k}\sum_{j=1}^{n}\left[Y_{ij}-Y_{1:n}^{(i)}\right]$	$N-k$	$\frac{WSSp}{N-k}$	

should be able to detect gross departures from the null hypothesis.

3 Alternative results on external and internal studentization in exponential populations

The broad concept of studentization as defined in David and Nagaraja (2003) inspires alternative approaches to inference on the location parameter of an exponential population $Y \frown Exponential(\lambda,\delta)$. Brilhante *et al.* (2001) summarize previous work by the authors, namely, in the one-sample case, the study of the studentized minimum using the range as the estimator of a simple function of the nuisance scale parameter,

$$\frac{Y_{1:n}-\lambda}{Y_{n:n}-Y_{1:n}} \stackrel{\mathrm{d}}{=} \frac{X_{1:n}}{X_{n:n}-X_{1:n}},$$

with $X=\frac{Y-\lambda}{\delta}$ is the standard exponential.

As $Y_{1:n}$ and $Y_{n:n}-Y_{1:n}$ are independent, it is once again external studentization. Due to the memoryless of exponential populations, $X_{n:n}-X_{1:n} \stackrel{\mathrm{d}}{=} X_{n-1:n-1}$, with probability density function $f_{X_{n:n}-X_{1:n}}(x)=(n-1)\,\mathrm{e}^{-nx}\left(1-\mathrm{e}^{-nx}\right)^{n-2}\,\mathbb{I}_{(0,\infty)}(x)$.

Using standard methods, the probability density function of $W=\frac{Y_{1:n}-\lambda}{Y_{n:n}-Y_{1:n}}$ is

$$f_W(x)=-n\left[B(n,nt)+nt\,\frac{\partial B(n,nt)}{\partial(nt)}\right]\mathbb{I}_{(0,\infty)}(x)=n\,B(n,nt)\sum_{k=1}^{n-1}\frac{nt}{(n-k)+nt}\,\mathbb{I}_{(0,\infty)}(x),$$

where $B(p,q)=\int_0^1 x^{p-1}(1-x)^{q-1}\mathrm{d}x$, $p,q>0$ is Euler's integral of the first kind.

Explicit formulas for the probability density function for $n\le 30$, and a table of high quantiles, can be requested from the authors. For larger values, observe that $n\,\ln(n)\frac{Y_{1:n}-\lambda}{Y_{n:n}-Y_{1:n}}$ converges in distribution to a standard exponential random variables.

Internal studentization may be an interesting alternative; for instance,

$$\tau^*{}_{(n-1;i,k)}=\frac{\overline{Y}_n-\lambda}{Y_{k:n}-Y_{i:n}}$$

$(1 \le i < k \le n)$ for appropriate choices of i and k has a smaller breaking point than any of the studentized variables considered so far. At first sight the problem looks analytically intractable, since $\overline{Y}_n - \lambda$ and $Y_{k:n} - Y_{i:n}$ have a very intricate dependence structure. But rewriting the above expression as

$$Y_{k:n} - Y_{i:n} = \frac{\overline{Y}n}{{\tau^*}_{(n-1;i,k)}},$$

in view of Basu's theorem, and with the standard notations for the Laplace transform $\mathcal{L}$ and inverse Laplace transforms $\mathcal{L}^{-1}$ at given points, we get that

$$\mathcal{L}\left(y^n f^*_{\tau_{(n-1;i,k)}}(y); nx\right) = \frac{\Gamma(n)\,\Gamma(n+1-i)}{n^n\,\Gamma(k-i)\,\Gamma(n+1-k)}\;\frac{\mathrm{e}^{-x(n+1-k)}\,(1-\mathrm{e}^{-x})^{k-i-1}}{x^{n-1}}$$

or

$$\mathcal{L}\left(y^n f^*_{\tau_{(n-1;i,k)}}(y); x\right) = \frac{\Gamma(n)\,\Gamma(n+1-i)}{n\,\Gamma(k-i)\,\Gamma(n+1-k)}\;\frac{\mathrm{e}^{-\frac{x}{n}\;(n+1-k)}\left(1-\mathrm{e}^{-\frac{x}{n}}\right)^{k-i-1}}{x^{n-1}}$$

and hence

$$f^*_{\tau_{(n-1;i,k)}}(y) = \frac{\Gamma(n)\,\Gamma(n+1-i)}{n\,\Gamma(k-i)\,\Gamma(n+1-k)}\;\frac{1}{y^n}\;\mathcal{L}^{-1}\left(\frac{\mathrm{e}^{-\frac{x}{n}\;(n+1-k)}\,\left(1-\mathrm{e}^{-\frac{x}{n}}\right)^{k-i-1}}{x^{n-1}}; y\right).$$

Choosing i and k so that

n	i	k
$3j-1$	$j-1$	$2j$
$3j$	j	$2j+1$
$3j+1$	$j+1$	$2j+2$

we have the specially simple expression

$$f_{\tau^*_{(n-1)}}(y) = \frac{\Gamma(n)\Gamma(2j+1)}{n\,\Gamma^2(j)}\;\frac{1}{y^2}\;\mathcal{L}^{-1}\left(\frac{1}{x^{k-2}}\left(\frac{\mathrm{e}^{-\frac{x}{n}}\,\left(1-\mathrm{e}^{-\frac{x}{n}}\right)}{x}\right)^j; y\right).$$

Since

$$\mathcal{L}^{-1}\left(\frac{\mathrm{e}^{-\frac{x}{n}}\,\left(1-\mathrm{e}^{-\frac{x}{n}}\right)}{x}; y\right) = 1\,\mathbb{I}_{\left(\frac{1}{n},\frac{2}{n}\right)}(y)$$

and

$$\mathcal{L}^{-1}\left(\frac{1}{s^n}; y\right) = \frac{y^{n-1}}{\Gamma(n)},$$

remembering how convolution of densities and products of Laplace transforms are related we get explicit solutions such as, for $n=3$, $i=1$, $k=3$,

$$f_{\tau^*_{(2)}}(x) = \begin{cases} 0 & x < \frac{1}{3} \\ \frac{4(x-\frac{1}{3})}{3\,x^3} & \frac{1}{3} \leq x < \frac{2}{3} \\ \frac{4}{9\,x^3} & x \geq \frac{2}{3} \end{cases}$$

or, for $n = 4$, $i = 2$, $k = 4$,

$$f_{\tau^*_{(3)}}(x) = \begin{cases} 0 & x < \frac{1}{4} \\ \frac{3}{32x^4}\,(4x-1)^2 & \frac{1}{4} \leq x < \frac{1}{2} \\ \frac{3\,(8x-3)}{32\,x^4} & x \geq \frac{1}{2} \end{cases} .$$

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