PENULTIMATE APPROXIMATIONS IN STATISTICS OF EXTREMES AND RELIABILITY OF LARGE COHERENT SYSTEMS

PAULA REIS, LUÍSA CANTO E CASTRO, SANDRA DIAS, AND M. IVETTE GOMES

Abstract. In reliability theory any coherent system can be represented as either a series-parallel or a parallel-series system. Its lifetime can thus be written as the minimum of maxima or the maximum of minima. For large-scale coherent systems it is sensible to assume that the number of system components goes to infinity. Then, the possible non-degenerate extreme value laws either for maxima or for minima are eligible candidates for the system reliability or at least for the finding of adequate lower and upper bounds for the reliability. The identification of the possible limit laws for the system reliability of homogeneous series-parallel (or parallel-series) systems has already been done under different frameworks. However, it is well-known that in most situations such non-degenerate limit laws are better approximated by an adequate penultimate distribution. Dealing with regular and homogeneous parallel-series systems, we assess both theoretically and through Monte-Carlo simulations the gain in accuracy when a penultimate approximation is used instead of the ultimate one.

1. Introduction

The main motivation for this paper lies on the fact that the study of R_T, the exact *reliability function* (RF) of a complex technological system with lifetime T and lifetime distribution function F_T, given by

$$R_T(t) := \mathbb{P}(T > t) = 1 - F_T(t),$$

can be an intractable problem due to its large number of components and to the way the operating process uses such components. Among several other examples, we can

All authors are supported by National Funds through **FCT** – Fundação para a Ciência e a Tecnologia, projects PEst-OE/MAT/UI0006/2011, PEst-OE/MAT/UI4080/2011, and EXTREMA, PTDC/FEDER.

mention transport networks of oil and water, telecommunication systems, electrical energy distribution networks, and charge and discharge networks.

Assuming that the number of components of a system **S** goes to infinity, asymptotic *extreme value* (EV) ultimate or limiting models often provide a good interpretion of the RF of **S**. Considering a fixed large number of components, pre-asymptotic or penultimate models provide an improvement of the convergence rate and a better approximation to the RF of **S**.

After a brief discussion, in Section 2, on the role of *order statistics* (OS) in reliability, we provide in Section 3 the main results in *extreme value theory* (EVT), needed for the derivation of ultimate and penultimate behaviour of a *parallel-series* (**PS**) system—a parallel structure with components connected in series. Such a penultimate behaviour will be discussed in Section 4. In Section 5 we present the results of a Monte-Carlo simulation study that enables us to understand the gain in accuracy when a penultimate approximation is used instead of the ultimate one. Finally, in Section 6, we prove the main results in Sections 3 and 4.

2. The Role of Order Statistics in Reliability

In reliability theory any coherent system can be represented as either a *series-parallel* (**SP**)—a series structure with components connected in parallel— or a **PS** system, and its lifetime can thus be written as the *minimum of maxima* or the *maximum of minima*. Just as mentioned above, let T denote the lifetime of a coherent structure with n components, with lifetimes $(T_1, \ldots, T_n)$, possibly with the same common *distribution function* (DF), F. Let us denote $(T_{1:n} \leq \cdots \leq T_{n:n})$ the sample of associated ascending OS, with $T_{1:n} = \min_{1 \leq i \leq n} T_i$, $T_{n:n} = \max_{1 \leq i \leq n} T_i$. The main importance of OS in reliability lies on the fact that the random variable (RV) T can always be written as a function of the OS associated to the RVs T_i, $1 \leq i \leq n$. See Gnedenko *et al.* (1969), among others, who identify as $T_{1:n}$ the lifetime of a series system, i.e. the one that works if and only if all its n components work, and as $T_{n:n}$ the lifetime of a parallel system, i.e. a structure that works if and only if at least one of its n components work. Indeed, we always have $T = T_{I:n}$, where I is a discrete RV with support $\{1, 2, \ldots, n\}$. The vector $\underline{s} := (s_1, s_2, \ldots, s_n)$, with $s_i := \mathbb{P}(I = i)$, $1 \leq i \leq n$, is the so-called signature of the system (Samaniego, 1985).

Moreover, we state the following relevant result in reliability theory.

Theorem 1 (Barlow and Proshan, 1975)**.** *Any coherent structure can be represented either as a* **SP** *or a* **PS** *structure.*

Example 2. Let us consider the simple bridge structure, represented in Figure 1.

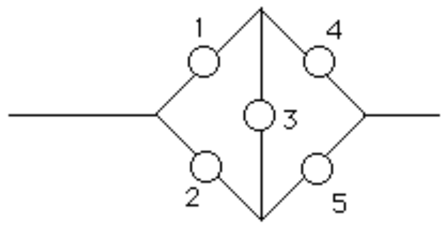

FIGURE 1. Representation of a simple bridge system

In order to find the aforementioned representations in Theorem 1 we need to identify the so-called *minimal paths*—paths without irrelevant components that enable the operation of the system, and the so-called *minimal cuts*—a set of relevant components that imply the failure of the system whenever removed.

In the bridge structure, we have the minimal paths, {1, 4}, {2, 5}, {1, 3, 5}, and {2, 3, 4}. Moreover, we have the minimal cuts, {1, 2}, {4, 5}, {1, 3, 5}, and {2, 3, 4}. Consequently, we show respectively in Figures 2 and 3, the **PS** and **SP** representations of the bridge system in Figure 1.

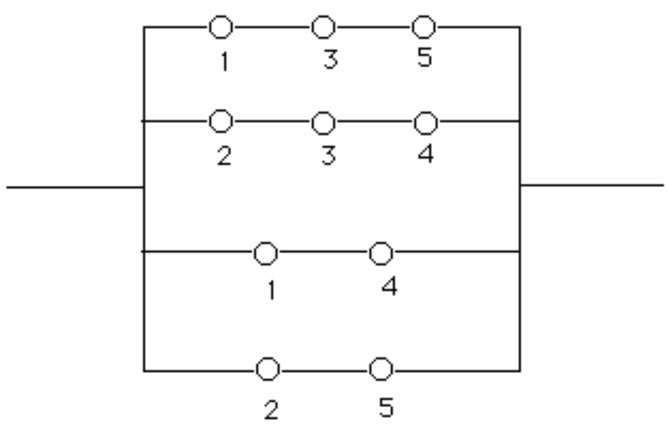

FIGURE 2. **PS** repreentation of the structure in Figure 1

We can thus write,

$$T = \max\Big(\min(T_1, T_3, T_5), \min(T_2, T_3, T_4), \min(T_1, T_4), \min(T_2, T_5)\Big)$$

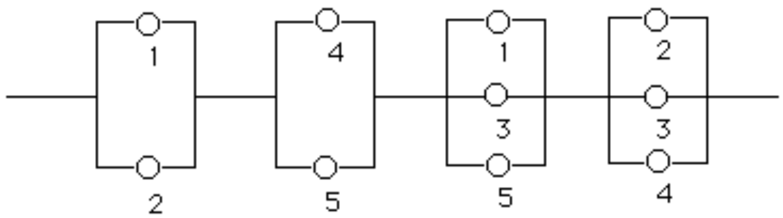

FIGURE 3. **SP** repreentation of the structure in Figure 1

as well as

$$T = \min\Big(\max(T_1, T_2), \max(T_4, T_5), \max(T_1, T_3, T_5), \max(T_2, T_3, T_4)\Big).$$

We obviously need to pay attention to the strong dependence of the different RVs either in the overall max or min operators. But we can build lower and upper bounds for the reliability on the basis of the minimal cuts (assuming they are independent) and minimal paths (assuming they are disjoint), respectively.

Generally speaking, let $P_j, 1 \le j \le p = p_n$, denote the minimal paths, and C_j, $1 \le j \le s = s_n$, the minimal cuts. Then, and for non-necessarily *identically distributed* (ID) components,

$$\prod_{j=1}^{s}\Big(1 - \prod_{i \in C_j} F_i(t)\Big) \le R_{_T}(t) \le 1 - \prod_{j=1}^{p}\Big(1 - \prod_{i \in P_j}(1 - F_i(t))\Big).$$

For sake of simplicity, we now assume that all minimal paths have the same size $l = l_n$ and that all minimal cuts have a size $r = r_n$ (the so-called *regular* system), and that $R_i(t) = R(t)$, $1 \le i \le n$ (the so-called *homogeneous* system). Then, we get

$$\Big(1 - (1 - R(t))^{r_n}\Big)^{s_n} \le R_{_T}(t) \le 1 - \Big(1 - R^{l_n}(t)\Big)^{p_n},$$

$n = r_n s_n = l_n p_n$.

3. LIMITING RESULTS IN EVT

For large-scale coherent systems it is sensible to assume that the number of system components goes to infinity. Then, the possible non-degenerate *extreme value distributions* (EVD) either for *maxima* (EV_MD) or for *minima* (EV_mD) are eligible candidates for the system reliability or at least for the finding of adequate lower and upper bounds for such a reliability.

3.1. **Max and Min-Stable Laws.** Let $\underline{\mathbf{X}}_n = (X_1, \ldots, X_n)$ be a sample of size n of *independent* ID (IID), or even weakly dependent RVs with DF F, and let $X_{i:n}$, $1 \leq i \leq n$, denote the associated ascending OS. EVT provides a great variety of limiting results that enable us to to deal with alternative approaches in the statistical analysis of extreme events. The main *limiting* result in EVT is due to Gnedenko (1943). In this seminal paper, Gnedenko has fully characterised the possible non-degenerate limiting distribution of the linearly normalised maximum, $(X_{n:n} - b_n^*)/a_n^*$, $a_n^* > 0,\ b_n^* \in \mathbb{R}$. Such a limit is of the type of the $\mathrm{EV_M D}$, the unique max-stable laws, in the sense that $\exists\ \alpha_r^* > 0, \beta_r^* \in \mathbb{R}$ with $H^r(x) = H(\alpha_r^* x + \beta_r^*)$, $\forall r \geq 1$. The $\mathrm{EV_M D}$ is given by

$$G(x) \equiv G_\gamma(x) := \begin{cases} \mathrm{e}^{-(1+\gamma x)^{-1/\gamma}},\ 1+\gamma x > 0 & \text{if} \quad \gamma \neq 0 \\ \mathrm{e}^{-\mathrm{e}^{-x}},\ x \in \mathbb{R} & \text{if} \quad \gamma = 0. \end{cases} \tag{1}$$

It was first introduced in von Mises (1936), and applied to the analysis of meteorological and hydrological data in Jenkinson (1956). We then say that F belongs to the *max-domain of attraction* of G_γ and denote this by $F \in \mathrm{D_M}(G_\gamma)$. The shape parameter γ, the so-called *extreme value index for maxima* ($\mathrm{EVI_M}$), measures the heaviness of the *right-tail* function (or RF), $\overline{F}(x) \equiv R(x) := 1 - F(x)$, as $x \to +\infty$ and the heavier the right-tail, the larger γ is.

The $\mathrm{EV_M D}$ is sometimes separated in the three following types,

$$\begin{array}{ll} \text{Type I (Gumbel) :} & \Lambda(x) = \exp(-\exp(-x)),\ x \in \mathbb{R} \\ \text{Type II (Fréchet) :} & \Phi_\alpha(x) = \exp(-x^{-\alpha}),\ x \geq 0 \\ \text{Type III (max-Weibull) :} & \Psi_\alpha(x) = \exp(-(-x)^{\alpha}),\ x \leq 0, \end{array}$$

with $\alpha > 0$, the types considered in Gnedenko (1943). We have

$$\Lambda(x) = G_0(x),\ \Phi_\alpha(x) = G_{1/\alpha}(\alpha(1-x)),\ \Psi_\alpha(x) = G_{-1/\alpha}(\alpha(x+1)),$$

with G_γ the $\mathrm{EV_M D}$, in (1).

Remark 3. Any result for maxima has its counterpart for minima due to the fact that

$$\min_{1\leq i\leq n} X_i = -\max_{1\leq i\leq n}(-X_i).$$

If the sequence $\max_{1\leq i\leq n}(-X_i)$ can be normalised, in order to admit a non degenerate limit Z, then the DF of Z will be of the same type as G_θ, the $\mathrm{EV_M D}$, for some $\theta \in \mathbb{R}$. Hence the possible limit laws for minima, conveniently normalised, will be such that $F_{-Z}(x) = \mathbb{P}(-Z \leq x) = \mathbb{P}(Z \geq -x) = 1 - G_\theta(-x) =: G_\theta^*(x)$.

We thus have the min-stable laws or $\text{EV}_\text{m}\text{D}$,

$$G^*(x) \equiv G^*_\theta(x) := \begin{cases} 1 - \mathrm{e}^{-(1-\theta x)^{-1/\theta}}, \ 1-\theta x > 0 & \text{if} \quad \theta \neq 0 \\ 1 - \mathrm{e}^{-\mathrm{e}^x}, \ x \in \mathbb{R} & \text{if} \quad \theta = 0. \end{cases} \tag{2}$$

We then say that F is in the min-domain of attraction of G^*_θ, using the notation $F \in \mathrm{D_m}(G^*_\theta)$ if the DF of $-X$ is in the max-domain of attraction of G_θ, i.e.

$$F \in \mathrm{D_m}(G^*_\theta) \quad \Longleftrightarrow \quad R = 1 - F \in \mathrm{D_M}(G_\theta).$$

In this case, there exist sequences $(a_n > 0, b_n \in \mathbb{R})$, such that

$$1 - (1 - F(a_n x + b_n))^n = 1 - R^n(a_n x + b_n) \underset{n\to\infty}{\longrightarrow} G^*_\theta(x).$$

The shape parameter θ, the so-called *extreme value index for minima* (EVI_m), measures the heaviness of the *left-tail function* of $F(x)$, as $x \to -\infty$, and the heavier the left-tail, the larger θ is.

Similarly to what happens in the max-scheme, and just as presented in Gnedenko (1943), the $\text{EV}_\text{m}\text{D}$ is sometimes separated in the three following types,

$$\begin{aligned} &\text{Type I (min-Gumbel)}: && \Lambda^*(x) = 1 - \exp(-\exp(x)), \ x \in \mathbb{R} \\ &\text{Type II (min-Fréchet)}: && \Phi^*_\alpha(x) = 1 - \exp(-(-x)^{-\alpha}), \ x \leq 0 \\ &\text{Type III (Weibull)}: && \Psi^*_\alpha(x) = 1 - \exp(-x^\alpha), \ x \geq 0. \end{aligned}$$

Remark 4. In most applications involving lifetimes the limit laws G^*_θ are restricted to the case $\theta \leq 0$. In fact, a lifetime T is always nonnegative. Thus $-T$ is a RV with a finite right endpoint and can only be in the max-domain of attraction of a max-Weibull or a Gumbel DF. However, since there are systems with large durability, we also often consider the case $\theta > 0$.

3.2. **Rates of convergence and penultimate approximations.** Another important problem in EVT concerns the rate of convergence of $F^n(a^*_n x + b^*_n)$ towards $G_\gamma(x)$ or, equivalently, the finding of estimates of the difference

$$d_n(F, G_\gamma, x) := F^n(a^*_n x + b^*_n) - G_\gamma(x).$$

Indeed, parametric inference on the right-tail of F, usually unknown, is done on the basis of the identification of $F^n(a^*_n x + b^*_n)$ and of $G_\gamma(x)$, replacing $F^n(x)$ by $G_\gamma((x - b^*_n)/a^*_n)$, with b^*_n and $a^*_n > 0$ being unknown parameters to be estimated from an adequate sample. The rate of convergence is thus important because it may validate or not the most usual models in *statistics of extremes*, and this was also already detected by Gnedenko. In EVT there exists no analogue of the Berry-Esséen theorem that, under broad conditions, gives a rate of convergence of the order of $1/\sqrt{n}$ in the central limit theorem. The rate of convergence depends here strongly on the right-tail of F, on the choice of the attraction coefficients, and can be rather slow, as first detected by Fisher and Tippett (1928). These authors were indeed the first ones to provide a so-called max-Weibull penultimate approximation for $\Phi^n(x)$, with Φ the normal DF.

The modern theory of rates of convergence in EVT began with Anderson (1971) and Galambos (1978). For papers on the subject prior to 1992, we refer the review in Gomes (1994). Developments have followed different directions that can be found in a recent paper by Beirlant *et al.* (2012). We refer here only the study of the structure of the remainder $d_n(F, G_\gamma, x)$, with $F \in \mathrm{D_M}(G_\gamma)$, $\gamma \in \mathbb{R}$, i.e. the finding of $d_n \to 0$, as $n \to \infty$, and $\varphi(x)$ such that

$$F^n(a_n^* x + b_n^*) - G_\gamma(x) = d_n \varphi(x) + o(d_n).$$

We then say that the rate of convergence of $F^n(a_n^* x + b_n^*)$ towards $G_\gamma(x)$ is of the order of d_n. In this same framework, the possible penultimate behaviour of $F^n(a_n^* x + b_n^*)$ has been studied, i.e. the possibility of finding $H(x) = H_n(x)$, perhaps a max-stable DF, such that

$$F^n(a_n^* x + b_n^*) - H_n(x) = O(r_n), \quad r_n = o(d_n).$$

Among others, we refer Gomes (1984), Gomes and Pestana (1987), and Gomes and de Haan (1999), who derived, for all $\gamma \in \mathbb{R}$, exact penultimate approximation rates, under von Mises-type conditions and some extra differentiability assumptions. Kaufmann (2000) proved a similar result, but under weaker conditions. This penultimate or pre-asymptotic behaviour has further been studied by Raoult and Worms (2003), and by Diebolt and Guillou (2005), among others.

3.3. **First, second order conditions and penultimate approximations.** Another crucial topic in the field of statistics of extremes is the establishment of the so-called first and second-order conditions. The first-order conditions are necessary and sufficient conditions (or sufficient conditions) to have $F \in \mathrm{D_M}(G_\gamma)$, with G_γ defined in (1), or similarly, $F \in \mathrm{D_m}(G^*_\theta)$, with G^*_θ defined in (2). Let us assume that von Mises' sufficient condition for convergence of $F^n(a_n^* x + b_n^*)$ to $G_\gamma(x)$ is fulfilled. This means that, with the notation

$$u(x) := -\ln(-\ln F(x)), \quad v(t) := u^{\leftarrow}(t) \quad \text{and} \quad \eta(t) := v''(t)/v'(t), \tag{3}$$

where $u^{\leftarrow}(t) := \inf\{x : F(x) \geq t\}$ denotes the generalised inverse function of u, we assume the validity of the first-order condition,

$$\lim_{t \uparrow x^F} \eta(t) = \gamma,$$

with x^F the right endpoint of the underlying model F.

The second-order conditions essentially measure the rate of convergence in the first-order conditions and depend upon a second-order parameter ρ (≤ 0), that can appear as the non-positive parameter in the limiting relation,

$$\lim_{t \to \infty} \frac{v'''(t) - \gamma v''(t)}{v''(t) - \gamma v'(t)} = \gamma + \rho.$$

Now we come to one of the questions answered in Gomes and de Haan (1999): in the max-scheme, under what circumstances, i.e. for which combination of γ and ρ, can the convergence rate be improved by the use of penultimate approximations? In brief words, the answer is the following: to get any improvement we need to have $\rho = 0$ and to choose $\gamma(t) := \eta(t)$, with η given in (3). We will next state a particular result of the main theorem in Gomes and de Haan (1999), which is adapted in the Theorem 5 below, just for the Ĝumbel case ($\gamma = 0$).

Theorem 5 (Gomes and de Haan, 1999, $\gamma = 0$)**.** *Suppose that*

$$\lim_{x \to \infty} \eta(x) = 0 \ (\textit{von Mises' first order condition}), \tag{4}$$

$$\lim_{x \to \infty} \frac{v'''(x)}{v''(x)} = 0 \ (\textit{von Mises second order condition with } \rho = 0), \tag{5}$$

and

$$\lim_{x \to \infty} \frac{\eta''(x)}{\eta'(x)} = 0 \ (\textit{von Mises type penultimate condition}). \tag{6}$$

Then, there exists a sequence $\{\gamma_n\}_{n \geq 1}$ *such that*

$$F^n(a_n^* x + b_n^*) - G_{\gamma_n}(x) = o\big(F^n(a_n^* x + b_n^*) - G_0(x)\big),$$

uniformly over $x \in \mathbb{R}$. More specifically, with $a_n^* := v'(\ln n)$, $b_n^* := v(\ln n)$ and $\gamma_n := \eta(\ln n)$

$$\lim_{n\to\infty} \frac{F^n(a_n^* x + b_n^*) - G_{\gamma_n}(x)}{\eta'(\ln n)} = \frac{x^3 G_0'(x)}{6},$$

uniformly for all $x \in \mathbb{R}$.

3.4. **Ultimate Models for the Sequence of RFs of a Regular and Homogeneous PS System.** The identification of the possible ultimate limit laws for the system reliability of homogeneous **PS** systems was performed in Reis and Canto e Castro (2009). We first state as a proposition, a result proved in the aforementioned article.

Proposition 1 (Reis and Canto e Castro, 2009)**.** *Any stable law for minima, i.e. G_θ^*, in (2), belongs to $\mathrm{D_M}(G_0)$, i.e. there exist sequences $\{a_n^* > 0\}_{n\geq 1}$ and $\{b_n^* \in \mathbb{R}\}_{n\geq 1}$ such that*

$$(G_\theta^*(a_n^* x + b_n^*))^n \underset{n\to\infty}{\longrightarrow} G_0(x),$$

uniformly in $\mathbb{R}$. With

$$c_n := -\ln(1-\exp(-1/n)),$$

we can choose

$$a_n^* = \frac{\mathrm{e}^{c_n} - 1}{n\ c_n^{\theta+1}}, \qquad b_n^* = \begin{cases} \frac{1-c_n^{-\theta}}{\theta} & \text{if} \quad \theta \neq 0 \\ \ln c_n & \text{if} \quad \theta = 0. \end{cases} \tag{7}$$

We next provide a result similar to the one in Theorem 3.1 of Reis and Canto e Castro (2009), but under slightly different assumptions.

Theorem 6. *Let $F \in \mathrm{D_m}(G_\theta^*)$, the min-domain of attraction of G_θ^*, i.e. let us assume that there exist sequences $\{a_n > 0\}_{n\geq 1}$ and $\{b_n \in \mathbb{R}\}_{n\geq 1}$ such that*

$$F_{1:n}(a_n x + b_n) := 1 - (1 - F(a_n x + b_n))^n \underset{n\to\infty}{\longrightarrow} G_\theta^*(x) = 1 - G_\theta(-x), \tag{8}$$

for all $x \in \mathbb{R}$ and where G_θ and G_θ^* are the $\mathrm{EV_MD}$ and the $\mathrm{EV_mD}$, in (1) and (2), respectively. Given a sequence of integers $p_n \to \infty$, such that

$$p_n e_{l_n} = o(1), \quad \text{with} \quad e_n := \sup_{x\in\mathbb{R}} |F_{1:n}(a_n x + b_n) - G_\theta^*(x)|, \tag{9}$$

and $l_n \to \infty$, with $l_n p_n = n$, there exist sequences $\{\alpha_n > 0\}_{n\geq 1}$ and $\{\beta_n \in \mathbb{R}\}_{n\geq 1}$ such that

$$F_n(\alpha_n x+\beta_n) := \Big(1-(1-F(\alpha_n x+\beta_n))^{l_n}\Big)^{p_n} \underset{n\to\infty}{\longrightarrow} \Lambda(x) \equiv G_0(x), \tag{10}$$

for all $x \in \mathbb{R}$. We can further consider $\alpha_n = a_{l_n} a^*_{p_n}$ and $\beta_n = b_{l_n} + a_{l_n} b^*_{p_n}$, with (a^*_n, b^*_n) and (a_n, b_n) defined in (7) and (8), respectively. Consequently, for a regular homogeneous **PS** *system, composed by p_n parallel subsystems with l_n components in series, the sequence of RFs, suitably normalised is such that,*

$$R_n(\alpha_n x+\beta_n) := 1-\Big(1-(1-F(\alpha_n x+\beta_n))^{l_n}\Big)^{p_n} \underset{n\to\infty}{\longrightarrow} 1-G_0(x), \tag{11}$$

for all $x \in \mathbb{R}$.

4. Penultimate Models for the Sequence of RFs of a Regular and Homogeneous **PS** System

Now, apart from identifying the right tail penultimate behaviour of min-stable laws, in Theorem 7, showing that in most situations such a non-degenerate limit law is better approximated by an adequate penultimate distribution, we provide, in Theorem 10, the penultimate behaviour of regular and homogeneous **PS** systems.

Theorem 7. *For all $\theta \neq -1$, a min-stable law G^*_θ is under the conditions of* Theorem 5. *Consequently,*

$$\lim_{n\to\infty} \frac{(G^*_\theta(a^*_n x+b^*_n))^n - G_{\gamma_n}(x)}{(\theta+1)/\ln^2 n} = \frac{x^3 G'_0(x)}{6}, \tag{12}$$

uniformly for all $x \in \mathbb{R}$, with (a^*_n, b^*_n) given in (7), and where γ_n is asymptotically given by

$$\gamma_n \equiv \gamma_n(\theta) = -\frac{\theta+1}{\ln n} + O\left(\tfrac{1}{n}\right). \tag{13}$$

We further have

$$(G^*_\theta(a^*_n x+b^*_n))^n - G_0(x) = O(1/\ln n).$$

Remark 8. Note that if $\theta = -1$, von Mises first-order condition in (4) holds, but the second-order condition, in (5), is not valid since $\lim_{x\to\infty} v'''(x)/v''(x) = -1$. Thus, the second-order parameter is $\rho = -1$ and not zero, as needed for the existence of a penultimate approximation. The asymptotic approximation $\big(G^*_{-1}(a_n x+b_n)\big)^n \approx$

$G_0(x)$ can thus not be improved. If $\theta < -1$, $\gamma_n > 0$, and G_{γ_n} is a penultimate sequence of Fréchet distributions for $\left(G_\theta^*\right)^n$. If $\theta > -1$, $\gamma_n < 0$ and G_{γ_n} is a penultimate sequence of max-Weibull distributions for $\left(G_\theta^*\right)^n$.

Let $k_{U,n}$ ($k_{P,n}$) denote the Kolmogorov-Smirnov distance between $\left(G_\theta^*(a_n^* x + b_n^*)\right)^n$ and $G_0(x)$ ($G_{\gamma_n}(x)$), with G_γ, G_θ^*, (a_n^*, b_n^*) and γ_n given in (1), (2), (7) and (13), respectively. We thus have $k_{U,n} := \sup_{x\in\mathbb{R}} |G_\theta^*(a_n^* x + b_n^*) - G_0(x)|$ and $k_{P,n} := \sup_{x\in\mathbb{R}} |G_\theta^*(a_n^* x + b_n^*) - G_{\gamma_n}(x)|$. Let us further consider the indicator $I_n := |k_{U,n} - k_{P,n}|/\max(k_{U,n}, k_{P,n})$. In Table 8, in fully agreement with the theoretical results, we provide the values of γ_n $k_{U,n}$, $k_{P,n}$ and I_n for $n = 10^j$, $1 \le j \le 5$ and $\theta = -2, -1, 0, 1$.

	n	10	10^2	10^3	10^4	10^5
	θ_n	0.434	0.217	0.145	0.109	0.087
	$k_{U,n}$	0.076948	0.043251	0.030276	0.023395	0.019085
$\theta = -2$	$k_{P,n}$	0.017408	0.006408	0.003326	0.002012	0.001343
	I_n	77.376	85.183	89.014	91.399	92.961
	θ_n	0	0	0	0	0
	$k_{U,n}$	0.007092	0.000677	0.000068	0.000007	0.000001
$\theta = -1$	$k_{P,n}$	0.007092	0.000677	0.000068	0.000007	0.000001
	I_n	0.000	0.000	0.000	0.000	0.000
	θ_n	-0.434	-0.217	-0.145	-0.109	-0.087
	$k_{U,n}$	0.078177	0.047254	0.032863	0.025035	0.020195
$\theta = 0$	$k_{P,n}$	0.044689	0.009650	0.004155	0.002351	0.001521
	I_n	42.836	79.578	87.357	90.607	92.471
	θ_n	-0.869	-0.434	-0.290	-0.217	-0.174
	$k_{U,n}$	0.891991	0.100544	0.068669	0.051800	0.041533
$\theta = 1$	$k_{P,n}$	0.840499	0.022204	0.009231	0.005108	0.003248
	I_n	5.773	77.917	86.557	90.140	92.179

TABLE 1. Comparison of ultimate and penultimate approximations for $\left(G_\theta^*(a_n^* x + b_n^*)\right)^n$

In Figure 4, we represent graphically the Kolmogorov-Smirnov distances $k_{U,n}$ and $k_{P,n}$ as a function of θ, for several values of n.

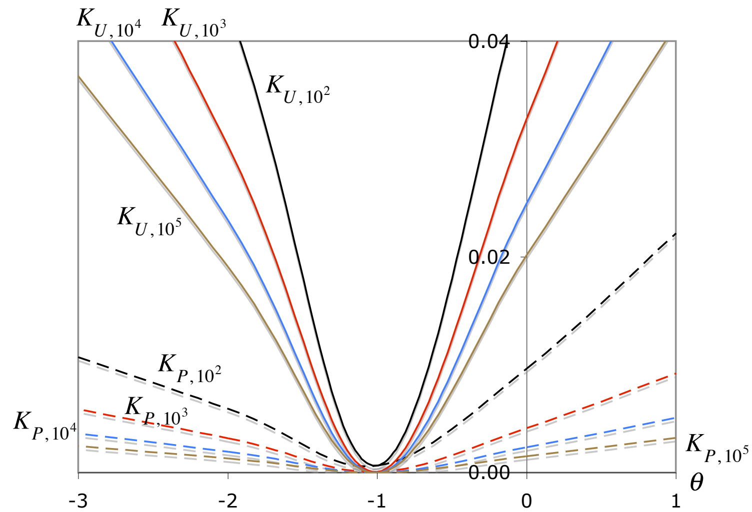

FIGURE 4. Kolmogorov-Smirnov distances, as a function of θ, for $n = 10^2,\ 10^3,\ 10^4$ and 10^5

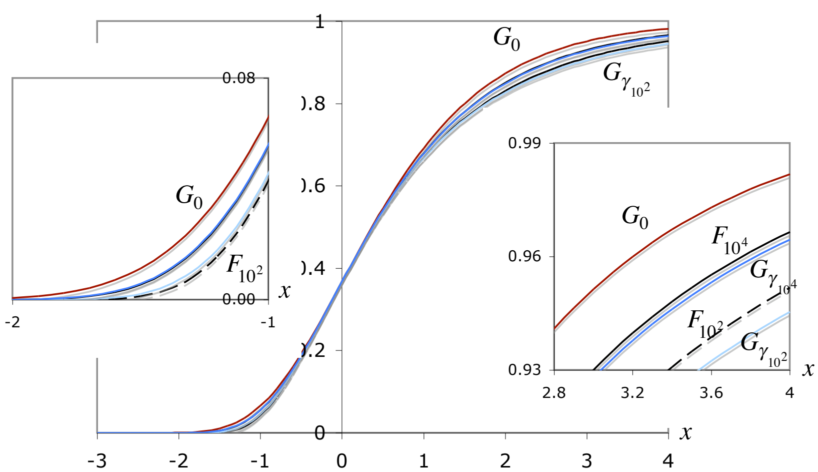

FIGURE 5. Penultimate and ultimate DFs associated to the DF $\left(G_\theta^*(a_n^* x + b_n^*)\right)^n$, $\theta = -2$, $n = 10^2,\ 10^4$

In Figures 5 and 6, we illustrate graphically the penultimate and ultimate bahaviour of $\left(G_\theta^*(a_n^* x + b_n^*)\right)^n$, for $\theta = -2$ and $\theta = 1$, respectively, and $n = 10^2,\ 10^4$.

Remark 9. When n goes to infinity, θ_n goes to $\theta = 0$ and we obtain the limit distribution $G_0(x)$ with a known convergence rate of order $v''(\ln n)/v'(\ln n) = -(\gamma+1)/(\ln n)$, as mentioned in the theorem above. Details can be seen in Gomes and de Haan (1999).

We finally present a theorem related to the penultimate behaviour of the reliability of a regular and homogeneous **PS** system.

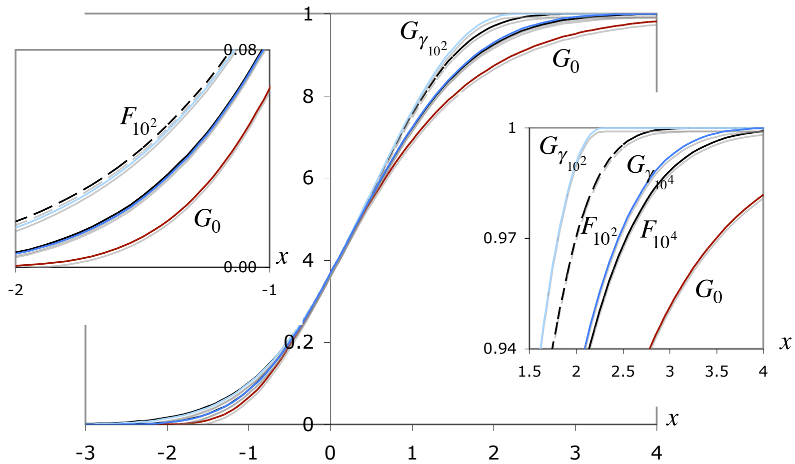

FIGURE 6. Penultimate and ultimate DFs associated to the DF $\left(G^*_\theta(a^*_n x + b^*_n)\right)^n$, $\theta = 1$, $n = 10^2,\ 10^4$

Theorem 10. *Under the conditions of* Theorem 6, *let us consider a sequence of integers* $p_n \to \infty$, *such that, with* $e_n := \sup_{x\in\mathbb{R}} |F_{1:n}(a_n x + b_n) - G^*_\theta(x)|$, $\theta \in \mathbb{R}$,

$$p_n(\ln^2 p_n)e_{l_n} = o(1). \tag{14}$$

Let $p_n l_n = n$, *and* $l_n \to \infty$, *as* $n \to \infty$. *Then, for all* $\theta \neq -1$, *there exist sequences* $\{\alpha_n > 0\}_{n\geq 1}$, $\{\beta_n \in \mathbb{R}\}_{n\geq 1}$ *and a sequence* $\{\gamma_n\}_{n\geq 1}$ *such that*

$$\lim_{n\to\infty} \frac{F_n(\alpha_n x + b_n) - G_{\gamma_n}(x)}{(\theta+1)/\ln^2 p_n} = \frac{x^3}{6}\ G_0'(x). \tag{15}$$

for all $x \in \mathbb{R}$. *Moreover, we can choose* γ_n *as in* (13), $\alpha_n = a_{l_n} a^*_{p_n}$ *and* $\beta_n = b_{l_n} + a_{l_n} b^*_{p_n}$, (a^*_n, b^*_n), (a_n, b_n) *and* $F_n(\alpha_n x + \beta_n)$ *defined in* (7), (8) *and* (10) *respectively.*

Consequently, for a regular homogeneous **PS** *system, composed by* p_n *parallel subsystems with* l_n *components in series, the sequence of* RFs $R_n(\alpha_n x + \beta_n)$, *defined in* (11), *is such that, for all* $x \in \mathbb{R}$,

$$\lim_{n\to\infty} \frac{R_n(\alpha_n x + b_n) - (1 - G_{\gamma_n}(x))}{(\theta+1)/\ln^2 p_n} = \frac{x^3}{6}\ G_0'(x).$$

Remark 11. Under the conditions imposed in Theorem 10, it follows that, as $n \to \infty$, $\gamma_n \to 0$ and $F_n(\alpha_n x + \beta_n) \to G_0(x)$, $\forall x \in \mathbb{R}$, with a rate of convergence of order $1/(\ln n)$. The conclusion of this theorem is thus that when $\theta > -1$, $G_{\gamma_n}(x)$ is a sequence of max-Weibull penultimate distributions for $F_n(\alpha_n x+\beta_n)$ and when $\theta < -1$, $G_{\gamma_n}(x)$ is a sequence of Fréchet penultimate distributions for $F_n(\alpha_n x + \beta_n)$. For

$\theta = -1$, the approximation of $F_n(\alpha_n x + \beta_n)$ to the Gumbel law $G_0(x)$ cannot be improved.

5. A Small-Scale Simulation Study

Dealing with regular and homogeneous **PS** systems, we assess the gain in accuracy when a penultimate approximation is used instead of the ultimate one. We have simulated **PS** systems with lifetime components from different models, including the $\mathrm{EV_m D}(\theta)$, and the $\mathrm{GP_m D}(\theta) = -\ln(1 - \mathrm{EV_m D}(\theta))$, for $\theta = -2(0.5)1$, p_n =20, 50, 100 and l_n = 10, 20, 50, 100. In the simulation study we have used system lifetimes samples of size 100 and have performed $R = 1000$ Monte Carlo replications.

5.1. Testing the EV Condition.

We have first tested, under a semi-parametric framework, the hypothesis

$$\mathcal{H}_0^* : G_n^* := 1 - (1 - F)^{l_n} \in \mathcal{D}_{\mathcal{M}}(G_\gamma), \text{ for some } \gamma \in \mathbb{R}.$$

We have used the test statistics developed in Dietrich *et al.* (2002) and in Drees *et al.* (2006). We have further followed the algorithm proposed in Hüsler and Li (2006). As a general conclusion, we can say that $\mathcal{H}_0^*$ was not rejected and there is no typical behaviour on the variation of l_n.

5.2. Goodness of Fit Test for the Gumbel Law.

We have also tested the ultimate law, i.e. we have considered the hypothesis

$$\mathcal{H}_0 : \; F_n(x) = \left(1 - (1 - F(x))^{l_n}\right)^{p_n} = G_0((x - \lambda)/\delta),$$

with $F_n(x)$ the DF of the lifetime of a **PS** system, already defined in (10), and $(\lambda, \delta) \in \mathbb{R} \times \mathbb{R}^+$ a vector of unknown (location and scale) parameters. For the same set-up described above, we have used the test statistic proposed in Laio (2004), a modified Anderson-Darling statistic. The main conclusions are sketched in the following:

(1) The null hypothesis was rejected except for $\theta = -1$ (showing consistency between simulated and theoretical results).
(2) Estimated type I error increases as θ moves away from -1.
(3) Results are not much affected by a variation of l_n (particularly for $\mathrm{EV_m}$ components' lifetimes) and the estimated type I error decreases as p_n increases.

5.3. **Gain in Accuracy.** In order to answer to the question whether the estimates of γ are closer to the penultimate parameter γ_n rather than to the ultimate parameter zero, we have considered the theoretical value $\gamma_n = -(\gamma+1)/\ln n$ and have computed $\widehat{\gamma}$ through maximum likelihood (ML). On the basis of the $R = 1000$ runs, we have simulated the *root mean square error* (RMSE) and the *bias* (BIAS)-values,

$$\mathrm{RMSE}_P = \sqrt{(1/R)\sum_{i=1}^{R}(\widehat{\gamma}_i - \gamma_n)^2}, \qquad \mathrm{RMSE}_U = \sqrt{(1/R)\sum_{i=1}^{R}\widehat{\gamma}_i^2}$$

$$\mathrm{BIAS}_P = (1/R)\sum_{i=1}^{R}\widehat{\gamma}_i - \gamma_n, \qquad \mathrm{BIAS}_U = (1/R)\sum_{i=1}^{R}\widehat{\gamma}_i.$$

As can be seen in Figure 7, where, as an illustration, we consider the estimates obtained for components with an $\mathrm{EV_mD}(\theta)$ (left) and $\mathrm{GP_mD}(\theta)$ (right), $l_n = 50$ and $p_n = 20$, we have got for $\theta \neq -1$, $\mathrm{RMSE}_P < \mathrm{RMSE}_U$ and $|\mathrm{Bias}_P| < |\mathrm{Bias}_U|$. This leads us to the adoption of max-Weibull or Fréchet models for F_n. Only for $\theta = -1$ were we led to the adoption of a Gumbel model for the RV T.

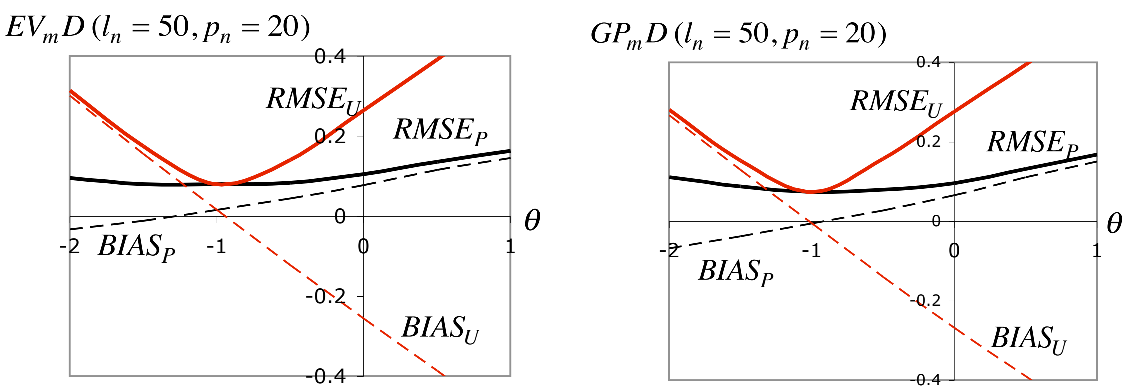

FIGURE 7. Ultimate and penultimate BIAS and RMSE of ML γ-estimates, as a function of θ

Similar results, out of the scope of this article, can be obtained for **SP** systems. Moreover, we have placed ourselves in a classical set-up, but more general frameworks can be considered.

6. Proofs

Proof. [Theorem 6] We know that

$$1 - (1 - F(a_n x + b_n))^n - G^*_\theta(x) = e_n(x),$$

with $e_n(x) \to 0$, as $n \to \infty$, uniformly over $x \in \mathbb{R}$. Also

$$(G^*_\theta(a^*_n x + b^*_n))^n = G_0(x) + e^*_n(x),$$

with $e^*_n(x) \to 0$, as $n \to \infty$. And we want to find conditions on (l_n, p_n), $l_n p_n = n$, such that there exist $(\alpha_n,\ \beta_n) \in \Re^+ \times \Re$ and

$$\{1 - (1 - F(\alpha_n x + \beta_n))^{l_n}\}^{p_n} \underset{n\to\infty}{\longrightarrow} G_0(x).$$

Let us consider $\alpha_n = a_{l_n} a^*_{p_n}$ and $\beta_n = a_{l_n} b^*_{p_n} + b_{l_n}$.
Then

$$\begin{aligned}
\left\{1 - \left(1 - F(\alpha_n x + \beta_n)\right)^{l_n}\right\}^{p_n} &= \left\{1 - \left(1 - F(a_{l_n}(a^*_{p_n} x + b^*_{p_n}) + b_{l_n})\right)^{l_n}\right\}^{p_n} \\
&= \left\{G^*_\theta(a^*_{p_n} x + b^*_{p_n}) + e_{l_n}(a^*_{p_n} x + b^*_{p_n})\right\}^{p_n} \\
&= \left\{G^*_\theta(a^*_{p_n} x + b^*_{p_n})\right\}^{p_n} + \rho_n(x) \\
&= G_0(x) + e^*_{p_n}(x) + \rho_n(x).
\end{aligned}$$

We can further write

$$\rho_n(x) = p_n e_{l_n}(a^*_{p_n} x + b^*_{p_n}) \left\{G^*_\theta(a^*_{p_n} x + b^*_{p_n})\right\}^{p_n - 1}(1 + o(1)).$$

We thus just need to assume that $p_n e_{l_n} = o(1)$, the condition in (9), in order to have the validity of (10) and (11).

□

Proof. [Theorem 7] To prove von Mises' conditions in Theorem 5, we will take $u(x) \equiv u_\theta(x) = -\ln(-\ln G^*_\theta(x))$ and the respective inverse function,

$$v(x) \equiv v_\theta(x) := \begin{cases} \left\{1 - (-\ln(1 - \exp(-\exp(-x))))^{-\theta}\right\}/\theta & \text{if} \quad \theta \neq 0 \\ \ln\left(-\ln\left(1 - \exp(-\exp(-x))\right)\right) & \text{if} \quad \theta = 0. \end{cases}$$

With the notation,

$$g(x) := -\ln(1 - \exp(-\exp(-x))),$$

the first and second derivatives of $v(x)$ are given by

$$v'(x) = g'(x) g^{-\theta-1}(x)$$

and

$$v''(x) = \left\{g''(x) - \tfrac{(\theta+1)(g'(x))^2}{g(x)}\right\} g^{-\theta-1}(x). \tag{16}$$

Hence,

$$\eta(x) \equiv \eta_\theta(x) = \frac{v''(x)}{v'(x)} = \frac{g''(x)}{g'(x)} - (\theta+1)\frac{g'(x)}{g(x)}. \tag{17}$$

Now attending to the exponential series expansion, we obtain the following asymptotic result, as $x \to \infty$,

$$g(x) = x + e^{-x}/2 + O(e^{-2x}). \tag{18}$$

On the other hand, since

$$g'(x) = 1 - e^{-x}/2 + O(e^{-2x}) \quad \text{and} \quad g''(x) = e^{-x}/2 + O(e^{-2x}), \tag{19}$$

we get

$$\frac{g'(x)}{g(x)} = \frac{1}{x} - \frac{e^{-x}}{2x} + O(e^{-2x}) \quad \text{and} \quad \frac{g''(x)}{g'(x)} = \frac{e^{-x}}{2} + O(e^{-2x}).$$

Returning to (17), it follows that

$$\eta(x) = -\frac{\theta+1}{x} + O(e^{-x}), \tag{20}$$

and taking $x \to \infty$, we obtain the von Mises' first order condition in (4). We deduce, as a result of the previous calculations that, when $x \to \infty$,

$$g(x) \to \infty,\ g'(x) \to 1,\ g''(x) \to 0,\ \tfrac{g'(x)}{g(x)} \to 0, \text{ and } \tfrac{g''(x)}{g'(x)} \to 0. \tag{21}$$

In order to prove von Mises' second order condition in (5), first note that

$$v'''(x) = \left\{ g'''(x) - (\theta+1)\left(\tfrac{2g''(x)g'(x)}{g(x)} - \tfrac{(g'(x))^3}{g^2(x)}\right)\right\} g^{-\theta-1}(x) - (\theta+1)v''(x)\ \tfrac{g'(x)}{g(x)}.$$

So, attending to (16), we get

$$\frac{v'''(x)}{v''(x)} = \frac{g'''(x)g(x) - (\theta+1)\left(2g''(x)g'(x) - (g'(x))^3/g(x)\right)}{g''(x)g(x) - (\theta+1)(g'(x))^2} - (\theta+1)\ \frac{g'(x)}{g(x)}.$$

Since we have the limits in (21), to obtain (5) we must guarantee, in the expression above, that $g''(x)g(x) \to 0$ and $g'''(x)g(x) \to 0$, as $x \to \infty$. By (18) and (19) we have

$$g''(x)g(x) = \frac{x\ e^{-x}}{2} + O(xe^{-2x}) \underset{x\to\infty}{\longrightarrow} 0.$$

On the other hand, $g'''(x) = -e^{-x}/2 + O(e^{-2x})$. This implies that

$$g'''(x)g(x) = -xe^{-x}/2 + O(xe^{-2x}) \underset{x\to\infty}{\longrightarrow} 0,$$

and consequently von Mises' second order condition in (5) is valid. Finally, to check (6), we will analyse the asymptotic behaviour of $\eta'(x)$ and $\eta''(x)$. By (20), we have

$$\eta'(x) = \frac{\theta+1}{x^2} + O(e^{-x}) \quad \text{and} \quad \eta''(x) = -\frac{2(\theta+1)}{x^3} + O(e^{-x}).$$

Consequently, von Mises type penultimate condition holds, since now

$$\frac{\eta''(x)}{\eta'(x)} = -\frac{2}{x} + O(e^{-x}) \underset{x\to\infty}{\longrightarrow} 0.$$

Theorem 5 enables us to conclude that the right tail of G^*_θ allows for a penultimate approximation. More precisely, taking

$$\gamma_n \equiv \gamma_n(\theta) := \eta(\ln n) = -\frac{\theta+1}{\ln n} + O(1/n),$$

as already given in (13), the limiting result in (12) holds, where $a_n := v'(\ln n) = a^*_n$ and $b_n := v(\ln n) = b^*_n$ are the sequences defined in (7).

When n goes to infinity, γ_n goes to $\gamma = 0$ and we obtain the limit distribution $G_0(x)$ with a known convergence rate of order $v''(\ln n)/v'(\ln n) = -(\theta+1)/(\ln n)$, as shown in Gomes and de Haan (1999). □

Proof. [Theorem 10] Given the sequences $\{a_n\}$ and $\{b_n\}$ for which (8) is valid and taking $\{\alpha_n\}$ and $\{\beta_n\}$ such that $\alpha_n = a_{l_n} a^*_{p_n}$ and $\beta_n = a_{l_n} b^*_{p_n} + b_{l_n}$, Theorem 6 guarantees that

$$F_n(\alpha_n x + \beta_n) = \Big\{G^*_\theta(a^*_{p_n}x + b^*_{p_n}) + e_{l_n}(a^*_{l_n}x + b^*_{l_n})\Big\}^{p_n} =: \Big\{G^*_\theta(a^*_{p_n}x + b^*_{p_n})\Big\}^{p_n} + \rho_n(x),$$

where

$$\rho_n(x) = p_n e_{l_n}(a^*_{l_n}x + b^*_{l_n}) \Big\{G^*_\theta(a^*_{p_n}x + b^*_{p_n})\Big\}^{p_n-1}(1+o(1)).$$

In order to get the main result in the theorem, observe that we can write

$$\begin{aligned}
&\frac{F_n(\alpha_n x + \beta_n) - G_{\gamma_n}(x)}{(\theta+1)/\ln^2 p_n} \\
&= \frac{F_n(\alpha_n x + \beta_n) - \big(G^*_\theta(a^*_{p_n}x + b^*_{p_n})\big)^{p_n}}{(\theta+1)/\ln^2 p_n} + \frac{\big(G^*_\theta(a^*_{p_n}x + b^*_{p_n})\big)^{p_n} - G_{\gamma_n}(x)}{(\theta+1)/\ln^2 p_n} \\
&= \frac{\rho_n(x)}{(\theta+1)/\ln^2 p_n} + \frac{\big(G^*_\theta(a^*_{p_n}x + b^*_{p_n})\big)^{p_n} - G_{\gamma_n}(x)}{(\theta+1)/\ln^2 p_n}.
\end{aligned}$$

The limiting result in (15) and the remaining of the theorem come thus from the results in Theorem 7, jointly with the condition in (14). □

References

[1] Anderson, C.W. (1971), *Contributions to the Asymptotic Theory of Extreme Values*, Ph.D. Thesis, University of London.

[2] Barlow A.A. and Proschan, F. (1975). *Statistical Theory of Reliability and Life Testing: Probability Models*, Holt, Rinehart and Winston, Inc., USA.

[3] Beirlant, J., Caeiro, F. and Gomes, M.I. (2012). An overview and open researh topics in statistics of univariate extremes, *Revstat* **10**:1, 1–31.

[4] Diebolt, J. and Guillou, A. (2005). Asymptotic behaviour of regular estimators, *Revstat* **3**:1, 19–44.

[5] Dietrich, D., de Haan, L. and Hüsler, J. (2002). Testing extreme value conditions, *Extremes*, 5, 71–85.

[6] Drees, H., de Haan, L., Li, D. and Pereira, T.T. (2006). Approximations to the tail empirical distribution function with applications to testing extreme value conditions, *Journal of Statistical Planning and Inference*, 136, 3498–3538.

[7] Fisher, R.A. and Tippett, L.H.C. (1928). Limiting forms of the frequency distributions of the largest or smallest member of a sample, *Proc. Camb. Phil. Soc.* **24**, 180–190.

[8] Galambos, J. (1978; 1987). *The Asymptotic Theory of Extreme Order Statistics*, 1st edition: Wiley, New York; 2nd edition: Krieger, Malabar, Florida.

[9] Gnedenko, B.V. (1943). Sur la distribution limite du terme maximum d'une série aléatoire, *Ann. Math.* **44**, 423–453.

[10] Gomes, M.I. (1984). Penultimate limiting forms in extreme value theory, *Ann. Inst. Statist. Math.* **36**, 71–85.

[11] Gomes M.I. (1994) Penultimate behaviour of the extremes. In: J. Galambos *et al.* (eds.) *Extreme Value Theory and Applications.* Kluwer Academic Publishers, 403–418.

[12] Gomes, M.I. and de Haan, L. (1999). Approximations by extreme value distributions, *Extremes* **2**:1, 71–85.

[13] Gomes, M.I. and Pestana, D.D. (1987). Nonstandard domains of attraction and rates of convergence. In L. Puri *et al.* (eds.), *New Perspectives in Theoretical and Applied Statistics*, Wiley, 467–477.

[14] Hüsler, J. and Li, D. (2006). On testing extreme value conditions, *Extremes* 9, 69–86.

[15] Jenkinson, A.F. (1955). The frequency distribution of the annual maximum (or minimum) values of meteorological elements, *Quart. J. Royal Meteorol. Society* **81**, 158–171.

[16] Kaufmann, E. (2000). Penultimate approximations in extreme value theory, *Extremes* **3**:1, 39–55.

[17] Laio, F. (2004). Cramer-von Mises and Anderson-Darling goodness of fit tests for extreme value distributions with unknown parameters, *Water Resources Research* 40, W09308, doi:10.1029/2004WR003204.

[18] Raoult, J.P. and Worms, R. (2003). Rate of convergence for the generalized Pareto approximation of the excesses, *Adv. Appl. Probab.* **35**:4, 1007–1027.

[19] Mises, R. von (1936). La distribution de la plus grande de n valeurs, *Revue Math. Union Interbalcanique* **1**, 141-160. Reprinted in *Selected Papers of Richard von Mises*, Amer. Math. Soc. **2** (1964), 271–294.

[20] Reis, P. and Canto e Castro, L. (2009). Limit model for the reliability of a regular and homogeneous series-parallel system, *Revstat* **7**:3, 227–243.

[21] Samaniego, F. (1985). On closure of the IFR class under formation of coherent systems, *IEEE Trans. Reliab.* **34**, 69–72.

CEAUL and Escola Superior de Tecnologia de Setúbal

E-mail address: paula.reis@estsetubal.ips.pt

CEAUL and DEIO, FCUL, Universidade de Lisboa

E-mail address: luisa.loura@fc.ul.pt

CM and UTAD–Universidade de Trás os Montes e Alto Douro

E-mail address: sdias@utad.pt

CEAUL and DEIO, FCUL, Universidade de Lisboa

E-mail address: ivette.gomes@fc.ul.pt