Music with Repetitive Structures, Complexity of Music Scores and Algorithms in Computational Music Composition, Using Philip Glass' *Façades* as Case Study

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> *The new musical style that Glass was evolving was eventually dubbed minimalism. Glass himself never liked the term and preferred to speak of himself as a composer of music with repetitive structures. Much of his early work was based on the extended reiteration of brief, elegant melodic fragments that wove in and out of an aural tapestry. Or, to put it another way, it immersed a listener in a sort of sonic weather that twists, turns, surrounds, develops.*
>
> in Biography, [5].

Abstract. Repetitive structures in the soprano saxophones, violas/synthesizer and violoncelli in the full score of Philip Glass' *Façades* [4] are investigated assimilating similar music bars with the axial-diagonal self-affine cartoons as defined by Mandelbrot [7], [8] to construct very general multi-fractals.
Transition matrices of order 1 Markov chains are used to simulate surrogates of phrases of the same piece, to investigate whether in this minimalist setting we obtain (i) an exact reproduction of the original, or (ii) something that though different sounds a pleasant variation.
Keywords: superposition of fractals, multifractals, complexity, self-affine cartoons and self-affine bars.

1 Introduction

Schröder [15], p. 109, boldly presents the key ideas of Birkhoff's *theory of aesthetic value*: an aesthetic creation is pleasing and interesting when it is neither too regular and predictable like a boring brown noise with a frequency dependence f^{-2}, nor a pack of too many surprises like an unpredictable white noise with a frequency dependence f^{-0}.
Multifractal measures — for an early overview of the field cf. Evertsz and

Mandelbrot [2] — is a candidate tool to analyze the complexity of musical scores, since a single similarity exponent characterizing a monofractal set is hardly appropriate to render the rich complexity of even minimalist compositions.

The idea of achieving aesthetic value blending harmoniously repetition with innovation and contrast — in Platzer's [14] definition of the classical rondo, for instance, a principal theme (sometimes called the "refrain") alternates with one or more contrasting themes, generally called "episodes," but also occasionally referred to as "digressions" or "couplets" — will be further discussed on an appendix, where Platzner's specialist description of the rondo is contrasted to the "amateur"[1], but eventually more eloquent, description in Sorti and Monaldi's *Imprimatur*. Proust's lyrical description of the *"petite phrase de Vinteul"*, and Poe's *The Philosophy of Composition* where he expresses the essential structuring role of the refrain (limited to the word "nevermore" in his masterpiece *The Raven*, that nevertheless has some dose of consonance with the name of the dead Leonore, and whose component *never* is phonetically the reversion of "raven") are two famous examples of the "repetitive structures" used in literature, and indeed examples of the structuring role of repetition/variation can be given in almost any art disicpline.

Philip Glass' *Façades*, with interpertrations ranging from strings, piano, flute/saxophone and oboe to piano and flute (or even an initial 42s section fingerpicked in guitar) is used as a case study on the appropriateness of multifractal tools and of Markov chain transition matrices in the description and/or analysis of musical complexity.

In a first attempt on using such tools, Pestana and Pestana [13] assimilated the musical notion of a generator bar (or measure) with the "cartoons" construction of multifractals used by Mandelbrot [8], namely chapter N1, or chapter E6 in [7], analyzing bars 19–39 on a Glass' page score publicly available in the www. In this sequel we use the full score of *Façades*, p. 49–55 of *Glassworks* [4], that unfortunately is not available in the *world wide web* and that we cannot reproduce, since we do not intend copyright infringement. Observe however that the full score of the *Opening* of *Glassworks* can be found in `http://www.glasspages.org/score-opening.pdf`, and the interested reader can experiment our analysis, or develop alternative analises, using this online bonus.

[1] Dr. Francesco Sorti has a background in musicology, so here please attach to the word "amateur" the original meaning "the one who loves".

2 Façades

Although *Façades* first appeared on Philip Glass' album *Glassworks* [4][2], it was conceived as part of the soundtrack to Godfrey Reggio's *Koyaanisqatsi*, see `http://www.youtube.com/watch?v=vz_R2y1oAzw&feature=related` or `http://www.youtube.com/watch?v=GQsoMIGuPD8` for the stream introducing a similar musical theme. Originally scored for an orchestral string section and two saxophones, it is often performed using two flutes instead of saxophones, or scored for 2 soprano sax, viola, cello and a synthesizer which doubles the viola and cello[3]. In fact, as Patrick Gary observed in *MusicWeb International*,

> "*Philip Glass is a composer whose body of work readily lends itself to re-orchestration. In fact, many of his early works were written with intentionally vague orchestrations to allow for greater ease in performance.*"

On the appraisal published in *Gramophone Magazine*[4], the reviewer wrote

> "*the Glass works gathered together on Glassworks make an excellent introduction to the sharp, hard sonorities, densely packed, slowly changing patterns and seemingly unstoppable linear flow of this important aspect of contemporary music.*"

We cannot adhere to the expression "*linear flow*", that certainly is written to convey the more general idea of smoothness. In fact, Glass himself noted

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[2] The musical score is also available from www.ChesterNovello.com, that stores interesting information on Glass achievements and works at `http://www.chesternovello.com/default.aspx?TabId=2431&State_2905=2&composerId_2905=540`.

[3] In the full score [4] we used, stating copyright dated 1982, there is the precise indication "synthesizer DX7", although the Yamaha DX7 has been marketed since the fall of 1983.

[4] We also quote some comments by Philip Glass himself:

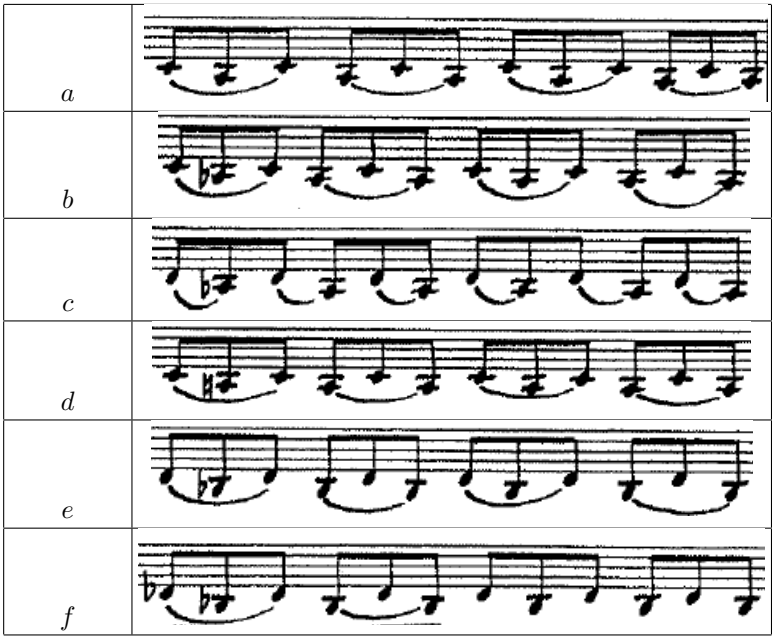
> "*Although I quite liked the way it turned out, it was not used for the film and ended up on my 1982 album for CBS, Glassworks. It also has become a staple of the live performances of the Philip Glass Ensemble and was included in Glasspieces, the production put on at the New York City Ballet in the spring of 1990, choreograp hed by Jerome Robbins.*"
> ...
> "*GLASSWORKS was intended to introduce my music to a more general audience than had been familiar with it up to then.*"
> ...
> "*I'm very pleased with it, the way it's received in performance. The pieces seem to have an emotional quality that everyone responds to, and they work very well as performance pieces.*"

23 “musical phrases”, with a variable number of bars each (something that we can assimilate with the concepts of first and second articulation from linguistics), as described in Table 1. Observe that the instruction *Da Capo* is marked after the two initial bars of phrase 5, and hence only the bars 5.1-5.2 of music phrase 5 are repeated with only one soprano saxophone playing in 5b. In fact, this unusual location of the *Da Capo* mark is done so that the first half of music phrase 5 followed by the repetition of phrase 1 reconstructs a 4 bar structural element (described in more detail below), and only in the repetition phrase 2b does the soprano saxophone begin to play.

Table 1. Musical phrases in the full score of *Façades*

phrase	# bars	instrumentation
1	2	viola, DX7 / violoncelli
2	2	viola, DX7 / violoncelli
3	4×2	viola, DX7 / violoncelli
4	4×2	viola, DX7 / violoncelli
5+1b	4	viola, DX7 / violoncelli
2b	2	soprano saxophone / viola, DX7 / violoncelli
3b	4×2	soprano saxophone / viola, DX7 / violoncelli
4b	4×2	soprano saxophone / viola, DX7 / violoncelli
5b	4	soprano saxophone / viola, DX7 / violoncelli
6	2	soprano saxophone / viola, DX7 / violoncelli
7	4×2	soprano saxophone / viola, DX7 / violoncelli
8	4×2	soprano saxophone / viola, DX7 / violoncelli
9	4	soprano saxophone / viola, DX7 / violoncelli
10	2	soprano saxophone 1 / soprano saxophone 2 / viola, DX7 / violoncelli
11	4×2	soprano saxophone 1 / soprano saxophone 2 / viola, DX7 / violoncelli
12	4×2	soprano saxophone 1 / soprano saxophone 2 / viola, DX7 / violoncelli
13	4	soprano saxophone 1 / soprano saxophone 2 / viola, DX7 / violoncelli
14	2	soprano saxophone 1 / soprano saxophone 2 / viola, DX7 / violoncelli
15	4×2	soprano saxophone 1 / soprano saxophone 2 / viola, DX7 / violoncelli
16	4×2	soprano saxophone 1 / soprano saxophone 2 / viola, DX7 / violoncelli
17	4	soprano saxophone 1 / soprano saxophone 2 / viola, DX7 / violoncelli
18	2	soprano saxophone / viola, DX7 / violoncelli
19	4×2	soprano saxophone / viola, DX7 / violoncelli
20	4×2	soprano saxophone / viola, DX7 / violoncelli
21	4	soprano saxophone / viola, DX7 / violoncelli
22	4×2	soprano saxophone / viola, DX7 / violoncelli
23	2×4	soprano saxophone / viola, DX7 / violoncelli

We shall also focus on the viola plus synthesizer part, since it illustrates quite well some similarities to the self-affine cartoons Mandelbrot's description of multifractals. To do so, we use sequential letters of the alphabet to tag its bars:



From herein, we shall call an isolated bar (ex.: a, b,...) a *gramma*, a sequential pair of bars (ex.: aa, bc, dd,...) a *digram*, a sequential set of 4 bars (ex.: $bcdd$, $efaa$, ...) a *tetragram*. We shall consider only digrams and tetragrams inside each of the 23 music phrases, i.e. we shall not count the sequence of digrams (aa) in phrases 1 and 2, or at the end of phrase 17 and in phrase 18, as a tetragram $(aaaa)$. Moreover (contrarily to the use in Linguistics or Encryption Theory, to identify digrams we advance in pairs inside each phrase (something like the "genetic fork", but advancing in pairs instead of triads), i.e., for instance, in phrase 3 we consider digrams (bc),(dd), but we do not consider cd or db digrams.

Observe that with the decision — justified by the location of the *Da Capo* instruction — of merging the first half of phrase 5 with the repetition 1b, we get the structural tetragram $(efaa)$, and preceded by the double tetragram $(efaa)(efaa)$ and followed by the digram (aa), the most frequently repeated macro-structure in this score.

The distribution of the 150 grammas and of the 75 digrams inside the music phrases is recorded in Table 3.

With the conventions described above, the structure of the viola plus synthesizer score is indeed "minimalist": 9 of the music phrases are digrams — 7 (aa), 6 are tetragrams $(efaa)$, and 14 are repeated tetragrams — 6 $(bcdd)$, 6 $(efaa)$, 1 $(aabc)$, 1 $(aaaa)$. Or more precisely:

Table 2. Viola plus synthesizer part bar structure of *Façades*

phrase	bars
1	aa
2	aa
3	$bcddbcdd$
4	$efaaefaa$
5+1b	$efaa$
2b	aa
3b	$bcddbcdd$
4b	$efaaefaa$
5b	$efaa$
6	aa
7	$bcddbcdd$
8	$efaaefaa$
9	$efaa$
10	aa
11	$bcddbcdd$
12	$efaaefaa$
13	$efaa$
14	aa
15	$bcddbcdd$
16	$efaaefaa$
17	$efaa$
18	aa
19	$bcddbcdd$
20	$efaaefaa$
21	$efaa$
22	$aabcaabc$
23	$aaaaaaaa$

Table 3. Distribution of grammas and of digrams inside music phrases (viola plus synthesizer part) in *Façades*

a	62
b	14
c	14
d	24
e	18
f	18
(aa)	31
(bc)	14
(dd)	12
(ef)	18

- opening: (aa) (phrases 1);
- main: 6 repetitions of the macro-structure (aa)–$(bcdd)(bcdd)$–$(efaa)(efaa)$–$(efaa)$;
- closing: $(aabc)(aabc)$–$(aaaa)(aaaa)$.

Hence, we can also claim that the finale $(aaaa)$ mirrors the beggining (aa)–(aa), and that in fact only phrases 22 and 23 depart slightly from the general structure.

3 Self-Affine Cartoons, Self-Affine Bars, and an Alternative Approach Using Markov Chains

One of the pathways described by Mandelbrot [7], [8] is via diagonal or diagonal-and-axial self-affine cartoons, cf. for instance figures N1-6 and N1-7, pp. 33–34 in [7], repeated in Figures E6-4 and E6-5, pp. 179–180 of [8]. This inspired us to assimilate the idea of self-affine cartoons and of self-affine bars.

In fact, considering the bar a the generator, in Mandelbrot's sense, the bars b, c, d, e and f are simply obtained from a via very moderate modifications:

- If the Low Note of a goes down a semi-tone ($L \downarrow 1$), we obtain b.
- If the Low Note of a goes down one semi-tone ($L \downarrow 1$) and the High Note goes up one tone ($H \uparrow 2$), we obtain c.
- If the Low Note of a goes up one semi-tone ($L \uparrow 1$), we obtain d.
- If the Low Note of a goes up one semi-tone ($L \uparrow 1$) and the High Note goes up one tone ($H \uparrow 2$), we obtain e.
- If he Low Note of a goes up one semi-tone ($L \uparrow 1$) and the High Note goes up one semitone ($H \uparrow 1$), we obtain f.

Hence, denoting L the Low Note and H the High Note of each bar, the transition $a - -bcdd$ is characterized, in number of semitones, by

$$L \downarrow 1;\ H \uparrow 2;\ L \uparrow 2 \text{ and } H \downarrow 2;\ -,$$

while the "inner" the transition $d - -b$ is characterized by

$$L \downarrow 2,$$

the transition $d - -efaa$ is characterized by

$$H \uparrow 2;\ H \downarrow 1;\ L \downarrow 1 \text{ and } H \downarrow 1;\ -,$$

and the "inner" transition $a - -e$ is obtained by

$$L \uparrow 1 \text{ and } H \uparrow 2.$$

Hence, the only extra complexity, when compared to Mandelbrot's cartoon construction, is that the generator must have a two-fold variation, one in what concerns L and the other in what concerns H. A simple way of dealing with this is to use different colors to represent the Low Note and High Note variation (or dashed lines for the variation of L, and solid lines for the variation of H), with slopes showing the variation of 1 or 2 semitones. Emulating Mandelbrot's cartoons generator style, Fig. 1 shows the variation of Low and High Notes for the $a - bcdd - b...$ sequence, and the $d - efaa - e...$

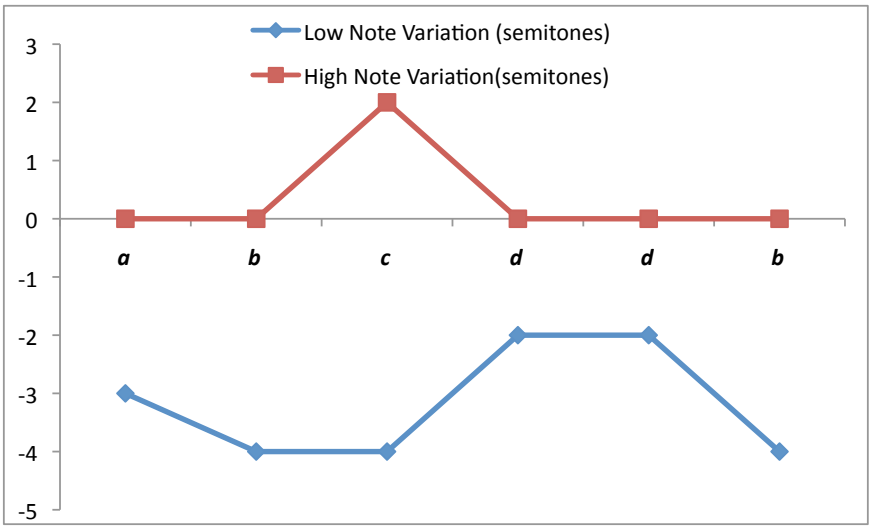


Fig. 1. Diagonal and axial chart for the Low and High Note variation of the $a - bcdd - b...$ sequence.

sequence Low and High Notes variation is shown in Fig. 2.

An illustration of the "bivariate" repeated macro structure $aa - bcdd - bcdd - efaa - efaa - efaa$ is shown, in an equivalent but eventually more expressive form, in Fig. 3.

Observe however that in any practical human made artifact, or in other practical applications, an important difference does exist: while in the construction of multifractals infinite iteration is conceived, in practical applications a rather limited number of iterations is mandatory, and hence some stopping rule has to be defined, see Pestana and Aleixo [11] and Aleixo *et al.* [1] on stuttering Cantor sets. Mandelbrot's illustration of the digonal or axial-and -diagonal cartoons construction of multifractals, itself, is only taken up till a very low order iteration.

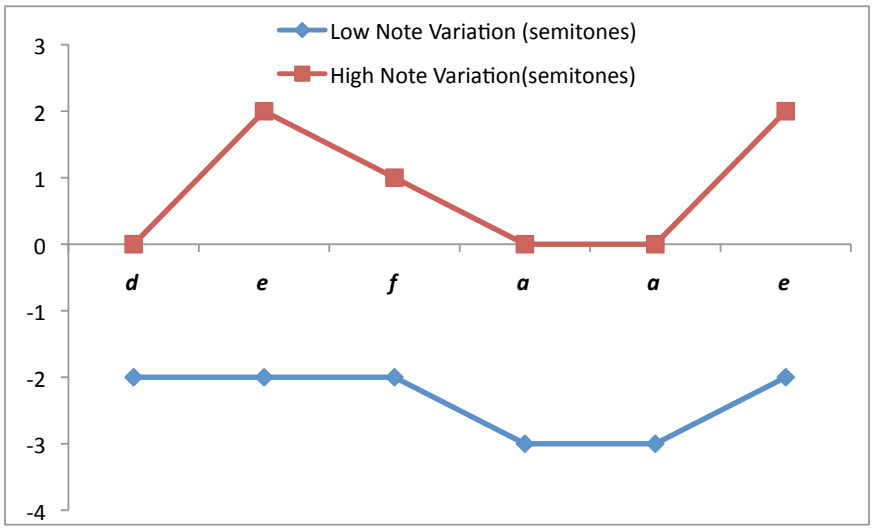


Fig. 2. Diagonal and axial chart for the Low and High Note variation of the $d-efaa-e...$ sequence.

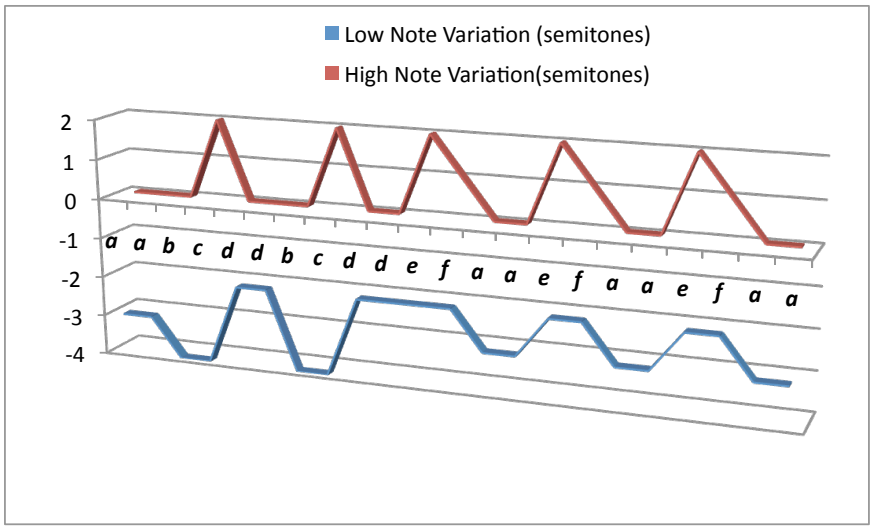


Fig. 3. Diagonal and axial chart for the Low and High Note variation of the repeated macro structure $aa-bcdd-bcdd-efaa-efaa-efaa$ from the viola plus synthesizer part of *Façades*.

Other sensible variations are under investigation, as well as a comparison with more linear and iterative procedures to generate musical scores, as for instance the Lindenmayer systems described in Pestana [12].

Markov chain transition matrices are an alternative way of investigating this type of repetitive structures, important pathbreaking work in the field being [10], [6]. In what regards *Façades*, we observe that the repeated macro structure is $aa-bcdd-bcdd-efaa-efaa-efaa$, and hence that the first order transition matrix in this block is

	a	b	c	d	e	f
a	4/7	1/7	0	0	3/7	0
b	0	0	1	0	0	0
c	0	0	0	1	0	0
d	0	1/2	0	0	1/2	0
e	0	0	0	0	0	1
f	1	0	0	0	0	0

We used this to compose random blocks of 22 bars to insert between the opening (aa) and the finale $(aabc)(aabc)$–$(aaaa)(aaaa)$, so generating

- 1000 variations where a single macro structureof 22 bars was produced using the above transition matrix, and repeated 6 times between the opening and the finale;
- 1000 variations where 6 independently generated blocks of 22 bars produced using the transition matrix were sequentially inserted between the opening and the finale — this, of course, largely increased the degrees of freedom, augmenting the probability of obtaining much less structured pieces.

4 Conclusions

A random selection of the random music computationally produced using *Façades* macro repeated structure as a basis via de use of the associated Markov chain transition matrix has been evaluated as follows:

- subjective and partial evaluation: we asked colleagues and students to listen to *Façades*, and then to mark (marks ranging from 0 to 10) 10 randomly generated pieces (5 produced by each of the methods described using the Markov chain transition matrix) in what concerns
  - pleasantness of the result;
  - closeness to *Façades*.

Table 4. Pleasantness and closeness of the result to *Façades*

	Markov chains, 1 structure used six-fold	Markov chain, 6 independent structures
pleasantness	9.2 ± 0.43	5.6 ± 0.76
closeness to *Façades*	7.4 ± 0.68	4.4 ± 1.16

Table 4 show the results.
The free software *MuseScore*, [16], greatly simplified the evaluation tasks in what regards this partial subjective evaluation.
This result is presented solely as a curiosity, since the validity of results using convenience samples instead of random samples is arguable. On the other hand, as this subjective appraisal has been just a matter of idle curiosity, the number of subjects has been very low, and hence the number of samples graded has been very small. There is however a strong indication that the use of 6 independently generated structures to insert between opening and finale — that almost certainly lowers "repeated structures" — has a strong effect on the subjective evaluation of their pleasantness and closeness to the original they are trying to recreate.
We also observe that a large proportion of the random scores produced using the transition matrix so roughly defined is rather pleasing. Romanticism brought in a respect for the artist and the idea of inspiration that tends to convince us that any modification of an inspired masterpiece cannot but spoil it. On his authoritative *Le Mythe de Rimbaud*, Etiemble [3] reports that in many occasions he recited *Le Bateau Ivre* purposely interchanging lines and blocks of the poem — with no complains from none of the many specialists that fiercely claim that not a single word can be changed in this immortal "chef-d'oeuvre"...

- Objective evaluation: Matching degree of randomly produced scores to the original score $aa-bcdd-bcdd-efaa-efaa-efaa$ has been evaluated. The use of tags $a, \ldots, f$ to represented bars greatly simplified our work, since the comparison of strings of text is a simple task.
  In what concerns our case study — the viola plus synthesizer part of *Façades* — the Markov chain transition matrix is a very stringent structuring element, since randomness only plays a role on the transition from either bar a or bar d.
  Hence, it is not surprising that in the crude simulation we made of $1000 + 1000 \times 6$ pieces, 4102 exactly matched the original score, a slightly higher proportion (58.6%) than we expected, but within tolerance bounds for random fluctuation in this moderate scale simulation experiment.

As in the analysis of 21 bars of the Trilogy Sonata for Piano [13], imposing less stringent conditions — for instance, using the Markov chain transition matrix to produce phrases with a random number of bars, using as stoping rule the first occurrence of the tetragram $aaaa$, that in the central development of Glass' score is the transition from $aa-bcdd-bcdd-efaa-efaa-efaa$ to another $aa-bcdd-bcdd-efaa-efaa-efaa$ — and evaluating the prevalence of such random phrases containing subsets matching the model in at least 4 of the tetagrams (for instance $aa-bcdd-bcdd-bcdd-efaa-efaa$, or $aa-bcdd-bcdd-efaa-bcdd-efaa$), the prevalence of approximate matches rises up to 69.82%.

More detailed studies are indeed needed to reach more reliable conclusions. So far, our opinion is that computationally generated music can match human music when this is rigidly structured as in the case study we used, but that the human intervention is arguably indispensable in the very crucial choices of the basic structure, and also on the repetition stoping rule, and appropriate finale.

Observe also that we didn't address matters such as instrumentation, playing instructions (the bar tagged c is different from all the others in other ways than Low and High Notes), repeated structures composition rules of the different parts, just to enumerate a few matters. Computational composition has arguably attained a stage where given a set of complex instructions and a starting structure, with a reasonable probability some simple pleasant samples can result. But the balance complexity of instructions / simplicity of the product in our opinion shows that the human factor in music composition is still a key issue for quality. In this match, even taking for granted the provisory (and eventually optimistic) computer $\frac{4102}{7000} = 0.586$ / Glass 1, Hurrah for Glass!

5 Appendix A: The IBM Glass Engine

The IBM Glass Engine enables deep navigation of the music of Philip Glass. Personal interests, associations, and impulses guide the listener through an expanding selection of over sixty Glass works.

The glass engine was developed at the IBM T.J. Watson Research Center in 2001. It can be downloaded from Phip Glass page `http://www.philipglass.com/music/compositions/facades.php`.

The answers to two of the Frequently Asked Questions deserve to be recorded:

Q: Who decided how to assign the subjective values (such as JOY) to the tracks? Was this done by a computer?

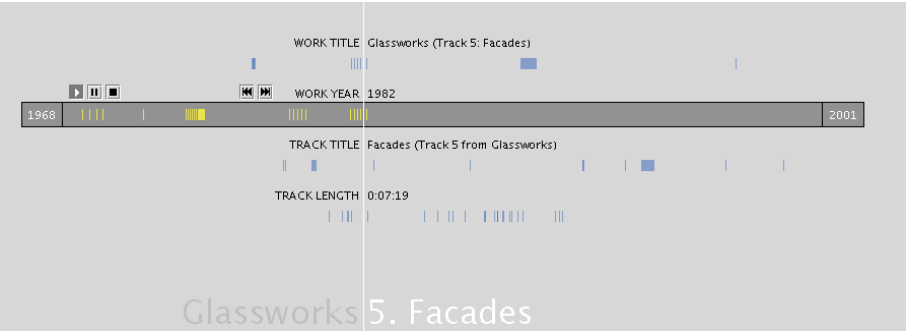


Fig. 4. glassengine — locating the track *Façades* — slide to work year 1982.

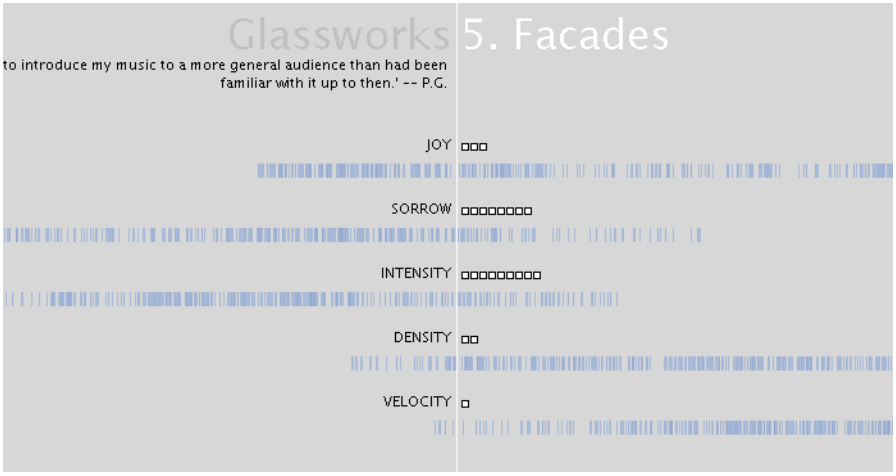


Fig. 5. glassengine — assignement of Joy, Sorrow, Intensity, Density and Velocity to the track *Façades*. The figure has been cut to enhance the low assignemnt of Joy, Density and Velocity, and high assignement of Sorrow and Density.

A: These values were assigned by Philip Glass's longtime producer and sound designer, Kurt Munkacsi, while eating several pounds of chocolate chip cookies.

Q: How is it possible for a track to have high amounts of both joy and sorrow?

A: Music can contain two conflicting emotions. Really.

6 Appendix B: Koyaanisqatsi

Koyaanisqatsi, Life Out Balance, 1982, directed by Godfrey Reggio, music by Philip Glass, "is the first film of the QATSI trilogy. The title is a Hopi Indian word meaning 'life out of balance'. Created between 1975 and 1982, the film is

an apocalyptic vision of the collision of two different worlds — urban life and technology versus the environment. [...] *Koyaanisqatsi* attempts to reveal the beauty of the beast!".

Aside from the MGM release presented by Francis Ford Coppola, (Credits: Music: Philip Glass. Philip Glass Music: Produced & Recorded by Kurt Munkacsi. Conducted by Michael Riesman), you may be interested in *KOYAANISQATSI — Godfrey Reggio — making of*, retrieved in `http://www.youtube.com/watch?v=_Mr26_m5rGQ`.

`http://www.youtube.com/watch?v=GQsoMIGuPD8` is Part 1/9 uploaded by schipflingerfred in youtube (some other parts have been blocked on copyright infringement rights). Other *url* addresses where parts of the movie can be watched:

`http://www.youtube.com/watch?v=Me7QaFMcQ9A&feature=relmfu`,
`http://www.youtube.com/watch?v=DlFg1MgATu4&feature=related`,
`http://www.youtube.com/watch?v=-iNJ8u4ewD8&feature=relmfu`,
`http://www.youtube.com/watch?v=M27874iHwpg&feature=relmfu`.

For more information, cf. also the section `http://www.philipglass.com/music/films/koyaanisqatsi.php` in Philip Glass' webpage.

7 Appendix C: Repetition and Innovation in the Philosophy of Music Composition

In the Introduction we mention Birkhoff's theory of aesthetic value, and present a formal description of the rondo — a musical form blending repetition and innovation — by Platzer [14].

In a deservly successful book, *Imprimatur*, Monaldi and Sorti (who has musicology background) describe the emotions and aesthetic rapture caused by the audition of a rondo, and they eloquently convey how the equilibrium of innovation and repetition enhance each other in the build up of beauty. Aside from being a very appropriate framework to look at minimalism, or, as Glass prefers, at repetitive structures, it may serve as an invitation to visit Monaldi and Sorti world (their second novel, Secretum, in the edition we bought, contains a CD with recording of period music described in the novel). The following excerpt (translated to English from the Portuguese edition by P. Pestana) comes from the beggining of the chapter *Second Day, 12th September 1683*, pp. 51-53 of the Portuguese edition.

[...] Suddenly, from his fingers sprang, more than a song, a marvelous architecture of sounds [. . .]. Initially it was a simple and innocent motif, that, like a dance, would skip from root to dominant [. . .], and back again, followed by a surprising leap avoiding a cadenza, before it started over again. But this was just the first of a rich and astonishing collection of gems, [. . .] called a

rondo and consisted of a first stanza that is repeated several times, and embellished every so often by a new and precious bliss, completely unexpected and shimmering in light.

As any other rondo, that one [...] was crowned by the extreme and conclusive repetition of the first stanza, providing meaning, plenitude and rest to the whole set. But the innocence and simplicity of that first stanza, though delightful, would amount to nothing if deprived of the sublime company of the others, which, chorus after chorus. scaled the admirable construction, freer and freer, unpredictable, refined and audacious. The final one dared wit and ear with a sweet duel [...]. The last arpeggio, after wandering with sobriety and near timidity over the low register, would perform a sudden ascent to the high notes, transforming their tortuous and shy pattern into a crystal clear torrent of beauty, flowing its harmonious abundance into an admirable downwards spiraling progression. It would then stop, reveling in ineffable and mysterious harmonies, [...], finally slowing down to give way to the extreme repetition of the initial cadenza.

[...] the rondo was so pleasing because, while the chorus was written according to the sound and old rules of consonance, the alternating stanzas always had new harmonic risk, which concluded unexpectedly, almost deviating from good musical doctrine. And after reaching its apex, the rondo would move briskly into the finale [...]

R. Monaldi and F. Sorti, *Imprimatur*

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