Location invariant reduced-bias tail index estimation under a third-order framework*

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Abstract

Under a convenient third-order framework, the asymptotic distributional behavior of a class of location invariant reduced-bias tail index estimators is derived. Such a class is based on the PORT methodology, with PORT standing for peaks over random thresholds, and combines a PORT-version of one of the pioneering classes of minimum-variance reduced bias tail index estimators with two classes of location invariant estimators of adequate second-order parameters, recently introduced in the literature. An application to the log-exchange rates of the Euro against USA Dollar and Euro against GB Pound is also provided.

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1 Introduction and scope of the article

Let $\underline{X}_n = (X_1, \ldots, X_n)$ denote a random sample of n independent, identically distributed (i.i.d.) random variables (r.v.'s) with distribution function (d.f.) F. Let us denote the associated ascending order statistics (o.s.'s) by $X_{1:n} \leq X_{2:n} \leq \ldots \leq X_{n:n}$ and let us assume that there exist sequences of real constants $\{a_n > 0\}$ and $\{b_n \in \mathbb{R}\}$ such that the linearly normalized maximum, i.e. $(X_{n:n} - b_n)/a_n$, converges in distribution to a non-degenerate r.v. Then the limit distribution is necessarily of the type of the *extreme value* (EV) d.f.,

$$G_\xi(x) := \begin{cases} \exp\left(-(1+\xi\ x)^{-1/\xi}\right), & 1+\xi\ x > 0, \quad \text{if} \quad \xi \neq 0, \\ \exp(-\exp(-x)), & x \in \mathbb{R}, \quad\quad\quad\quad \text{if} \quad \xi = 0. \end{cases} \tag{1.1}$$

The d.f. F is said to belong to the max-domain of attraction of G_ξ, and we write $F \in \mathcal{D}_{\mathcal{M}}(G_\xi)$. The parameter ξ, in (1.1), is the *extreme-value index* (EVI), one of the primary parameters of extreme events. We are interested in heavy-tailed models, i.e. in d.f.'s with a regularly varying right-tail. This means that $\xi > 0$, and the right tail-function

$$\overline{F}(x) := 1 - F(x)$$

is such that

$$\lim_{t\to\infty} \overline{F}(tx)/\overline{F}(t) \ = \ x^{-1/\xi}, \quad \text{for all } x > 0. \tag{1.2}$$

We then say that $\overline{F}$ is of regular variation at infinity with an index equal to $-1/\xi$. This type of heavy-tailed models appears often in practice, in fields like biometry, ecology, economics, finance and telecommunication traffic, among others.

Let $F^{\leftarrow}$ denote the generalized inverse function of F, defined by

$$F^{\leftarrow}(t) := \inf\left\{x : F(x) \geq t\right\}, \tag{1.3}$$

and let U be the associated (reciprocal) right tail quantile function, defined as

$$U(t) := F^{\leftarrow}(1 - 1/t), \qquad t \geq 1. \tag{1.4}$$

1.1 First, second, and third-order conditions for heavy-tailed models

In a heavy-tailed framework, i.e. if (1.2) holds, with the notation RV_a for the class of regularly varying functions at infinity with an index $a \in \mathbb{R}$, and on the basis of the results in Gnedenko

(1943), for the right-tail function $\overline{F} = 1 - F$, and in de Haan (1984), for U in (1.4), we assume the validity of any of the first-order conditions below:

$$F \in \mathcal{D}_{\mathcal{M}}(G_{\xi>0}) \quad \Longleftrightarrow \quad \overline{F} \in RV_{-1/\xi} \quad \Longleftrightarrow \quad U \in RV_{\xi}. \tag{1.5}$$

For several technical proofs in the field of extreme value theory we further need information about the rate of convergence in (1.5), assuming that for every $x > 0$,

$$\lim_{t\to\infty} \frac{\ln U(tx) - \ln U(t) - \xi \ln x}{A(t)} = \psi_\rho(x) := \begin{cases} \frac{x^\rho - 1}{\rho}, & \text{if} \quad \rho < 0, \\ \ln x, & \text{if} \quad \rho = 0, \end{cases} \tag{1.6}$$

where $|A|$ must then be in RV_ρ (Geluk and de Haan, 1987). Often, we further need information on the rate of convergence in (1.6), and assume that for all $x > 0$,

$$\lim_{t\to\infty} \frac{\frac{\ln U(tx) - \ln U(t) - \xi \ln x}{A(t)} - \psi_\rho(x)}{B(t)} = \begin{cases} \frac{x^{\rho+\rho'} - 1}{\rho+\rho'}, & \text{if} \quad \min(\rho, \rho') < 0, \\ \ln x, & \text{if} \quad \rho = \rho' = 0, \end{cases} \tag{1.7}$$

where $|B|$ must then be in $RV_{\rho'}$. Details on this precise third-order condition can be found in Gomes *et al.* (2002), Fraga Alves *et al.* (2003, 2006), Caeiro *et al.* (2009), and more generally in Wang and Cheng (2005).

For technical simplicity, we assume that $\rho < 0$ and that we can choose

$$U(t) = Ct^\xi(1 + o_p(1)), \qquad A(t) = \xi\beta t^\rho, \qquad B(t) = \beta' t^{\rho'}, \tag{1.8}$$

in (1.4), (1.6) and (1.7), respectively, with C a positive real number, β and β' non-null real numbers or even slowly varying functions, i.e. regularly varying functions with an index of regular variation equal to zero.

The pair of second-order parameters (β, ρ), in (1.8), rules the rate of convergence in (1.6) and is dependent on a possible shift in the data. More precisely, if we have a location or shift parameter $s \in \mathbb{R}$, not necessarily null, i.e. if $F(x) = F_s(x) = F_0(x - s)$, then $U(t) \equiv U_s(t) = U_0(t) + s$ and also $(\beta, \rho) =: (\beta_s^*, \rho_s^*)$ depends obviously on s, with

$$(\beta_s^*, \rho_s^*) := \begin{cases} (-s/C, -\xi), & \text{if} \quad \xi + \rho_0^* < 0 \wedge s \neq 0, \\ (\beta_0^* - s/C, \rho_0^*), & \text{if} \quad \xi + \rho_0^* = 0 \ \wedge \ s \neq 0, \\ (\beta_0^*, \rho_0^*), & \text{otherwise}, \end{cases} \tag{1.9}$$

where $\beta_0^* \equiv \beta_0$ and $\rho_0^* \equiv \rho_0$ are respectively the scale and shape second-order parameters associated with an unshifted model ($s = 0$). For further details on the influence of a shift $s \neq 0$ in the second-order parameters, see Henriques-Rodrigues *et al.* (2014, 2015).

1.2 MVRB-EVI estimation

We first refer the classical EVI-estimators, the Hill estimators, derived and studied in Hill (1975), defined for $k = 1, 2, \ldots, n-1$, and given by

$$H(k) \equiv H(k; \underline{\mathbf{X}}_n) := \frac{1}{k} \sum_{i=1}^{k} \{\ln X_{n-i+1:n} - \ln X_{n-k:n}\} \equiv M_n^{(1)}(k), \tag{1.10}$$

with $M_n^{(\alpha)}(k)$ the moment statistics used in Dekkers *et al.* (1989), Gomes *et al.* (2002), Fraga Alves *et al.* (2003) and Caeiro and Gomes (2006), among others, with the functional form

$$M_n^{(\alpha)}(k) \equiv M_n^{(\alpha)}(k; \underline{X}_n) := \tfrac{1}{k} \sum_{i=1}^{k} \left(\ln X_{n-i+1:n} - \ln X_{n-k:n}\right)^{\alpha}. \tag{1.11}$$

The Hill estimators are consistent for the estimation of $\xi \geq 0$ provided that k is *intermediate*, i.e. provided that

$$k \equiv k_n \to \infty \quad \text{and} \quad k/n \to 0, \text{ as } n \to \infty. \tag{1.12}$$

Under the second-order framework, in (1.6), the asymptotic distributional representation

$$H(k) \stackrel{d}{=} \xi + \frac{\xi\ Z_k}{\sqrt{k}} + \frac{A(n/k)}{1-\rho}(1 + o_p(1))$$

holds (de Haan and Peng, 1998), where with $\{Ei\}$ a sequence of i.i.d. standard exponential r.v.'s

$$Z_k = \sqrt{k}\Big(\sum_{i=1}^{k} E_i/k - 1\Big) \tag{1.13}$$

is an asymptotically standard normal r.v. If we further assume to be under the third-order framework in (1.7), we can write,

$$H(k) \stackrel{d}{=} \xi + \frac{\xi\ Z_k}{\sqrt{k}} + \Big(\frac{A(n/k)}{1-\rho} + \frac{A(n/k)B(n/k)}{1-\rho-\rho'}\Big)(1 + o_p(1)). \tag{1.14}$$

Due to the high bias of the Hill estimators, in (1.10), for moderate up to large k, several authors have been dealing with bias reduction in the field of extremes. We refer the pioneering articles by Beirlant *et al.* (1999), Feuerveger and Hall (1999) and Gomes *et al.* (2000), as well as the more recent second-order *minimum-variance reduced-bias* (MVRB) EVI-estimators in Caeiro *et al.* (2005), Gomes and Pestana (2007a) and Gomes *et al.* (2007a, 2008c). The simplest class of second-order MVRB EVI-estimators is the one introduced in Caeiro *et al.* (2005), further studied, under a third-order framework, in Caeiro *et al.* (2009). This class, denoted $\overline{H} \equiv \overline{H}(k)$, depends upon the estimation of the second-order parameters (β, ρ), $\beta \neq 0$, $\rho < 0$, in (1.8). The functional form of those EVI-estimators is

$$\overline{H}(k) \equiv \overline{H}(k; \hat{\beta}, \hat{\rho}) \equiv \overline{H}(k; \underline{\mathbf{X}}_n) := H(k)\big(1 - \hat{\beta}(n/k)^{\hat{\rho}}/(1-\hat{\rho})\big), \tag{1.15}$$

with $H(k)$ the Hill estimator in (1.10), and where $(\hat{\beta}, \hat{\rho})$ needs to be an adequate consistent estimator of (β, ρ). The adequate estimation of the second-order parameters β and ρ is of primordial importance in the adaptive choice of the best number of top o.s.'s to be considered in the EVI-estimation, as well as in the construction of second-order MVRB EVI-estimators. Overviews of the subject can be found in Gomes *et al.* (2007b), Chapter 6 of the book by Reiss and Thomas (2007), Gomes *et al.* (2008a), Beirlant *et al.* (2012) and Gomes and Guillou (2015). However, despite being scale invariant, these classes of EVI-estimators are not location-invariant, as often desired, due to the fact that the EVI itself enjoys such a property, i.e. it is both location and scale invariant (see Gomes *et al.*, 2016b, for further details).

1.3 The PORT methodology

The class of estimators suggested here is a function of the sample of excesses over a random threshold $X_{\lfloor nq \rfloor+1:n}$, with $n^{(q)} := n - \lfloor nq \rfloor - 1$ the size of the sample of excesses, and where $\lfloor x \rfloor$ stands for the integer part of x. Such a sample is denoted by

$$\underline{X}_n^{(q)} := \big(X_{n:n} - X_{\lfloor nq \rfloor+1:n}, X_{n-1:n} - X_{\lfloor nq \rfloor+1:n}, \ldots, X_{\lfloor nq \rfloor+2:n} - X_{\lfloor nq \rfloor+1:n}\big)\,, \tag{1.16}$$

where, we can have

- $0 < q < 1$, for any $F_0 \in D_{\mathcal{M}}(G_{\xi>0})$ (the random threshold is an empirical quantile);
- $q = 0$, for d.f.'s with a finite left endpoint $x_{_F} := \inf\{x : F_0(x) > 0\}$ (the random threshold is the minimum, $X_{1:n}$).

Any statistical inference methodology based on the sample of excesses $\underline{X}_n^{(q)}$, $0 \le q < 1$, defined in (1.16), is called a PORT-methodology. This methodology enabled the introduction and study of classical location/scale invariant EVI-estimators, like the PORT-Hill and the PORT-moment estimators in Araújo Santos *et al.* (2006). The PORT-Hill estimators have the functional form,

$$H^{(q)}(k) \equiv H(k; \underline{X}_n^{(q)}) := \frac{1}{k} \sum_{i=1}^{k} \ln \frac{X_{n-i+1:n} - X_{\lfloor nq \rfloor+1:n}}{X_{n-k:n} - X_{\lfloor nq \rfloor+1:n}} \equiv M_n^{(1,q)}(k), \tag{1.17}$$

with $M_n^{(\alpha,q)}(k)$ the location-invariant statistics,

$$M_n^{(\alpha,q)}(k) \equiv M_n^{(\alpha)}(k; \underline{X}_n^{(q)}) := \frac{1}{k} \sum_{i=1}^{k} \left(\ln \frac{X_{n-i+1:n} - X_{\lfloor nq \rfloor+1:n}}{X_{n-k:n} - X_{\lfloor nq \rfloor+1:n}} \right)^{\alpha}, \tag{1.18}$$

defined for $k < n^{(q)}$, with $M_n^{(\alpha)}(k; \underline{X}_n) \equiv M_n^{(\alpha)}(k)$ given in (1.11), $\alpha > 0$.

These PORT EVI-estimators were further studied for finite-samples in Gomes *et al.* (2008b). This methodology was also applied in the estimation of high quantiles in Henriques-Rodrigues and Gomes (2009). Classes of PORT-MVRB EVI-estimators have already been studied for finite samples and by Monte-Carlo simulation in Gomes *et al.* (2011a, 2013) and Gomes and Henriques-Rodrigues (2012), where PORT-estimators of the scale second-order parameters based on scaled log-spacings have been considered.

The PORT methodology leads to location-invariant estimation, where the unshifted model F_0 plays the central role. In what follows, we use the notation χ_q for the q-quantile of the d.f. F_0, i.e. the value

$$\chi_q := F_0^{\leftarrow}(q) \tag{1.19}$$

(by convention $\chi_0 := x_F$, the left endpoint of F_0), with $F^{\leftarrow}$ defined in (1.3). Since $(\lfloor nq \rfloor + 1)/n \to q$, as $n \to \infty$, we then know that the o.s. $X_{\lfloor nq \rfloor+1:n}$, associated with a sample from F_0, is a consistent estimator for $F_0^{\leftarrow}(q)$ (van der Vaart, 1998, p.308), i.e. we have the following convergence in probability:

$$X_{\lfloor nq \rfloor+1:n} \underset{n\to\infty}{\overset{p}{\longrightarrow}} \chi_q, \quad \text{for} \quad 0 \le q < 1 \quad \big(\chi_0 = x_F\big).$$

When applying the PORT-methodology, we are working with the sample of excesses in (1.16), and we can assume that we are dealing with an unshifted d.f. F_0 underlying the r.v. X_0, to which we are inducing a random shift, strictly related to χ_q, in (1.19). More precisely, we have a shift $s = -\chi_q$, i.e. we are working with $X_q := X_0 - \chi_q$, and use from now on the simpler notation (β_q, ρ_q) for $(\beta^*_{-\chi_q}, \rho^*_{-\chi_q})$, with (β^*_s, ρ^*_s) defined in (1.9). Hence

$$(\beta_q, \rho_q) := \begin{cases} (\chi_q/C, -\xi), & \text{if } \xi + \rho_0 < 0 \wedge \chi_q \neq 0, \\ (\beta_0 + \chi_q/C, \rho_0), & \text{if } \xi + \rho_0 = 0 \ \wedge \ \chi_q \neq 0, \\ (\beta_0, \rho_0), & \text{otherwise.} \end{cases} \tag{1.20}$$

Remark 1.1. *For technical simplicity, and even when we use the* PORT *methodology, we often keep the notation* n *for the size of the sample of the excesses, instead of* $n^{(q)} = n - \lfloor nq \rfloor - 1$, $(0 \leq q < 1)$, *and* n^+ *for the number of positive elements in the sample.*

Remark 1.2. *The value* $q = -1/n$, *in* (1.16), *and the notation* $X_{0:n} = 0$, *will enable the recovering of the original sample and the non-*PORT *method. However, in the algorithm of* Section 5, *and for sake of simplicity, the value* $q = 1$ *is used for the non-*PORT *methodology.*

1.4 Scope of the paper

Classes of location-invariant semi-parametric estimators of the also called PORT-ρ second-order parameter, ρ_q, and PORT-β second-order parameter, β_q, both defined in (1.20), were recently introduced and studied in Henriques-Rodrigues and Gomes (2012) and Henriques-Rodrigues *et al.* (2014, 2015). These authors mention that the main motivation for the theoretical study of a class of estimators of the shape second-order parameter ρ_q is related to its possible use, concomitantly with a class of PORT estimators of the functional A, in (1.6), also dependent on q, or at least of an adequate location-invariant estimator of the scale parameter, β_q, of such an A-function, in the study of the asymptotic behaviour of second-order PORT-MVRB EVI-estimators, invariant for changes in location. With the same motivation, we are now interested in the asymptotic behaviour of the PORT-version of the MVRB EVI-estimators in (1.15). Note that a class of the also called PORT-MVRB EVI-estimators was already considered in Gomes *et al.* (2011a, 2013) and Gomes and Henriques-Rodrigues (2012), but due to some technicalities associated with the PORT-version of a class of β-estimators based on the *scaled log-spacings*, the authors only studied the finite sample behavior of the estimators through Monte Carlo experiments.

In this article, and for sake of technical simplicity, we shall work with a new class of PORT-MVRB estimators, which uses a class of PORT-β estimators based in the *log-excesses* (see Caeiro and Gomes, 2006, Henriques-Rodrigues *et al.*, 2015, and Section 2.1.1 for further details on these estimators). Such a new class has the same functional form of the MVRB estimators in (1.15), but with the original sample $\underline{\mathbf{X}}_n$ replaced everywhere by the sample of excesses $\underline{\mathbf{X}}_n^{(q)}$, in (1.16). Consequently, with $(\hat{\beta}^{(q)}, \hat{\rho}^{(q)})$, adequate estimators of the second-order parameters (β_q, ρ_q), in (1.20), such estimators are given by the functional equation,

$$\overline{H}^{(q)}(k) \equiv \overline{H}^{(q)}(k; \hat{\beta}^{(q)}, \hat{\rho}^{(q)}) \equiv \overline{H}(k; \underline{\mathbf{X}}_n^{(q)}) := H^{(q)}(k)\Big(1 - \hat{\beta}^{(q)}\ (n/k)^{\hat{\rho}^{(q)}}/(1 - \hat{\rho}^{(q)})\Big), \quad (1.21)$$

with $\overline{H}(k; \underline{\mathbf{X}}_n)$, $\underline{\mathbf{X}}_n^{(q)}$ and $H^{(q)}(k)$ given in (1.15), (1.16) and (1.17), respectively. These estimators are now invariant for both changes of location and scale, and depend on the *tuning parameter* q, which only influences their asymptotic bias, making them highly flexible, and able to compare favorably with the MVRB estimators in (1.15), for a large variety of underlying models in the domain of attraction for maxima of the EV d.f., in (1.1). In this paper we present and derive the asymptotic behavior of these new PORT-MVRB EVI-estimators, based on the PORT-ρ estimators of the shape second-order parameter introduced in Henriques-Rodrigues and Gomes (2012) and Henriques-Rodrigues *et al.* (2014), and on the PORT-β estimators of the scale second-order parameter recently introduced in Henriques-Rodrigues *et al.* (2015). The use of such a class of the second-order scale parameter enable us to overcome the technicality of these estimators and to fully derive the asymptotic behavior of the new class of PORT-MVRB EVI-estimtors.

In Section 2, we present, under a third-order framework, a few basic results on the topic. The asymptotic behaviour obtained by Caeiro *et al.* (2009) for the classical MVRB EVI-estimators, in (1.15), is sketched in Section 2.1. Next, in Section 2.2, we address the second and third-order frameworks for heavy-tailed models under a PORT-methodology. The asymptotic behaviour of the PORT-Hill estimator, in (1.17), is presented in Section 2.3. The PORT-estimators of the scale and shape second-order parameters, (β_q, ρ_q), in (1.20), are presented in Section 3. In this Section we also address the choice of the high level for the estimation of the second-order parameters and the choice of the tuning parameters under play. In Section 4 we provide the non-degenerate asymptotic behaviour of the new class of PORT-MVRB EVI-estimators. In Section 5, we provide an illustration of the behaviour of the estimators through the analysis of two data sets in the field of finance. Some final remarks are drawn in Section 6. In Section 7, the non-degenerate asymptotic behaviour of the PORT-estimators of the scale and shape second-order parameters, (β_q, ρ_q), in (1.20), is presented. Finally, in Section 8, we provide the proof of the new result stated in Section 4.

2 Basic results under a third-order framework

2.1 Asymptotic behaviour of the classical MVRB EVI-estimators

If we assume that the second-order parameters (β_0, ρ_0) associated with the standard model F_0 are known we have the validity of the following theorem, proved in Caeiro *et al.* (2009).

Theorem 2.1 (Caeiro *et al.*, 2009)**.** *Under* (1.7) *and* (1.8)*, with* $(U, A, B, \rho, \rho', \beta, \beta')$ *replaced by* $(U_0, A_0, B_0, \rho_0, \rho'_0, \beta_0, \beta'_0)$*, and for levels* k *such that* (1.12) *holds, the r.v.* $\overline{H}(k; \beta_0, \rho_0)$*, with* $\overline{H}(k; \hat\beta, \hat\rho)$ *defined in* (1.15)*, has an asymptotic distributional representation of the type,*

$$\overline{H}(k; \beta_0, \rho_0) \stackrel{d}{=} \xi + \frac{\xi\ Z_k}{\sqrt{k}} + R(k), \tag{2.1}$$

with Z_k *given in* (1.13)*, and*

$$R(k) = A_0(n/k)\left(uA_0(n/k) + vB_0(n/k) + O_p\left(\frac{1}{\sqrt{k}}\right)\right)(1 + o_p(1)), \tag{2.2}$$

where,

$$u = -\frac{1}{\xi(1-\rho_0)^2} \quad \textit{and} \quad v = \frac{1}{1-\rho_0-\rho'_0}. \tag{2.3}$$

Consequently, even if $\sqrt{k}A_0(n/k) \to \infty$*, with* $\sqrt{k}A_0^2(n/k) \to \lambda_A$ *and* $\sqrt{k}A_0(n/k)B_0(n/k) \to \lambda_B$*, both finite,* $\sqrt{k}\left(\overline{H}(k; \beta_0, \rho_0) - \xi\right) \xrightarrow[n\to\infty]{d} \mathcal{N}(b := \lambda_A u + \lambda_B v, \xi^2)$*, where* $\mathcal{N}(\mu, \sigma^2)$ *denotes a normal r.v. with mean value* μ *and variance* σ^2*. If* $\sqrt{k}A_0^2(n/k) \to \infty$ *or* $\sqrt{k}A_0(n/k)B_0(n/k) \to \infty$*, then* $\sqrt{k}\left(\overline{H}(k; \beta_0, \rho_0) - \xi\right)/A_0(n/k)$ *is either* $O_p(A_0(n/k))$ *or* $O_p(B_0(n/k))$*, the one of highest order.*

Remark 2.1. *More generally, and like in* Caeiro *et al.* (2016)*, we could have considered the* RB *mean-of-order-p* (MO_p) EVI-*estimator,*

$$\overline{H}_p(k) \equiv \overline{H}_p(k;\hat\beta,\hat\rho) \equiv \mathrm{RBMO}_p(k) := H_p(k)\Big(1-\frac{\hat\beta(1-pH_p(k))}{1-\hat\rho-pH_p(k)}\Big(\frac{n}{k}\Big)^{\hat\rho}\Big),$$

$$H_p(k) := \begin{cases} \left(1-\left(\frac{1}{k}\sum_{i=1}^{k}\left(X_{n-i+1:n}/X_{n-k:n}\right)^p\right)^{-1}\right)/p, & \text{if} \quad p<1/\xi,\ p\neq 0,\\ H(k), & \text{if} \quad p=0. \end{cases}$$

which can be seen as a generalization to a real p of the $\overline{H}\equiv\overline{H}_0$ *class of* EVI-*estimators in* (1.15)*. Then,* Theorem 2.1 *holds with* $R(k) =: R_0(k)$ *replaced by* $R_p(k)$*, where* $\xi =: \sigma_0$*,* $Z_k =: Z_k^{(0)}$*,* $u =: u_0^*$ *and* $v =: v_0^*$*, are respectively replaced by adequate* σ_p*,* $Z_k^{(p)}$*,* u_p^* *and* v_p^**.*

2.1.1 Estimation of β_0 and ρ_0 at the same high level k_1

The class of semi-parametric estimators of the second-order shape parameter ρ_0 used in the MVRB-EVI estimators is the one proposed by Fraga Alves *et al.* (2003), but parameterized in a real tuning parameter τ (Caeiro and Gomes, 2006), and defined as

$$\widehat\rho(k) \equiv \widehat\rho^{(\tau)}(k) := -\big|(3\big(T_n^{(\tau)}(k)-1\big)/(T_n^{(\tau)}(k)-3)\big| \tag{2.4}$$

where

$$T_n^{(\tau)}(k) \equiv T_n^{(\tau)}(k;\underline{X_n}) = \frac{\Big(M_n^{(1)}(k)\Big)^{\tau}-\Big(M_n^{(2)}(k)/2\Big)^{\tau/2}}{\Big(M_n^{(2)}(k)/2\Big)^{\tau/2}-\Big(M_n^{(3)}(k)/6\Big)^{\tau/3}}, \quad \tau\in\mathbb{R}, \tag{2.5}$$

with $M_n^{(\alpha)}(k)$ given in (1.11) and the notation $a^{b\tau} = b\ln a$, whenever $\tau = 0$.
For the estimation of the scale second-order parameter β_0, and given the sample $\underline{\mathbf{X}}_n$, we shall consider the estimator in Gomes and Martins (2002),

$$\hat\beta_V(k;\hat\rho) \equiv \hat\beta_V(k;\hat\rho,\ \underline{\mathbf{X}}_n) := \left(\frac{k}{n}\right)^{\hat\rho}\frac{d_{\hat\rho}(k)\ D_0(k)-D_{\hat\rho}(k)}{d_{\hat\rho}(k)\ D_{\hat\rho}(k)-D_{2\hat\rho}(k)}, \tag{2.6}$$

dependent on an adequate ρ_0-estimator, $\hat\rho$, and where, for any $\alpha\le 0$,

$$d_\alpha(k) := \frac{1}{k}\sum_{i=1}^{k}\left(\frac{i}{k}\right)^{-\alpha} \quad \text{and} \quad D_\alpha(k) := \frac{1}{k}\sum_{i=1}^{k}\left(\frac{i}{k}\right)^{-\alpha} V_i,$$

with

$$V_i := i\Big(\ln X_{n-i+1:n} - \ln X_{n-i:n}\Big), \qquad 1 \le i \le k,$$

the *scaled log-spacings* associated with the sample $\underline{\mathbf{X}}_n$.

The asymptotic behaviour of the estimators in (2.4) and (2.6) was derived, under an adequate third-order framework, in Caeiro *et al.* (2009) (see their Proposition 2.1 and Theorem 2.1, for further details), and the associated MVRB-EVI estimators have the non-degenerate asymptotic behaviour stated in the following theorem (Theorem 3.2 of the aforementioned paper).

Theorem 2.2 (Caeiro *et al.*, 2009). *Let us assume that* (1.7) *holds, and let* $\hat\beta$ *and* $\hat\rho$ *be any consistent estimators of the second-order parameters* β_0 *and* ρ_0*, both computed at a high level,* $k_1 := O(n^{-2(\rho_0+\rho_0')/(1-2(\rho_0+\rho_0'))})$*, such that*

$$\hat\rho - \rho_0 = o_p(1/\ln n), \quad \textit{as} \quad n \to \infty. \tag{2.7}$$

Then, whenever $\sqrt{k}A_0(n/k) \to \lambda$*, finite,* $\sqrt{k}\left(\overline{H}(k;\hat\beta,\hat\rho) - \xi\right)$ *is asymptotically normal with null mean and variance,* $\sigma^2 = \xi^2$*. We can still get this same limiting result for levels* k *such that* $\sqrt{k}A_0(n/k) \to \infty$ *provided that* $k = o(k_1)$*, as* $n \to \infty$*, and we choose* k_1 *optimal for the estimation of* ρ_0*, i.e., such that* $\sqrt{k_1}A_0^2(n/k_1) \to \lambda_{A_1}$ *and* $\sqrt{k_1}A_0(n/k_1)B_0(n/k_1) \to \lambda_{B_1}$*, finite. If for* k *such that* $\sqrt{k}A_0^2(n/k) \to \lambda_A$ *and* $\sqrt{k}A_0(n/k)B_0(n/k) \to \lambda_B$*, both finite, we assume the validity of the following heuristic condition:*

> C_U*: There exist* τ_U *and* k_1*, with* $\sqrt{k_1}A_0(n/k_1)B_0(n/k_1) \to \infty$ *and/or* $\sqrt{k_1}A_0^2(n/k_1) \to \infty$*, such that, with* $\hat\rho^{(\tau)}(k)$*, defined in* (2.4)*,* $\hat\rho_U - \rho_0 = \hat\rho^{(\tau_U)}(k_1) - \rho_0 = O_p(1/\left(\sqrt{k_1}A_0(n/k_1)\right))$*, for a level* $k_1 = O(n/\ln\ln n)$*.*

Considering $\hat\beta_U \equiv \hat\beta_{U|V} = \hat\beta_V(k_1;\hat\rho_U)$*, the tail index estimators* $\overline{H}(k;\hat\beta_U,\hat\rho_U)$ *have an asymptotic variance still equal to* ξ^2 *and an asymptotic bias, still given by* $\lambda_A u + \lambda_B v$*, with* u *and* v *given in* (2.3)*. This same result holds for any* $\overline{H}(k;\hat\beta,\hat\rho)$ *provided that* $\hat\rho - \rho_0 = o_p(\ln(n/k)/(\sqrt{k}A_0(n/k))$*.*

2.2 Second and third-order PORT-framework for heavy-tailed models

Under the aforementioned set-up in Section 1.3, the transformed r.v., $X_q = X_0 - \chi_q$, with χ_q given in (1.19), has an associated quantile function given by $U_q(t) = U_0(t) - \chi_q$. The second-order condition in (1.6) translates as

$$\lim_{t\to\infty} \frac{\ln U_q(tx) - \ln U_q(t) - \xi \ln x}{A_q(t)} = \psi_{\rho_q}(x) := \begin{cases} \frac{x^{\rho_q}-1}{\rho_q}, & \text{if} \quad \rho_q < 0, \\ \ln x, & \text{if} \quad \rho_q = 0, \end{cases} \tag{2.8}$$

for all $x > 0$. Moreover, $|A_q| \in RV_{\rho_q}$, $\rho_q \le 0$, and A_q relates to A_0 according to the following lemma.

Lemma 2.1 (Henriques-Rodrigues *et al.*, 2014)**.** *Assume that* $U_0 \in RV_\xi$ *is such that the second-order condition in* (2.8) *holds. Then* $U_q(t) = U_0(t) - \chi_q$*, with* χ_q *defined in* (1.19)*, is such that* $U_q \in RV_\xi$ *and* (2.8) *holds with* (β_q, ρ_q) *given in* (1.20) *and*

$$A_q(t) := \begin{cases} \xi\chi_q/U_0(t), & \text{if} \quad \xi+\rho_0 < 0 \ \wedge \ \chi_q \neq 0, \\ A_0(t) + \xi\chi_q/U_0(t), & \text{if} \quad \xi+\rho_0 = 0 \ \wedge \ \chi_q \neq 0, \\ A_0(t), & \text{if} \quad \xi+\rho_0 > 0 \vee \chi_q = 0. \end{cases} \tag{2.9}$$

To obtain information on the order of a possibly non null asymptotic bias of the second-order RB tail index estimators considered in this article, we shall further assume an adequate third-order condition, ruling now the rate of convergence in (2.8), and which guarantees that, for all $x > 0$,

$$\lim_{t\to\infty} \frac{\frac{\ln U_q(tx)-\ln U_q(t)-\xi\ln x}{A_q(t)} - \psi_{\rho_q}(x)}{B_q(t)} = \begin{cases} \frac{x^{\rho_q+\rho'_q}-1}{\rho_q+\rho'_q}, & \text{if} \quad \min(\rho_q, \rho'_q) < 0, \\ \ln x, & \text{if} \quad \rho_q = \rho'_q = 0, \end{cases} \tag{2.10}$$

where $|B_q|$ must then be in $RV_{\rho'_q}$. For technical simplicity, we shall assume that $\rho_q,\ \rho'_q < 0$.

2.3 Asymptotic behaviour of the PORT-Hill estimator under a third-order framework

The asymptotic behaviour of the PORT-Hill estimator, in (1.17), can be derived taking into account the results presented in Proposition 2.2 of Henriques-Rodrigues *et al.* (2014), related to the distributional representation of the statistics $M_n^{(\alpha,q)}(k)$, in (1.18), for $\alpha = 1$.

Theorem 2.3 (Henriques-Rodrigues *et al.*, 2014)**.** *Let us assume that* (1.12) *holds, as well as the third-order condition in* (2.10)*, for* $\rho_0, \rho'_0 < 0$*. We then get for* $H^{(q)}(k) \equiv M_n^{(1,q)}(k)$*, in*

(1.17), $k < n^{(q)}$, $0 \le q < 1$, *the distributional representation,*

$$H^{(q)}(k) \stackrel{d}{=} \xi + \frac{\xi Z_k}{\sqrt{k}} + \frac{A_q(n/k)}{1-\rho_q}$$
$$+ \Big(\frac{A_0(n/k)B_0(n/k)}{1-\rho_0-\rho_0'} + \frac{\chi_q \; A_0(n/k)}{(1+\xi)(1+\xi-\rho_0)U_0(n/k)} + \frac{\xi \; \chi_q^2}{(1+2\xi) \; U_0^2(n/k)}\Big)(1+o_p(1)), \quad (2.11)$$

with Z_k, χ_q *and* $A_q(n/k)$ *given in* (1.13), (1.19) *and* (2.9), *respectively.*

Remark 2.2. *Note that the distributional representation of the* PORT-Hill *estimator, given in* (2.11), *is a generalization of the classical representation of the* Hill *estimator, given in* (1.14).

The dominant component of the right hand-side of (2.11) depends on the relative behaviour of the functions $A_0(t)$ and $1/U_0(t)$, and as usual whenever working in a PORT set-up, we shall thus consider three different regions related to χ_q, in (1.19), and the vector (ξ, ρ_0) of the unshifted model F_0 associated with the available data:

- $\mathcal{R}_1 := \{F_0 : \xi + \rho_0 < 0 \wedge \chi_q \neq 0\}$,
- $\mathcal{R}_2 := \{F_0 : \xi + \rho_0 > 0 \vee \chi_q = 0\}$,
- $\mathcal{R}_3 := \{F_0 : \xi + \rho_0 = 0 \wedge \chi_q \neq 0\}$.

Corollary 2.1. *Under the conditions of* **Theorem 2.3**, *the following results hold:*

i) *In* $\mathcal{R}_1$,

$$H^{(q)}(k) \stackrel{d}{=} \xi + \frac{\xi Z_k}{\sqrt{k}} + \frac{\xi \; \chi_q}{(1+\xi)U_0(n/k)} + \frac{\xi \; \chi_q^2}{(1+2\xi) \; U_0^2(n/k)}(1+o_p(1)).$$

ii) *In* $\mathcal{R}_2$,

$$H^{(q)}(k) \stackrel{d}{=} \xi + \frac{\xi Z_k}{\sqrt{k}} + \frac{A_0(n/k)}{1-\rho_0} + \frac{A_0(n/k)B_0(n/k)}{1-\rho_0-\rho_0'}(1+o_p(1)),$$

i.e., we recover the representation of the classical Hill *estimator, presented in* (1.14), *with* (A, B) *replaced by* (A_0, B_0).

iii) *In* $\mathcal{R}_3$,

$$H^{(q)}(k) \stackrel{d}{=} \xi + \frac{\xi Z_k}{\sqrt{k}} + \Big(\frac{A_0(n/k)}{1-\rho_0} + \frac{\xi \; \chi_q}{(1+\xi)U_0(n/k)}\Big)$$
$$+ \Big(\frac{A_0(n/k)B_0(n/k)}{1-\rho_0-\rho_0'} + \frac{\chi_q \; A_0(n/k)}{(1+\xi)(1+\xi-\rho_0)U_0(n/k)} + \frac{\xi \; \chi_q^2}{(1+2\xi) \; U_0^2(n/k)}\Big)(1+o_p(1)).$$

3 PORT-estimators of the second-order parameters

3.1 The PORT-ρ estimators

The class of location invariant estimators of the shape second-order parameter considered here, named PORT-ρ class of estimators, was introduced and validated under a second-order framework in Henriques-Rodrigues and Gomes (2012), and further studied under a third-order framework in Henriques-Rodrigues *et al.* (2014). It is explicitly given by

$$\widehat{\rho}^{(\tau_q,q)}(k) := \frac{3\big(T_n^{(\tau_q,q)}(k)-1\big)}{T_n^{(\tau_q,q)}(k)-3}\, \mathbb{1}\big\{T_n^{(\tau_q,q)}(k)\in(1,3)\big\}, \quad \tau_q\in\mathbb{R} \tag{3.1}$$

where $T_n^{(\tau_q,q)}(k)\equiv T_n^{(\tau=\tau_q)}(k;\underline{X}_n^{(q)})$, with $T_n^{(\tau)}(k)$ given in (2.5), and $M_n^{(\alpha,q)}(k)\equiv M_n^{(\alpha)}(k;\underline{X}_n^{(q)})$, given in (1.18), with $M_n^{(\alpha)}(k)\equiv M_n^{(\alpha)}(k;\underline{X}_n)$ and $\underline{X}_n^{(q)}$ respectively given in (1.11) and (1.16). The asymptotic non-degenerate behaviour of this class is provided in Section 7 (Theorem 7.1).

3.1.1 A few remarks on the choice the high level for the estimation of ρ_q

We now rephrase for the PORT-ρ-estimators described in this article, the comments made in Caeiro *et al.* (2009) related to the choice of the high value, now named $k_1^{(q)}$, that should be used for the estimation of ρ_q.

1. The optimal choice of the threshold $k_1^{(q)}$ should enable us to guarantee the asymptotic normality of the ρ_q-estimators, $\hat{\rho}^{(q)} := \widehat{\rho}^{(\tau_q,q)}(k_1^{(q)})$, for any of the ρ_q-estimators in (3.1) computed at $k_1^{(q)}$, with a non null asymptotic bias. The level $k_1^{(q)}$ is then such that:
   i) In $\mathcal{R}_1$, if $\lim_{n\to\infty}\sqrt{k}A_0(n/k_1^{(q)})=\lambda_1$ and $\lim_{n\to\infty}\sqrt{k}/U_0^2(n/k_1^{(q)})=\lambda_{U_1}$, both finite, with at least one of them non null, let us say λ_{U_1}, we get $k_1^{(q)}=O(n^{4\xi/(1+4\xi)})$. Then, $\hat{\rho}^{(q)}-\rho_q = O_p\Big(1/(\sqrt{k_1^{(q)}}/U_0(n/k_1^{(q)}))\Big)=O_p\left(n^{-\xi/(1+4\xi)}\right)$.
   ii) In $\mathcal{R}_2$, if $\lim_{n\to\infty}\sqrt{k}A_0^2(n/k)=\lambda_A$, $\lim_{n\to\infty}\sqrt{k}A_0(n/k)B_0(n/k)=\lambda_B$ and $\lim_{n\to\infty}\sqrt{k}/U_0(n/k)=\lambda'$, finite, with at least one of them non null, let us say λ_B, we get $k_1^{(q)}=O(n^{-2(\rho_0+\rho_0')/(1-2(\rho_0+\rho_0'))})$. Then, $\hat{\rho}^{(q)}-\rho_q=O_p\Big(1/(\sqrt{k_1^{(q)}}A_0(n/k_1^{(q)}))\Big)= O_p\left(n^{\rho_0'/(1-2(\rho_0+\rho_0'))}\right)$.
   iii) In $\mathcal{R}_3$, if $\lim_{n\to\infty}\sqrt{k}A_0^2(n/k)=\lambda_A$, $\lim_{n\to\infty}\sqrt{k}A_0(n/k)B_0(n/k)=\lambda_B$, $\lim_{n\to\infty}\sqrt{k}A_0(n/k)/U_0(n/k)=\lambda_{AU}$, with at least one of them non null, let us say λ_{AU},

and $\widetilde{\lambda} = \lim_{n\to\infty} 1/\big(A_0(n/k)U_0(n/k)\big) \neq 0$ finite, we get $k_1^{(q)} = O(n^{-2(\rho_0-\xi)/(1+2(\rho_0-\xi))})$. Then, $\hat{\rho}^{(q)} - \rho_q = O_p\Big(1/(\sqrt{k_1^{(q)}}A_0(n/k_1^{(q)}))\Big) = O_p\left(n^{-\xi/(1+2(\rho_0-\xi))}\right)$.

Thence,

$$\hat{\rho}^{(q)} - \rho_q = o_p(1/\ln n), \quad \text{as} \quad n \to \infty, \tag{3.2}$$

in the three regions $\mathcal{R}_1$, $\mathcal{R}_2$ and $\mathcal{R}_3$, a condition needed in Section 4, and similar to condition (2.7), related to the classical ρ-estimation.

2. Note that for most of the common heavy-tailed models ($\xi > 0$), the parameter ρ' in (1.7) is equal to the parameter ρ in (1.6). Among them we mention, the Fréchet, the generalized Pareto, the Student's t_ν-model and the Burr model. In this case, is possible to find a pair (τ_q, q) such that the asymptotic bias of the PORT-ρ estimator is null. Such a claim is made essentially on the basis of the high versatility of the PORT-ρ estimators, and given the fact that the sample paths of the ρ_q-estimators associated with "optimal" (τ_q, q) exhibit high stability for large k. The use of a value $k_1^{(q)}$ larger then the "optimal" for the PORT-ρ estimation, but intermediate, like $k_1^{(q)}$ (see Gomes and Martins, 2002, for details on the choice of k_1 for the classical ρ-estimation), enable us to guarantee condition (3.2). The choice of the "optimal" $(\tau_q, k_1^{(q)}, q)$ is put forward in the following heuristic condition $C^\star_{U_\bullet}$, written accordingly to the region $\mathcal{R}_\bullet$ in the (ξ, ρ)-plane, $\bullet = 1, 2, 3$, and associated with the underlying parent:
   i) $C^\star_{U_1}$: In $\mathcal{R}_1$, there exists $(\tau_{q;U_1}, k_1^{(q)}, q)$, with $\lim_{n\to\infty} \sqrt{k}A_0(n/k_1^{(q)}) = \infty$ and/or $\lim_{n\to\infty} \sqrt{k}/U_0^2(n/k_1^{(q)}) = \infty$, such that, with $\widehat{\rho}^{(\tau,q)}(k)$ defined in (3.1), $\widehat{\rho}_{U_1}^{(q)} - \rho_q = \widehat{\rho}^{(\tau_{q;U_1},q)}(k_1^{(q)}) - \rho_q = O_p\Big(1/(\sqrt{k_1^{(q)}}/U_0(n/k_1^{(q)}))\Big)$.
   ii) $C^\star_{U_2}$: In $\mathcal{R}_2$, there exists $(\tau_{q;U_2}, k_1^{(q)}, q)$, with $\lim_{n\to\infty} \sqrt{k}A_0^2(n/k) = \infty$ and/or $\lim_{n\to\infty} \sqrt{k}A_0(n/k)B_0(n/k) = \infty$ and/or $\lim_{n\to\infty} \sqrt{k}/U_0(n/k) = \infty$, such that, with $\widehat{\rho}^{(\tau,q)}(k)$ defined in (3.1), $\widehat{\rho}_{U_2}^{(q)} - \rho_q = \widehat{\rho}^{(\tau_{q;U_2},q)}(k_1^{(q)}) - \rho_q = O_p\Big(1/(\sqrt{k_1^{(q)}}A_0(n/k_1^{(q)}))\Big)$.
   iii) $C^\star_{U_3}$: In $\mathcal{R}_3$, there exists $(\tau_{q;U_3}, k_1^{(q)}, q)$, with $\lim_{n\to\infty} \sqrt{k}A_0^2(n/k) = \infty$ and/or $\lim_{n\to\infty} \sqrt{k}A_0(n/k)B_0(n/k) = \infty$ and/or $\lim_{n\to\infty} \sqrt{k}A_0(n/k)/U_0(n/k) = \infty$, such that, with $\widehat{\rho}^{(\tau,q)}(k)$ defined in (3.1), $\widehat{\rho}_{U_3}^{(q)} - \rho_q = \widehat{\rho}^{(\tau_{q;U_3},q)}(k_1^{(q)}) - \rho_q = O_p\Big(1/(\sqrt{k_1^{(q)}}A_0(n/k_1^{(q)}))\Big)$.

3. Note that if we consider a level $k_1^{(q)} = \lfloor n^{1-\epsilon} \rfloor$, with $\epsilon > 0$, small, we can also guarantee (3.2) for a large class of models without the need to assume the stronger condition $C^{\star}_{U_\bullet}$. This is the reason why we advise in practice the consideration of such a level. Notice also that the choice of ϵ is not crucial.

3.1.2 A few remarks on the choice of τ_q for the estimation of ρ_q

The theoretical and simulation results in Fraga Alves *et al.* (2003) and in Gomes and Martins (2002) for the classic estimation of the second-order parameter ρ have led several authors to advise in practice the consideration of the *tuning* parameter $\tau = 0$ for the region $\rho \in [-1, 0)$ and $\tau = 1$ for the region $\rho \in (-\infty, -1)$, and to consider the ρ-estimators associated with a high level k_1, in order to estimate the tail index ξ, through the use of second-order RB estimators, like the ones introduced in Caeiro *et al.* (2005), Gomes *et al.* (2007a) and Gomes *et al.* (2008c). We now present the PORT-version of this empirical criterion, to be used in Section 5.

$$\tau_q = \begin{cases} 0, & \text{if} \quad \rho_q \geq -1 \\ 1, & \text{if} \quad \rho_q < -1. \end{cases}$$

3.2 The PORT-β estimators

The class of location invariant estimators of the second-order scale parameter considered here, named PORT-β class of estimators, was introduced and validated under a third-order framework in Henriques-Rodrigues *et al.* (2015), it is based on the classical β-estimators introduced in Caeiro and Gomes (2006), dependent on a real *tuning* parameter η_q, being explicitly given by

$$\widehat{\beta}^{(\eta_q,q)}(k;\hat{\rho}^{(q)}) \quad := \quad \begin{cases} -\frac{2(2-\hat{\rho}^{(q)})^2}{\eta_q \hat{\rho}^{(q)}} \left(\frac{k}{n}\right)^{\hat{\rho}^{(q)}} \frac{\left[\left(M_n^{(1,q)}(k)\right)^{\eta_q} - \left(M_n^{(2,q)}(k)/2\right)^{\eta_q/2}\right]^2}{\left(M_n^{(2,q)}(k)/2\right)^{\eta_q} - \left(M_n^{(4,q)}(k)/24\right)^{\eta_q/2}}, & \text{if} \quad \eta_q \neq 0, \\ -\frac{2(2-\hat{\rho}^{(q)})^2}{\hat{\rho}^{(q)}} \left(\frac{k}{n}\right)^{\hat{\rho}^{(q)}} \frac{\left[\ln\left(M_n^{(1,q)}(k)\right) - \frac{1}{2}\ln\left(M_n^{(2,q)}(k)/2\right)\right]^2}{\ln\left(M_n^{(2,q)}(k)/2\right) - \frac{1}{2}\ln\left(M_n^{(4,q)}(k)/24\right)}, & \text{if} \quad \eta_q = 0, \end{cases} \tag{3.3}$$

with $\hat{\rho}^{(q)}$ any consistent estimator of ρ_q, in (1.20), and $M_n^{(\alpha,q)}(k)$ given in (1.18). The asymptotic behaviour of the PORT-β estimators is presented in Section 7, Theorem 7.2 and Corollary 7.1.

4 Asymptotic behavior of the PORT-MVRB Hill-estimators

As mentioned previously, the class of PORT-MVRB EVI-estimators introduced in this paper is different from the class of PORT-MVRB EVI estimators introduced in Gomes *et al.* (2011a), where the authors considered the estimators in (1.21), but being $\hat{\beta}^{(q)} \equiv \hat{\beta}_V^{(q)}$ the PORT version of the estimator of the scale second-order parameter β, introduced in Gomes and Martins (2002), and given in (2.6). The asymptotic behavior of the class of PORT-MVRB EVI-estimators addressed in this paper will be derived taking into account the asymptotic behavior of the PORT classes of the second-order parameters' estimators defined in Sections 3.1 and 3.2.

4.1 Known β_q and ρ_q

We can now establish the non-degenerate behavior of the PORT-MVRB Hill-estimators assuming that the second-order parameters, β_q and ρ_q, in (1.20), are known, similarly to the set-up of Theorem 2.1. The sketch of the proof is similar to the one of Theorem 3.1 in Caeiro *et al.* (2009).

Theorem 4.1. *Under* (1.7)*, further assuming that* $\big(U_0(\cdot), A_0(\cdot), B_0(\cdot)\big)$ *can be chosen as* $\big(U(\cdot), A(\cdot), B(\cdot)\big)$*, in* (1.8)*, with* $(\rho, \beta, \rho', \beta')$ *replaced by* $(\rho_0, \beta_0, \rho_0', \beta_0')$*, and for levels* k *such that* (1.12) *holds, the r.v.* $\overline{H}^{(q)}(k; \beta_q, \rho_q)$*, with* $\overline{H}^{(q)}(k; \hat{\beta}^{(q)}, \hat{\rho}^{(q)})$ *defined in* (1.21)*, has an asymptotic distributional representation of the type,*

$$\overline{H}^{(q)}(k; \beta_q, \rho_q) \stackrel{d}{=} \xi + \frac{\xi\ Z_k}{\sqrt{k}} + R^{(q)}(k),$$

a generalization of (2.1)*, with* Z_k *given in* (1.13)*, and*

$$R^{(q)}(k) = A_0(n/k)\left(u_q A_0(n/k) + v B_0(n/k) + w_q \frac{\chi_q}{U_0(n/k)} + O_p\left(\frac{1}{\sqrt{k}}\right)\right)(1 + o_p(1)) + \frac{1}{U_0(n/k)}\left(y_q \frac{\chi_q^2}{U_0(n/k)} + O_p\left(\frac{1}{\sqrt{k}}\right)\right)(1 + o_p(1)),$$

where, with

$$u_q := -\frac{1}{\xi(1-\rho_q)^2} = \begin{cases} -\frac{1}{\xi(1+\xi)^2}, & \text{in } \mathcal{R}_1, \\ u \equiv -\frac{1}{\xi(1-\rho_0)^2}, & \text{in } \mathcal{R}_2 \cup \mathcal{R}_3, \end{cases} \qquad v := \frac{1}{1-\rho_0-\rho_0'},$$

$$w_q := \frac{1}{(1+\xi)(1+\xi-\rho_0)} - \frac{2}{(1-\rho_q)^2} = \begin{cases} \frac{1+\xi-2\rho_0}{(1+\xi)^2(1+\xi-\rho_0)}, & \text{in } \mathcal{R}_1, \\ \frac{(1-\rho_0)^2-2(1+\xi)^2+2\rho_0(1+\xi)}{(1+\xi)(1+\xi-\rho_0)(1-\rho_0)^2}, & \text{in } \mathcal{R}_2, \\ \frac{-1+3\rho_0}{(1-\rho_0)^2(1-2\rho_0)}, & \text{in } \mathcal{R}_3, \end{cases}$$

$$y_q := \frac{\xi}{1+2\xi} - \frac{\xi}{(1-\rho_q)^2} = \begin{cases} \frac{\xi^3}{(1+2\xi)(1+\xi)^2}, & \text{in } \mathcal{R}_1, \\ \frac{\xi\left(\rho_0^2-2(\rho_0+\xi)\right)}{(1+2\xi)(1-\rho_0)^2}, & \text{in } \mathcal{R}_2, \\ \frac{-\rho_0^3}{(1-2\rho_0)(1-\rho_0)^2}, & \text{in } \mathcal{R}_3, \end{cases} \tag{4.1}$$

we can write, with $R(k)$ *given in* (2.2),

$$R^{(q)}(k) \quad = \quad \begin{cases} \frac{\chi_q}{U_0(n/k)}\left(\frac{\xi^3}{(1+2\xi)(1+\xi)^2}\frac{\chi_q}{U_0(n/k)} + O_p\left(\frac{1}{\sqrt{k}}\right)\right)(1+o_p(1)), & \text{in } \mathcal{R}_1, \\ R(k), & \text{in } \mathcal{R}_2, \\ R(k) + A_0(n/k)\left(\frac{-1+3\rho_0}{(1-2\rho_0)(1-\rho_0)^2}\frac{\chi_q}{U_0(n/k)} + O_p\left(\frac{1}{\sqrt{k}}\right)\right)(1+o_p(1)) & \\ +\frac{\chi_q}{U_0(n/k)}\left(\frac{-\rho_0^3}{(1-2\rho_0)(1-\rho_0)^2}\frac{\chi_q}{U_0(n/k)} + O_p\left(\frac{1}{\sqrt{k}}\right)\right)(1+o_p(1)), & \text{in } \mathcal{R}_3. \end{cases}$$

Consequently:

i) *In* $\mathcal{R}_1$*, even if* $\sqrt{k}/U_0(n/k) \to \infty$*, with* $\sqrt{k}/U_0^2(n/k) \to \lambda_U$*, finite, then*

$$\sqrt{k}\left(\overline{H}^{(q)}(k;\beta_q,\rho_q) - \xi\right) \underset{n\to\infty}{\overset{d}{\longrightarrow}} \mathcal{N}(b_{\mathcal{R}_1}^{(q)} := \lambda_U y_q, \xi^2).$$

If $\sqrt{k}/U_0^2(n/k) \to \infty$*, then* $\sqrt{k}\ U_0(n/k)\left(\overline{H}^{(q)}(k;\beta_q,\rho_q) - \xi\right)$ *is* $O_p(1/U_0(n/k))$*.*

ii) *In* $\mathcal{R}_2$*, even if* $\sqrt{k}A_0(n/k) \to \infty$*, with* $\sqrt{k}A_0^2(n/k) \to \lambda_A$*,* $\sqrt{k}A_0(n/k)B_0(n/k) \to \lambda_B$*, both finite, then*

$$\sqrt{k}\left(\overline{H}^{(q)}(k;\beta_q,\rho_q) - \xi\right) \underset{n\to\infty}{\overset{d}{\longrightarrow}} \mathcal{N}(b_{\mathcal{R}_2}^{(q)} := \lambda_A u_q + \lambda_B v, \xi^2).$$

If $\sqrt{k}A_0^2(n/k) \to \infty$ *or* $\sqrt{k}A_0(n/k)B_0(n/k) \to \infty$*, then* $\sqrt{k}\left(\overline{H}^{(q)}(k;\beta_q,\rho_q) - \xi\right)/A_0(n/k)$ *is either* $O_p(A_0(n/k))$ *or* $O_p(B_0(n/k))$*, the one of highest order.*

iii) *In* $\mathcal{R}_3$*, even if* $\sqrt{k}A_0(n/k) \to \infty$ *or* $\sqrt{k}/U_0(n/k) \to \infty$*, with* $\sqrt{k}A_0^2(n/k) \to \lambda_A$*,* $\sqrt{k}A_0(n/k)B_0(n/k) \to \lambda_B$*,* $\sqrt{k}A_0(n/k)U_0(n/k) \to \lambda_{AU}$*, and* $\sqrt{k}/U_0^2(n/k) \to \lambda_U$*, all* *finite, then*

$$\sqrt{k}\left(\overline{H}^{(q)}(k;\beta_q,\rho_q)-\xi\right)\underset{n\to\infty}{\overset{d}{\longrightarrow}}\mathcal{N}(b_{\mathcal{R}_3}^{(q)}:=\lambda_A u_q+\lambda_B v+\lambda_{AU}w_q+\lambda_U y_q,\xi^2).$$

If $\sqrt{k}A_0^2(n/k)\to\infty$ *or* $\sqrt{k}A_0(n/k)B_0(n/k)\to\infty$ *or* $\sqrt{k}A_0(n/k)/U_0(n/k)\to\infty$, *then* $\sqrt{k}\left(\overline{H}^{(q)}(k;\beta_q,\rho_q)-\xi\right)/A_0(n/k)$ *is either* $O_p(A_0(n/k))=O_p(1/U_0(n/k))$ *or* $O_p(B_0(n/k))$, *the one of highest order. If* $\sqrt{k}A_0(n/k)/U_0(n/k)\to\infty$ *or* $\sqrt{k}/U_0^2(n/k)\to\infty$ *then* $\sqrt{k}U_0(n/k)\left(\overline{H}^{(q)}(k;\beta_q,\rho_q)-\xi\right)$ *is* $O_p(A_0(n/k))=O_p(1/U_0(n/k))$.

4.2 Estimation of β_q and ρ_q at the same high level $k_1^{(q)}$

If we estime β_q and ρ_q at the same high level $k_1^{(q)}$, as described in Section 3.1.1, we can state the following.

Theorem 4.2. *Under the third-order framework, as in* (1.7), *let us consider the tail index estimators,* $\overline{H}^{(q)}(k)\equiv\overline{H}^{(q)}(k;\hat\beta^{(q)},\hat\rho^{(q)})$, *given in* (1.21), *with* $\hat\rho^{(q)}$ *and* $\hat\beta^{(q)}$ *the pair of consistent estimators of the second-order parameters* ρ_q *and* β_q, *given in* (3.1) *and* (3.3), *respectively, both computed at the same high level* $k_1^{(q)}$ *such that* (3.2) *holds. Then, in* $\mathcal{R}_1$, *for levels* k *such that* $\sqrt{k}/U_0(n/k)\to\lambda'$, *finite, when* $n\to\infty$, *in* $\mathcal{R}_2$, *for levels* k *such that* $\sqrt{k}A_0(n/k)\to\lambda$, *finite, when* $n\to\infty$ *and in* $\mathcal{R}_3$, *for levels* k *such that* $\sqrt{k}/U_0(n/k)\to\lambda'$ *or* $\sqrt{k}A_0(n/k)\to\lambda$, *both finite, when* $n\to\infty$, *i.e., for levels* k *such that* $\sqrt{k}A_q(n/k)\to c$, *finite, as* $n\to\infty$, *then*

$$\sqrt{k}\left(\overline{H}^{(q)}(k)-\xi\right)\underset{n\to\infty}{\overset{d}{\longrightarrow}}\mathcal{N}(0,\xi^2).$$

We can guarantee the same limiting result in the three aforementioned regions $\mathcal{R}_1$, $\mathcal{R}_2$ *and* $\mathcal{R}_3$.

i) *In* $\mathcal{R}_1$, *consider levels* k *such that* $\sqrt{k}/U_0(n/k)\to\infty$, *provided that* $k=o(k_1^{(q)})$, *as* $n\to\infty$, *with* $k_1^{(q)}$ *optimal for the estimation of* ρ_q, *i.e., such that* $\lim_{n\to\infty}\sqrt{k_1^{(q)}}A_0(n/k_1^{(q)})=\lambda_1$ *and* $\lim_{n\to\infty}\sqrt{k_1^{(q)}}/U_0^2(n/k_1^{(q)})=\lambda_{U_1}$, *finite.*
Further considering levels k *such that* $\lim_{n\to\infty}\sqrt{k}A_0(n/k)=\lambda$ *and* $\lim_{n\to\infty}\sqrt{k}/U_0^2(n/k)=\lambda_U$, *both finite, assuming the validity of* $C_{U_1}^\star$ *for a level* $k_1^{(q)}=O(n/\ln\ln n)$ *and considering* $\hat\beta_{U_1}^{(q)}=\hat\beta^{(q)}(k_1^{(q)},\hat\rho_{U_1}^{(q)})$, *the tail index estimators* $\overline{H}^{(q)}(k;\hat\beta_{U_1}^{(q)},\hat\rho_{U_1}^{(q)})$ *have an asymptotic variance still equal to* ξ^2 *and asymptotic bias still given by* $b_{\mathcal{R}_1}^{(q)}:=\lambda_U y_q$, *with* y_q *given in* (4.1). *The same result holds for any* $\overline{H}^{(q)}(k;\hat\beta^{(q)},\hat\rho^{(q)})$ *provided that* $\hat\rho^{(q)}-\rho_q=o_p\Big(\ln(n/k)/\left(\sqrt{k}/U_0(n/k)\right)\Big)$.

ii) *In $\mathcal{R}_2$, consider levels k such that* $\sqrt{k}A_0(n/k) \to \infty$, *provided that* $k = o(k_1^{(q)})$, *as* $n \to \infty$, *with* $k_1^{(q)}$ *optimal for the estimation of* ρ_q, *i.e., such that* $\lim_{n\to\infty} \sqrt{k_1^{(q)}} A_0^2(n/k_1^{(q)}) = \lambda_{A_1}$ *and* $\lim_{n\to\infty} \sqrt{k_1^{(q)}} A_0(n/k_1^{(q)} B_0(n/k_1^{(q)}) = \lambda_{B_1}$, *finite.*
Further considering levels k *such that* $\lim_{n\to\infty} \sqrt{k}A_0^2(n/k) = \lambda_A$ *and* $\lim_{n\to\infty} \sqrt{k}A_0(n/k)B_0(n/k) = \lambda_B$, *both finite, assuming the validity of* $C^\star_{U_2}$ *for a level* $k_1^{(q)} = O(n/\ln\ln n)$ *and considering* $\hat\beta^{(q)}_{U_2} = \hat\beta^{(q)}(k_1^{(q)}, \hat\rho^{(q)}_{U_2})$, *the tail index estimators* $\overline{H}^{(q)}(k; \hat\beta^{(q)}_{U_2}, \hat\rho^{(q)}_{U_2})$ *have an asymptotic variance still equal to* ξ^2 *and asymptotic bias still given by* $b^{(q)}_{\mathcal{R}_2} := \lambda_A u_q + \lambda_B v$, *with* (u_q, v) *given in* (4.1). *The same result holds for any* $\overline{H}^{(q)}(k; \hat\beta^{(q)}, \hat\rho^{(q)})$ *provided that* $\hat\rho^{(q)} - \rho_q = o_p\Big(\ln(n/k)/\Big(\sqrt{k}A_0(n/k)\Big)\Big)$.

iii) *In $\mathcal{R}_3$, consider levels* k *such that* $\sqrt{k}A_0(n/k) \to \infty$ *or* $\sqrt{k}/U_0(n/k) \to \infty$, *provided that* $k = o(k_1^{(q)})$, *as* $n \to \infty$, *with* $k_1^{(q)}$ *optimal for the estimation of* ρ_q, *i.e., such that* $\lim_{n\to\infty} \sqrt{k_1^{(q)}} A_0^2(n/k_1^{(q)}) = \lambda_{A_1}$, $\lim_{n\to\infty} \sqrt{k_1^{(q)}} A_0(n/k_1^{(q)}) B_0(n/k_1^{(q)}) = \lambda_{AB_1}$, $\lim_{n\to\infty} \sqrt{k_1^{(q)}} A_0(n/k_1^{(q)}) U_0(n/k_1^{(q)}) = \lambda_{AU_1}$, *and* $\lim_{n\to\infty} \sqrt{k_1^{(q)}}/U_0^2(n/k_1^{(q)}) = \lambda_{U_1}$, *finite.*
Further considering levels k such that $\lim_{n\to\infty} \sqrt{k}A_0^2(n/k) = \lambda_A$, $\lim_{n\to\infty} \sqrt{k}A_0(n/k)B_0(n/k) = \lambda_{AB}$, $\lim_{n\to\infty} \sqrt{k}A_0(n/k)U_0(n/k) = \lambda_{AU}$, *and* $\lim_{n\to\infty} \sqrt{k}/U_0^2(n/k) = \lambda_U$, *all finite, assuming the validity of* $C^\star_{U_3}$ *for a level* $k_1^{(q)} = O(n/\ln\ln n)$ *and further considering* $\hat\beta^{(q)}_{U_3} = \hat\beta^{(q)}(k_1^{(q)}, \hat\rho^{(q)}_{U_3})$, *the tail index estimators* $\overline{H}^{(q)}(k; \hat\beta^{(q)}_{U_3}, \hat\rho^{(q)}_{U_3})$ *have an asymptotic variance still equal to* ξ^2 *and asymptotic bias still given by* $b^{(q)}_{\mathcal{R}_3} := \lambda_A u_q + \lambda_B v + \lambda_{AU} w_q + \lambda_U y_q$, *with* (u_q, v, w_q, y_q) *given in* (4.1). *The same result holds for any* $\overline{H}^{(q)}(k; \hat\beta^{(q)}, \hat\rho^{(q)}))$ *provided that* $\hat\rho^{(q)} - \rho_q = o_p\Big(\ln(n/k)/\Big(\sqrt{k}A_0(n/k)\Big)\Big)$.

5 Analysis of the performance of the estimators – an application to real data

To enhance the importance of the PORT-MVRB EVI-estimation, we shall consider in this section an application to the analysis of the positive log-returns $P_i = \ln(S_{i+1}/S_i) = -L_i$, $1 \le i \le n-1$, with S_i the set of financial data. Such data, collected over the period from January 4, 1999, until November 17, 2005, and with a size n = 1762, are the daily exchange rates of the Euro-USA Dollar (EUSD), considered in Gomes *et al.* (2007a). Additionally, we have considered over the same period the Euro-GB Pound (EGBP) daily exchange rates, one of the data sets analyzed in Gomes and Pestana (2007a) and in Gomes *et al.* (2011b; 2013), through the use of different algorithms. Although there is some increasing trend in the volatility of all these log-returns, stationarity and weak dependence is often assumed, under the same considerations as in Drees (2003). As stated in the aforementioned papers, the underlying model has heavy left and right tails leading us to eliminate the estimators associated with $q = 0$, due to their inconsistency (see Gomes *et al.*, 2008b, for details).

In Tables 1 and 2 we present for the classical and PORT-estimation with $q = 0.1(0.1)0.4$, the number of positive elements in the available sample of log-returns, n^+, the levels $k_1 = \lfloor (n^+)^{0.999} \rfloor$ used to obtain $(\hat{\rho}, \hat{\beta})$, the estimates of the second-order parameters ρ and β for the EUSD and EGBP data sets, respectively. The ρ-estimates were obtained considering $\tau = 0$ in (2.4) and $\tau_q = 0$ in (3.1). For the β-estimates we have considered $\eta_q = -1.2$ in (3.3) and we have also used the same η-value for the classical β-estimation. In Figures 1 and 2 we present the sample path of the second-order estimators under play.

Method of Estimation	n^+	k_1	$\hat{\rho}^{(q)}$	$\hat{\beta}^{(q)}$
Classical	867	861	-0.707	1.034
PORT with $q = 0.1$	1585	1573	-0.717	1.039
PORT with $q = 0.2$	1409	1398	-0.729	1.039
PORT with $q = 0.3$	1233	1224	-0.719	1.038
PORT with $q = 0.4$	1057	1049	-0.729	1.038

Table 1: Sample sizes, levels k_1 and estimates of the second-order parameters for the EUSD data set

Remark 5.1. *We have also computed the estimates of the* PORT *version of the β-estimator introduced in* Gomes and Martins (2002)*, given in* (2.6)*, and we have got an absolute difference between the two estimates equal to* 10^{-3} *for the classical estimation and in the range* $[7 \times 10^{-3}, 1.5 \times 10^{-3}]$*, for the* PORT*-estimation.*

Method of Estimation	n^+	k_1	$\hat\rho^{(q)}$	$\hat\beta^{(q)}$
Classical	835	829	-0.667	1.030
PORT with $q = 0.1$	1585	1573	-0.724	1.039
PORT with $q = 0.2$	1409	1398	-0.711	1.039
PORT with $q = 0.3$	1233	1224	-0.722	1.037
PORT with $q = 0.4$	1057	1049	-0.718	1.037

Table 2: Sample sizes, levels k_1 and estimates of the second-order parameters for the EGBP data set

The adaptive estimates of the EVI were obtained on the basis of a heuristic sample-path stability algorithm similar to the one presented in Gomes *et al.* (2013) and described in the following:

1. Given an observed sample $(x_1, x_2, \ldots, x_n)$, consider, for $q = 0.1(0.1)0.4$ and $q = 1$, the observed sample of excesses, $\underline{\mathbf{x}}_n^{(q)}$, with $\underline{\mathbf{X}}_n^{(q)}$ given in (1.16), and compute $\hat\rho \equiv \hat\rho^{(1)} \equiv \hat\rho^{(0,1)} = \hat\rho^{(0)}(k_1; \underline{\mathbf{x}})$ and $\hat\beta \equiv \hat\beta^{(1)} \equiv \hat\beta^{(\eta,1)} := \hat\beta^{(\eta,1)}(k_1, \hat\rho)$, $\hat\rho(k)$ and $\hat\beta(k)$ respectively given in (2.4) and (3.3), and $k_1 = \lfloor (n^+ - 1\rfloor^{0.999}\rceil$, and compute $\hat\rho^{(q)} \equiv \hat\rho^{(0,q)} = \hat\rho^{(0)}(k_1^{(q)}; \underline{\mathbf{x}}_n^{(q)})$ and $\hat\beta^{(q)} \equiv \hat\beta^{(\eta_q,q)} := \hat\beta^{(\eta_q,q)}(k_1^{(q)}, \hat\rho^{(q)})$, $\hat\rho^{(q)}(k)$, $\hat\beta(q)(k)$ given in (3.1), (3.3), respectively and $k_1^{(q)} = \left\lfloor \left(n^{(q)} - 1\right)^{0.999} \right\rfloor$, $n^{(q)} = n - \lfloor nq \rfloor - 1$ ($n^{(1)} = n$, by notation).

2. Next compute, for $k = 1, 2, \ldots, n^{(q)} - 1$, the observed values of $\overline{H}^{(q)}(k)$, given in (1.21), and for $k = 1, 2, \ldots, n^+ - 1$, compute the observed values of $\overline{H}(k) \equiv \overline{H}^{(1)}(k)$.

3. Obtain j_0, the minimum value of j, a non-negative integer, such that $a_k^{(q)}(j)$=round$(\overline{H}^{(q)}(k), j)$, $k = 1, 2, ..., n^{(q)} - 1$, has distinct elements (in this case we were led to $j_0 = 1$ for the two data sets considered).

4. Choose q in the following way: for each q consider as possible estimates of ξ the values $a_k^{(q)}(j_0) \equiv a^{(q)}(k)$, $k_{min}^{(q)} \le k \le k_{max}^{(q)}$, to which is associated the largest run, with a size $l_q = k_{max}^{(q)} - k_{min}^{(q)} + 1$. Choose $q^\star := \arg\max_q l_q$.

5. Consider all those estimates, $\overline{H}^{(q^\star)}(k)$, $k_{min}^{(q^\star)} \le k \le k_{max}^{(q^\star)}$, now with an extra decimal, i.e. $\overline{H}^{(q^\star)}(k) = a_k^{(q^\star)}(j_0 + 1)$. Count the frequencies associated with these estimates and

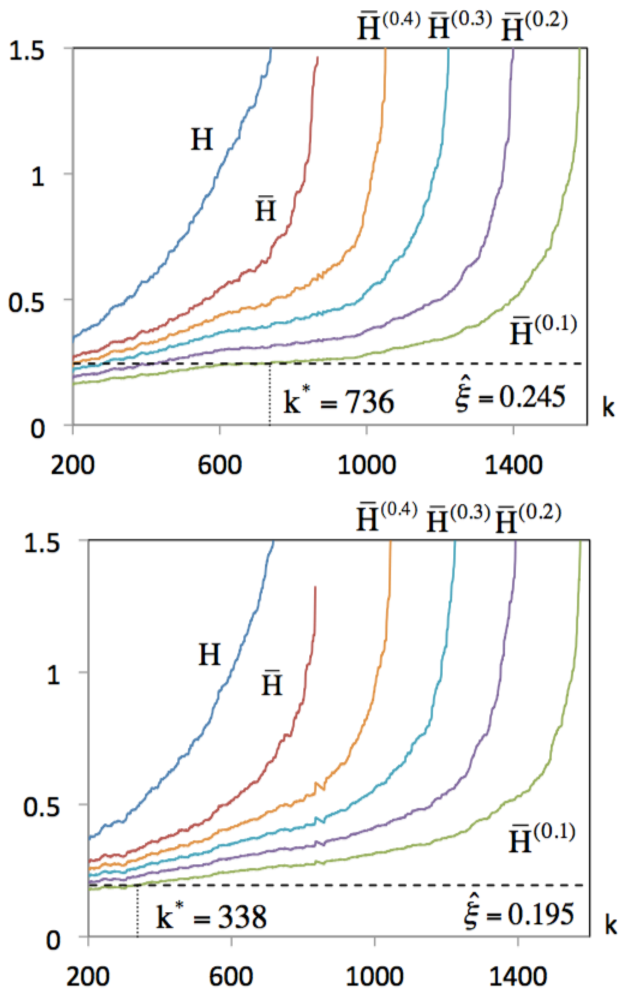

Figure 1: Estimates of the second-order parameters, ρ_q (*left*) and β_q (*right*) for the EUSD data

obtain the mode, ζ, of these values. Let us denote $\mathcal{K}^*$ the set of k-values corresponding to those estimates.

6. Take k^* as the maximum of $\mathcal{K}^*$ (in order to minimize the variance) and the adaptive EVI-estimate $\hat{\xi} = \overline{H}^{(q^\star)}(k^*)$.

On the basis of the asymptotic behaviour of the estimators under consideration, approximate confidence intervals (CI's) for ξ can be easily obtained (see, for instance, Gomes and Pestana, 2007b).

Remark 5.2. *If we consider a second-order* PORT-MVRB EVI-*estimator, and levels k such*

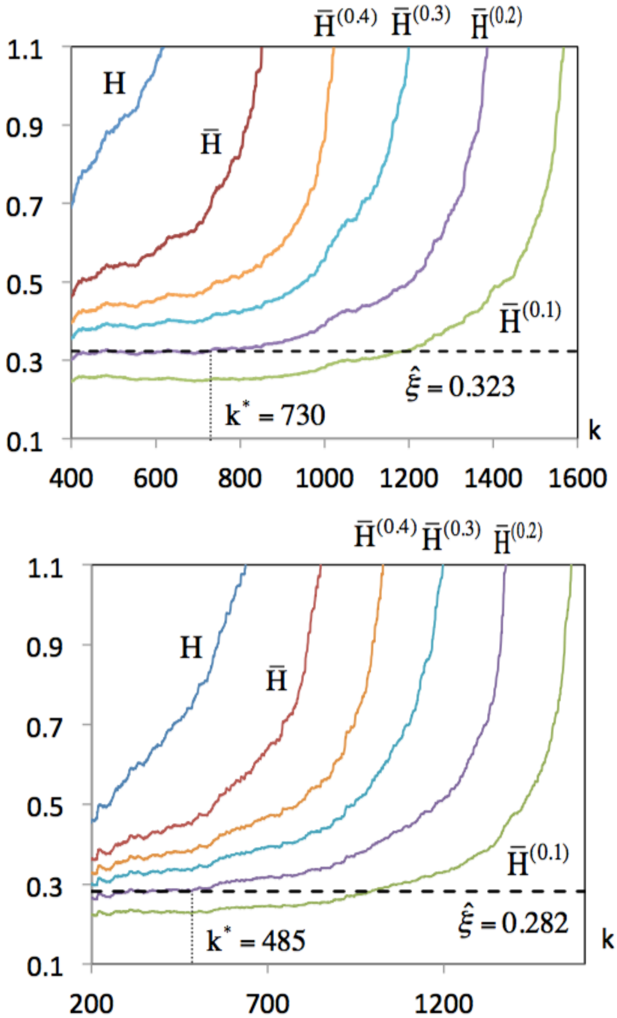

Figure 2: Estimates of the second-order parameters, ρ_q (*left*) and β_q (*right*) for the EGBP data

that $\sqrt{k}\ A_q(n/k) \to \lambda$, *finite, we can also easily get an approximate* $100(1-\alpha)\%$ CI *for* ξ. *On the basis of the statistic* $\overline{H}$ *in* (1.15), *or even* $\overline{H}^{(q)}$, *in* (1.21), *for adequate values of* τ_q, η_q *and* q *and for the same* k*-levels, we get the following* $100(1-\alpha)\%$ *approximate* CI *for* ξ,

$$\left(\frac{\overline{H}^{(q)}(k)}{1+\frac{z_{1-\alpha/2}}{\sqrt{k}}},\ \frac{\overline{H}^{(q)}(k)}{1-\frac{z_{1-\alpha/2}}{\sqrt{k}}}\right), \tag{5.1}$$

with z_α *the* α*-quantile of the standard normal distribution.*

In Figure 3 we present the non-adaptive estimates of ξ, provided by H, $\overline{H}$ and $\overline{H}^{(q)}$, with

$q = 0.1, 0.2, 0.3, 0.4$ and the adaptive EVI-estimates for the two data sets.

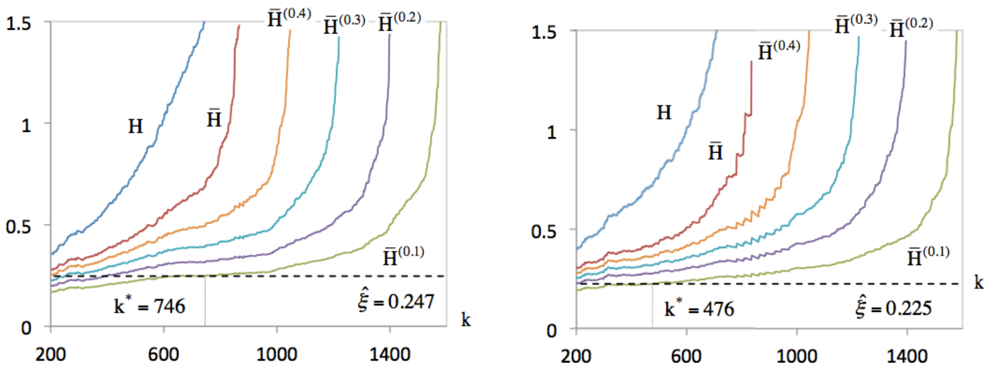

Figure 3: EVI-estimates for the EUSD data (*left*) and for the EGBP data (*right*)

- For the EUSD data set we were led in Step **4.** of the *Algorithm* to the choice of $q^* = 0.1$ associated with a run of size 712 for a ξ-estimate equal to 0.2. In Step **5.** we have got a mode $\zeta = 135$ associated with the ξ-estimate 0.25, $k^* = 746$ and the adaptive PORT-MVRB EVI-estimate $\hat{\xi} = 0.247$. The associated 99% MVRB- CI for ξ, based upon (5.1), is $(0.226, 0.273)$, with a size equal to 0.047.
- For the EGBP data set we were also led in Step **4.** of the *Algorithm* to the choice of $q^* = 0.1$ associated with a run of size 659 for a ξ-estimate equal to 0.2. In Step **5.** we have got a mode $\zeta = 162$ associated with the ξ-estimate 0.22, $k^* = 476$ and the adaptive PORT-MVRB EVI-estimate $\hat{\xi} = 0.225$. The associated 99% MVRB- CI for ξ, based upon (5.1), is $(0.201, 0.255)$, with a size equal to 0.054. Note that in Gomes *et al.* (2013) the authors were led to the same adaptive estimate and the same 99% MVRB- CI.

Further note that the choice of the tuning parameters (k, q) could be done by bootstrap algorithms of the type of the one in Gomes *et al.* (2016a) (see also Gomes *et al.*, 2011b, 2012, Brilhante *et al.*, 2013, and Caeiro and Gomes, 2015, where R-scripts are provided).

6 Final remarks

1. It is known, from a classical point of view, that the sample paths of the β-estimator in (2.6) are less volatile than the ones of the β-estimator introduced in Caeiro and Gomes

(2006). However, for a suitably chosen value of η the later estimator exhibits for large values of k, a quite stable sample path around the true value of β. This type of behavior is also true for the correspondent PORT-versions of the β-estimators and given an adequate choice of the *tuning* parameters k, η_q and τ_q the estimates obtained in (3.3) are quite close to the ones obtained by the PORT-version of the β-estimator in (2.6).

2. The proximity between the estimates of the two PORT-β estimators allow us to conclude that the finite sample behavior of the PORT-MVRB EVI-estimators under study is quite similar to the class of PORT-MVRB EVI-estimators studied in Gomes *et al.* (2011a), among others.
3. The use of the PORT-β estimators in (3.3), concomitantly with the class of PORT-ρ estimators in (3.1) enable us to overcome the technicality of this subject and derive the non-degenerate behavior of the class of PORT-MVRB EVI-estimators in (1.21).
4. We can now derive the asymptotic behavior of PORT-MVRB VaR-estimators as well as of other classes of RB location invariant estimators of the EVI, topics out of the scope of this article, but which are straightforward for some recently introduced classes of MVRB EVI-estimators, just as mentioned in Remark 2.1.

7 Asymptotic behaviour of the PORT-estimators of second-order parameters

7.1 The PORT-ρ estimators

The asymptotic non-degenerate behaviour of the PORT-ρ_q-estimators, defined in (3.1), is provided in the following theorem.

Theorem 7.1 (Henriques-Rodrigues *et al.*, 2014). *Under the validity of the third-order condition in* (2.10)*, with* $\rho = \rho_0$*,* $\rho' = \rho_0' < 0$*, we have the validity of the following asymptotic distributional representation for the* PORT-ρ *estimator,* $\widehat{\rho}^{(\tau_q,q)}(k)$*, in* (3.1)*.*

i) *In* $\mathcal{R}_1$*, let us consider the regions* $\mathcal{R}_{11} := \{\rho_0 < -2\xi \wedge \chi_q \neq 0\}$*,* $\mathcal{R}_{12} := \{\rho_0 = -2\xi \wedge \chi_q \neq 0\}$

and $\mathcal{R}_{13} := \{-2\xi < \rho_0 < -\xi \wedge \chi_q \neq 0\}$,

$$\begin{aligned}
\widehat{\rho}^{(\tau_q,q)}(k) &\stackrel{d}{=} \rho_q + \frac{\overset{\bullet}{\sigma}_{\rho_0,q}}{\sqrt{k}A_q(n/k)} W_k^{R_1} \\
&+ \begin{cases}
\dfrac{\chi_q\ y_{\rho_0}(\xi,\tau_q)}{U_0(n/k)}(1+o_p(1)), & in \quad \mathcal{R}_{11}, \\
\left(\dfrac{z_{\rho_0}(\xi,\rho_0)A_0(n/k)U_0(n/k)}{\chi_q} + \dfrac{\chi_q\ y_{\rho_0}(\xi,\tau_q)}{U_0(n/k)}\right)(1+o_p(1)), & in \quad \mathcal{R}_{12}, \\
\dfrac{z_{\rho_0}(\xi,\rho_0)A_0(n/k)U_0(n/k)}{\chi_q}(1+o_p(1)), & in \quad \mathcal{R}_{13},
\end{cases}
\end{aligned}$$

where $W_k^{R_1}$ *is asymptotically standard normal,*

$$y_{\rho_0}(\xi,\tau) = \frac{6\xi\left(-4+\xi\left(-13+2\xi\left(-3+2\xi(2+\xi)^2\right)\right)\right) - \xi(3+\xi)(1+2\xi)^3(3+2\xi)\tau}{12(1+\xi)^2(1+2\xi)^3}$$

$$z_{\rho_0}(\xi,\rho_0) = -\frac{(1+\xi)^3\rho_0(\xi+\rho_0)}{\xi^2(1-\rho_0)^3}$$

and $\overset{\bullet}{\sigma}{}^2_{\rho_0,q} = (1+\xi)^6\left(2\xi^2+2\xi+1\right)$.

ii) *In* $\mathcal{R}_2$, *let us consider the regions* $\mathcal{R}_{21} := \{-\xi < \rho_0 < -\frac{\xi}{2} \wedge \chi_q \neq 0\}$, $\mathcal{R}_{22} := \{\rho_0 = -\frac{\xi}{2} \wedge \chi_q \neq 0\}$ *and* $\mathcal{R}_{23} := \{\frac{\xi}{2} < \rho_0 < 0 \vee (\xi > -\rho_0 \wedge \chi_q = 0)\}$,

$$\begin{aligned}
\widehat{\rho}^{(\tau_q,q)}(k) &\stackrel{d}{=} \rho_q + \frac{\sigma_{\rho_0,q}}{\sqrt{k}A_q(n/k)} W_k^{R_2} \\
&+ \begin{cases}
\left(\dfrac{\chi_q f_{\rho_0}(\xi,\rho_0)}{A_0(n/k)U_0(n/k)}\right)(1+o_p(1)), & in \quad \mathcal{R}_{21}, \\
\left(m_{\rho_0,\rho_0'} + \dfrac{\chi_q f_{\rho_0}(\xi,\rho_0)}{A_0(n/k)U_0(n/k)}\right)(1+o_p(1)), & in \quad \mathcal{R}_{22}, \\
m_{\rho_0,\rho_0'}(1+o_p(1)), & in \quad \mathcal{R}_{23},
\end{cases}
\end{aligned}$$

where $m_{\rho,\rho'} = u_\rho A_0(n/k) + v_{\rho,\rho'}B_0(n/k)$, *with* $u_\rho \equiv u$ *and* $v_{\rho,\rho'} \equiv v$, *and* (u,v) *given in* (2.3), *respectively. Moreover,* $W_k^{R_2}$ *is asymptotically standard normal,*

$$\begin{aligned}
\sigma^2_{\rho_0,q} &\equiv \sigma^2_{\rho_0} = \xi^2(1-\rho_0)^6\left(2\rho_0^2 - 2\rho_0 + 1\right)/{\rho_0}^2, \\
f_{\rho_0}(\xi,\rho_0) &= \frac{\xi^2(1-\rho_0)^3(\xi+\rho_0)}{(1+\xi)^3\rho_0}.
\end{aligned}$$

iii) *In* $\mathcal{R}_3$, *and with* $\widetilde{\lambda} = \lim_{n\to\infty} 1/\big(A_0(n/k)U_0(n/k)\big) \neq 0$,

$$\begin{aligned}
\widehat{\rho}^{(\tau_q,q)}(k) &\stackrel{d}{=} \rho_q + \tfrac{\widetilde{\sigma}_{\rho_0,q}}{\sqrt{k}A_q(n/k)} W_k^{R_3} \\
&+ \left(\widetilde{u}_{\rho_0}A_0(n/k) + \widetilde{v}_{\rho_0,\rho_0'}B_0(n/k) + \tfrac{\xi\chi_q\left(\widetilde{g}_{\rho_0}(\xi,\rho_0) + \widetilde{\lambda}\chi_q\widetilde{w}_{\rho_0}(\xi,\rho_0,\tau_q)\right)}{U_0(n/k)}\right)(1+o_p(1)),
\end{aligned}$$

where $W_k^{R_3}$ *is an asymptotically standard normal r.v.,* $\widetilde{u}_\rho = u_\rho/(1+\xi\widetilde{\lambda}\chi_q)$*,* $\widetilde{v}_{\rho,\rho'} = v_{\rho,\rho'}/(1+\xi\widetilde{\lambda}\chi_q)$*, with* $u_\rho \equiv u$ *and* $v_{\rho,\rho'} \equiv v$*, and* (u,v) *defined in* (2.3)*, respectively, and* $\widetilde{\bullet}_{\rho_0} = \bullet_{\rho_0}/(1+\xi\widetilde{\lambda}\chi_q)$*, with* $\bullet = g, w$*, and*

$$\begin{aligned} g_{\rho_0}(\xi,\rho_0) = g_{\rho_0}(-\rho_0,\rho_0) \;&\equiv\; g_{\rho_0}(\rho_0) \\ &= -\frac{6\left(4+\rho_0\left(-13+2\rho_0\left(3+2\rho_0(2-\rho_0)^2\right)\right)\right)+(3-\rho_0)(3-2\rho_0)(1-2\rho_0)^3\tau}{6(1-\rho_0)^2(1-2\rho_0)^3}, \end{aligned}$$

$$w_{\rho_0}(\xi,\rho_0,\tau) = w_{\rho_0}(-\rho_0,\rho_0,\tau) \;\equiv\; w_{\rho_0}(\rho_0,\tau) = \tfrac{(3-\rho_0)(1-\rho_0)^3}{2\rho_0} b(\rho_0,\tau),$$

$$\begin{aligned} b(\rho,\tau) = &-\tfrac{(\rho-2)^2(\tau-2)}{4(1-\rho)^4} + \tfrac{\tau-1}{(1-\rho)^2} - \tfrac{2(1-\rho)}{(1-2\rho)^2} + \tfrac{2}{1-2\rho} - \tfrac{1}{1-\rho(3-2\rho)} \\ &+ \tfrac{(1-\rho)\rho\left\{-(\rho+3)(5\rho(\rho+3)+12)(2\rho+1)^3\tau - 6(6+\rho(3+2\rho)(4\rho^5+24\rho^4+42\rho^3+31\rho^2+14\rho+9))\right\}}{12(3-\rho)(1+\rho)^6(1+2\rho)^3} \end{aligned}$$

and $\widetilde{\sigma}^2_{\rho_0,q} = (1-\rho_0)^6\left(2\,{\rho_0}^2 - 2\,\rho_0 + 1\right)$*.*

7.2 The PORT-β estimators

The asymptotic behaviour of the PORT-β estimators, defined in (3.3), is presented in the next theorem.

Theorem 7.2 (Henriques-Rodrigues *et al.*, 2015)**.** *Under the validity of the third-order condition in* (2.10)*, with* $\rho_0,\ \rho_0' < 0$*, we have the validity of the following asymptotic distributional representation for* $\widehat{\beta}^{(\eta_q,q)}(k;\rho_q)$*, with* $\widehat{\beta}^{(\eta_q,q)}(k;\hat{\rho}^{(q)})$ *the* PORT-β *estimator in* (3.3)*.*

i) *In* $\mathcal{R}_1$*, with* $\beta_q = \chi_q/C$*,* C *given in* (1.4)*, and with the same notation as before for* $\mathcal{R}_{11}$*,* $\mathcal{R}_{12}$ *and* $\mathcal{R}_{13}$*,*

$$\widehat{\beta}^{(\eta_q,q)}(k;\rho_q) \;\stackrel{d}{=}\; \beta_q + \frac{\beta_q \overset{\bullet}{\sigma}_{\rho_q}}{\sqrt{k}A_q(n/k)} W_k^{R_1} + \beta_q V^{R_1}(1+o_p(1)), \tag{7.1}$$

where $W_k^{R_1}$ *is asymptotically standard normal and* V^{R_1}

$$V^{R_1} := \begin{cases} \frac{\chi_q y_{\beta_0}(-\xi,\xi)}{U_0(n/k)}, & \text{in } \mathcal{R}_{11}, \\ \frac{z_{\beta_0}(\rho_0,-\xi)}{\xi\chi_q} A_0(n/k)U_0(n/k) + \frac{\chi_q y_{\beta_0}(-\xi,\xi)}{U_0(n/k)}, & \text{in } \mathcal{R}_{12}, \\ \frac{z_{\beta_0}(\rho_0,-\xi)}{\xi\chi_q} A_0(n/k)U_0(n/k), & \text{in } \mathcal{R}_{13}, \end{cases} \tag{7.2}$$

with

$$A_\xi := 32\xi^8 + 212\xi^7 + 568\xi^6 + 780\xi^5 + 538\xi^4 + 93\xi^3 - 108\xi^2 - 68\xi - 12 \quad (7.3)$$

$$B_\xi := 6 + \xi(4+\xi) \quad (7.4)$$

$$C_{\xi,\rho} := \rho^3(21 + 5\xi B_\xi) - 2\rho^2(30 + 7\xi B_\xi) + 6\rho(9 + 2\xi B_\xi) - 2(6 + \xi B_\xi) \quad (7.5)$$

$$y_{\beta_0}(\xi, -\xi) = \frac{\xi(2+\xi)(8+18\xi+16\xi^2+5\xi^3)(1+2\xi)^4\eta+2\xi^2 A_\xi}{2(1+\xi)^2(2+\xi)^2(1+2\xi)^4(2+2\xi+\xi^2)}$$

$$z_{\beta_0}(\rho_0, -\xi) = -\frac{(1+\xi)^5(2-\rho_0)C_{\xi,\rho_0}}{(2+\xi)^2(2+2\xi+\xi^2)(3+3\xi+\xi^2)(1-\rho_0)^6}$$

and $\overset{\bullet}{\sigma}{}^2_{\rho_q} \equiv \overset{\bullet}{\sigma}{}^2_{-\xi} = \left(\frac{1+\xi}{2+\xi}\right)^2 (21\xi^4 + 68\xi^3 + 86\xi^2 + 68\xi + 33)$.

ii) *In* $\mathcal{R}_2$*, with* $\beta_q = \beta_0$*, and again with the same notation as before for* $\mathcal{R}_{21}$*,* $\mathcal{R}_{22}$ *and* $\mathcal{R}_{23}$*,*

$$\widehat{\beta}^{(\eta_q,q)}(k;\rho_q) \overset{d}{=} \beta_q + \frac{\beta_q \sigma_{\rho_q}}{\sqrt{k}A_q(n/k)} W_k^{R_2} + \beta_q V^{R_2}(1 + o_p(1)), \quad (7.6)$$

with V^{R_2} *given by*

$$V^{R_2} := \begin{cases} \frac{\xi \chi_q z_{\beta_0}(-\xi,\rho_0)}{A_0(n/k)U_0(n/k)}, & \text{in } \mathcal{R}_{21}, \\ \mu(\rho_0, \rho_0') + \frac{\xi \chi_q z_{\beta_0}(-\xi,\rho_0)}{A_0(n/k)U_0(n/k)}, & \text{in } \mathcal{R}_{22}, \\ \mu(\rho_0, \rho_0'), & \text{in } \mathcal{R}_{23}, \end{cases} \quad (7.7)$$

where $\mu(\rho,\rho') \equiv \mu_{\beta_0}(\rho,\rho') = u_{\beta_0}(\rho)A_0(n/k) + v_{\beta_0}(\rho,\rho')B_0(n/k)$*, with* u_{β_0} *and* v_{β_0}*, given by*

$$u_{\beta_0}(\rho) \equiv u_{\beta_0}(\eta_q,\rho) = -\frac{2\rho^2\Big(4-\rho(2-\rho)\big(16\rho^5-68\rho^4+116\rho^3-96\rho^2+33\rho+2\big)\Big)}{2\xi(2-\rho)^2(1-\rho)^2(1-2\rho)^3(2-2\rho+\rho^2)} - \frac{\eta_q(2-\rho)(1-2\rho)^3\big(5\rho^3-16\rho^2+18\rho-8\big)}{2\xi(2-\rho)^2(1-\rho)^2(1-2\rho)^3(2-2\rho+\rho^2)} \quad (7.8)$$

and

$$v_{\beta_0}(\rho,\rho') = \frac{(1-\rho)^3\big((2-\rho)^2(1-\rho)^3(2-2\rho+\rho^2)-2(2-\rho)^2(1-\rho)^2(5-4\rho+2\rho^2)\rho'\big)}{(2-\rho)^2(2-2\rho+\rho^2)(1-(\rho+\rho'))^6} + \frac{(1-\rho)^3\big(2(1-\rho)(29-55\rho+44\rho^2-18\rho^3+3\rho^4)\rho'^2-2(17-35\rho+29\rho^2-12\rho^3+2\rho^4)\rho'^3\big)}{(2-\rho)^2(2-2\rho+\rho^2)(1-(\rho+\rho'))^6} - \frac{(1-\rho)^3(-7+9\rho-5\rho^2+\rho^3)\rho'^4}{(2-\rho)^2(2-2\rho+\rho^2)(1-(\rho+\rho'))^6}, \quad (7.9)$$

respectively. Moreover, $W_k^{R_2}$ *is asymptotically standard normal,*

$$\begin{aligned}\sigma^2_{\rho_q} &\equiv \sigma^2_{\rho_0} = \left(\tfrac{\xi(1-\rho_0)}{2-\rho_0}\right)^2\left(\tfrac{21\rho_0^4-68\rho_0^3+86\rho_0^2-68\rho_0+33}{\rho_0^2}\right), \qquad (7.10)\\ z_{\beta_0}(-\xi,\rho_0) &= -\tfrac{(1-\rho_0)^5(2+\xi)C_{-\rho_0,\xi}}{(2-\rho_0)^2(2-2\rho_0+\rho_0^2)(3-3\rho_0+\rho_0^2)(1+\xi)^6},\end{aligned}$$

with B_ξ *and* $C_{\xi,\rho}$ *defined in* (7.4) *and* (7.5)*, respectively.*

iii) *In* $\mathcal{R}_3$*, with* $\beta_q = \beta_0 + \chi_q/C$*,* $\widetilde{\lambda} = \lim_{n\to\infty} 1/\big(A_0(n/k)U_0(n/k)\big) = (\xi\beta_0 C)^{-1} \neq 0$*, and* C *given in* (1.4)

$$\widehat{\beta}^{(\eta_q,q)}(k;\rho_q) \overset{d}{=} \beta_q + \frac{\beta_q\widetilde{\sigma}_{\rho_q}}{\sqrt{k}A_q(n/k)}W_k^{R_3} + \beta_q V^{R_3}(1+o_p(1)), \qquad (7.11)$$

where $W_k^{R_3}$ *is an asymptotically standard normal r.v. with* $\widetilde{\sigma}^2_{\rho_q} = \sigma^2_{\rho_q}$*, with* $\sigma^2_{\rho_q}$ *given in* (7.10)*, and* V^{R_3} *given by*

$$V^{R_3} := \frac{1}{1+\xi\widetilde{\lambda}\chi_q}\Bigg(u_{\beta_0}(\rho_0)A_0(n/k) + v_{\beta_0}(\rho_0,\rho_0')B_0(n/k) + \frac{\xi\chi_q\Big(w_{\beta_0}(\xi,\rho_0) + \chi_q\widetilde{\lambda}\; y_{\beta_0}(\xi,\rho_0)\Big)}{U_0(n/k)}\Bigg), \qquad (7.12)$$

with u_{β_0} *and* v_{β_0}*, defined in* (7.8) *and* (7.9)*, respectively, and*

$$\begin{aligned} w_{\beta_0}(\xi,\rho_0) = w_{\beta_0}(-\rho_0,\rho_0) &= -\tfrac{(-8+18\rho_0-16\rho_0^2+5\rho_0^3)\eta_q}{(2-\rho_0)(1-\rho_0)^2(2-2\rho_0+\rho_0^2)}\\ &\quad + \tfrac{2\rho_0\big(14-\rho_0(91-221\rho_0+216\rho_0^2+25\rho_0^3-252\rho_0^4+240\rho_0^5-100\rho_0^6+16\rho_0^7)\big)}{(1-2\rho_0)^4(2-\rho_0)^2(1-\rho_0)^2(2-2\rho_0+\rho_0^2)},\\ y_{\beta_0}(\xi,\rho_0) = y_{\beta_0}(-\rho_0,\rho_0) &= \tfrac{-\rho_0(2-\rho_0)(8-18\rho_0+16\rho_0^2-5\rho_0^3)(1-2\rho_0)^4\eta_q+2\rho_0^2A_{-\rho_0}}{2(1-\rho_0)^2(2-\rho_0)^2(1-2\rho_0)^4(2-2\rho_0+\rho_0^2))},\end{aligned}$$

and A_ξ *defined in* (7.3)*.*

Corollary 7.1. *Under the conditions of* **Theorem 7.2** *and assuming the validity of condition* $C^\star_{U_\bullet}$ *for a* $k_1^{(q)} = O(n/\ln\ln n)$ *these results are kept for* $\widehat{\beta}^{(\eta_q,q)}(k;\hat{\rho}^{(q)}_{U_\bullet})$*,* $\bullet = 1,2,3$*. If we consider* $\hat{\beta}^{(\eta_q,q)}(k;\hat{\rho}^{(\tau_q,q)}(k))$*, with* $\hat{\rho}^{(\tau_q,q)}(k)$ *given in* (3.1)*, the rate of convergence towards* β_q *is of the order of* $\ln(n/k)/(\sqrt{k}A_q(n/k))$*, in* $\mathcal{R}_1$*,* $\mathcal{R}_2$ *and* $\mathcal{R}_3$*, which must converge to zero. Moreover,*

$$\hat{\beta}^{(\eta_q,q)}(k;\hat{\rho}^{(\tau_q,q)}(k)) - \beta_q \overset{p}{\sim} -\beta_q\ln(n/k)(\hat{\rho}^{(\tau_q,q)}(k) - \rho_q). \qquad (7.13)$$

i) *In* $\mathcal{R}_1$*, if* $\sqrt{k}A_q(n/k)/\ln(n/k) \to \infty$*,* $\sqrt{k}A_0(n/k) \to \lambda$ *and* $\sqrt{k}/U_0^2(n/k) \to \lambda_U$*, both finite, then*

$$\sqrt{k}A_q(n/k)\frac{\beta_q-\hat\beta^{(\eta_q,q)}(k;\hat\rho^{(\tau_q,q)}(k))}{\beta_q\ln(n/k)} \overset{a}{\sim} \begin{cases} \mathcal{N}\left(\chi_q\ y_{\rho_0}(\xi,\tau_q)\lambda_U, \overset{\bullet}{\sigma}{}^2_{\rho_0,q}\right), & \text{in } \mathcal{R}_{11}, \\ \mathcal{N}\left(\frac{z_{\rho_0}(\xi,\rho_0)}{\chi_q}\lambda + \chi_q y_{\rho_0}(\xi,\tau_q)\lambda_U, \overset{\bullet}{\sigma}{}^2_{\rho_0,q}\right), & \text{in } \mathcal{R}_{12}, \\ \mathcal{N}\left(\frac{z_{\rho_0}(\xi,\rho_0)}{\chi_q}\lambda, \overset{\bullet}{\sigma}{}^2_{\rho_0,q}\right), & \text{in } \mathcal{R}_{13}, \end{cases}$$

with $y_{\rho_0}(\xi,\tau_q)$*,* $z_{\rho_0}(\xi,\rho_0)$ *and* $\overset{\bullet}{\sigma}{}^2_{\rho_0,q}$ *given in Step* i) *of* Theorem 7.1.

ii) *In* $\mathcal{R}_2$*, if* $\sqrt{k}A_q(n/k)/\ln(n/k) \to \infty$*,* $\sqrt{k}A_0^2(n/k) \to \lambda_A$*,* $\sqrt{k}A_0(n/k)B_0(n/k) \to \lambda_B$ *and* $\sqrt{k}/U_0(n/k) \to \lambda'$*, all finite, then*

$$\sqrt{k}A_q(n/k)\frac{\beta_q-\hat\beta^{(\eta_q,q)}(k;\hat\rho^{(\tau_q,q)}(k))}{\beta_q\ln(n/k)} \overset{a}{\sim} \begin{cases} \mathcal{N}\left(\chi_q f_{\rho_0}(\xi,\rho_0)\lambda', \sigma^2_{\rho_0,q}\right), & \text{in } \mathcal{R}_{21}, \\ \mathcal{N}\left(m^\star_{\rho_0,\rho_0'} + \chi_q f_{\rho_0}(\xi,\rho_0)\lambda', \sigma^2_{\rho_0,q}\right), & \text{in } \mathcal{R}_{22}, \\ \mathcal{N}\left(m^\star_{\rho_0,\rho_0'}, \sigma^2_{\rho_0,q}\right), & \text{in } \mathcal{R}_{23}, \end{cases}$$

where $m^\star_{\rho_0,\rho_0'} = u_{\rho_0}\lambda_A + v_{\rho_0,\rho_0'}\lambda_B$*, with* $u_\rho \equiv u$ *and* $v_{\rho,\rho'} \equiv v$*,* (u,v) *given in* (2.3)*, and* $f_{\rho_0}(\xi,\rho_0)$ *and* $\sigma^2_{\rho_0,q}$ *given in Step* ii) *of* Theorem 7.1.

iii) *In* $\mathcal{R}_3$*, if* $\sqrt{k}A_q(n/k)/\ln(n/k) \to \infty$*,* $\sqrt{k}A_0^2(n/k) \to \lambda_A$*,* $\sqrt{k}A_0(n/k)B_0(n/k) \to \lambda_B$ *and* $\sqrt{k}/A_0(n/k)U_0(n/k) \to \lambda_{AU}$*, all finite, then*

$$\sqrt{k}A_q(n/k)\frac{\beta_q-\hat\beta^{(\eta_q,q)}(k;\hat\rho^{(\tau_q,q)}(k))}{\beta_q\ln(n/k)} \overset{a}{\sim} \mathcal{N}\left(\frac{m^\star_{\rho_0,\rho_0'}+\xi\chi_q(g_{\rho_0}(\xi,\rho_0)+\widetilde\lambda\chi_q w_{\rho_0}(\xi,\rho_0,\tau_q))\lambda_{AU}}{1+\xi\widetilde\lambda\chi_q}, \widetilde\sigma^2_{\rho_0,q}\right),$$

with $g_{\rho_0}(\xi,\rho_0)$*,* $w_{\rho_0}(\xi,\rho_0,\tau_q)$ *and* $\widetilde\sigma^2_{\rho_0,q}$ *given in Step* iii) *of* Theorem 7.1.

Proof. From (3.3), we have

$$d\widehat\beta^{(\eta_q,q)}(k;\rho^{(q)})/d\rho^{(q)} = -\ln(n/k)\widehat\beta^{(\eta_q,q)}(k;\rho^{(q)})(1+o_p(1)),$$

and using delta's method,

$$\widehat\beta^{(\eta_q,q)}(k;\widehat\rho^{(q)}) = \widehat\beta^{(\eta_q,q)}(k;\rho^{(q)}) - \widehat\beta^{(\eta_q,q)}(k;\rho^{(q)})(\widehat\rho^{(q)}-\rho_q)\ln(n/k)(1+o_p(1)). \tag{7.14}$$

i) In $\mathcal{R}_1$, given (7.1) and with V^{R_1} given in (7.2),

$$\widehat\beta^{(\eta_q,q)}(k;\widehat\rho^{(q)}) = \beta_q + \frac{\beta_q\overset{\bullet}{\sigma}_{\rho_q}}{\sqrt{k}A_q(n/k)}W_k^{R_1} + \beta_q V^{R_1} - \beta_q(\widehat\rho^{(q)}-\rho_q)\ln(n/k)(1+o_p(1)), \tag{7.15}$$

provided that $(\widehat{\rho}^{(q)} - \rho_q)\ln(n/k) = o_p(1)$. In order to keep the same results of Theorem 7.2, we need to work with a level k such that $\sqrt{k}A_q(n/k) \to \infty$, i.e. $\sqrt{k}/U_0(n/k) \to \infty$, $\sqrt{k}A_0(n/k) \to \lambda$, $\sqrt{k}/U_0^2(n/k) \to \lambda_U$, both finite, and $\sqrt{k}A_q(n/k)\ln(n/k)(\widehat{\rho}^{(q)} - \rho_q) = o_p(1)$, as $n \to \infty$. This result is also true if we assume the validity of $C^\star_{U_1}$ for a level $k_1^{(q)} = O(n/\ln\ln n)$ and $\hat{\rho}^{(q)}_{U_1}$. If we estimate ρ_q in the level k, through $\hat{\rho}^{(\tau_q,q)}(k)$, in (3.1), then $(\hat{\rho}^{(\tau_q,q)}(k) - \rho_q)\ln(n/k)$ is the dominant term in (7.14), dependent on k. Then the asymptotic behaviour of $\hat{\beta}^{(\eta_q,q)}(k;\hat{\rho}^{(\tau_q,q)}(k))$ is related to the asymptotic behaviour of $\hat{\rho}^{(\tau_q,q)}(k) - \rho_q$, as stated in (7.13). Then, from Theorem 7.1, we can write,

$$\frac{\hat{\beta}^{(\eta_q,q)}(k;\hat{\rho}^{(\tau_q,q)}(k)) - \beta_q}{-\beta_q\ln(n/k)} \overset{d}{=} \frac{\overset{\bullet}{\sigma}_{\rho_0,q}}{\sqrt{k}A_q(n/k)} W_k^{R_1} + \begin{cases} \dfrac{\chi_q\; y_{\rho_0}(\xi,\tau_q)}{U_0(n/k)}(1+o_p(1)), & \text{in } \mathcal{R}_{11}, \\ \left(\dfrac{z_{\rho_0}(\xi,\rho_0)A_0(n/k)U_0(n/k)}{\chi_q} + \dfrac{\chi_q\; y_{\rho_0}(\xi,\tau_q)}{U_0(n/k)}\right)(1+o_p(1)), & \text{in } \mathcal{R}_{12}, \\ \dfrac{z_{\rho_0}(\xi,\rho_0)A_0(n/k)U_0(n/k)}{\chi_q}(1+o_p(1)), & \text{in } \mathcal{R}_{13}. \end{cases}$$

Consequently, we need to have $\ln(n/k)/(\sqrt{k}A_q(n/k)) \to 0$, in order to guarantee that $\hat{\beta}^{(\eta_q,q)}(k;\hat{\rho}^{(\tau_q,q)}(k))$ is consistent for the estimation of β_q. If $\sqrt{k}A_0(n/k) \to \lambda$ and $\sqrt{k}/U_0^2(n/k) \to \lambda_U$, both finite, the remain of the Corollary follows.

ii) In $\mathcal{R}_2$, given (7.6) and with V^{R_2} given in (7.7),

$$\widehat{\beta}_k^{(\eta_q,q)}(\widehat{\rho}^{(q)}) = \beta_q + \frac{\beta_q\sigma_{\rho_q}}{\sqrt{k}A_q(n/k)}W_k^{R_2} + \beta_q V^{R_2} - \beta_q(\widehat{\rho}^{(q)} - \rho_q)\ln(n/k)(1+o_p(1)),$$

provided that $(\widehat{\rho}^{(q)} - \rho_q)\ln(n/k) = o_p(1)$. In order to keep the same results of Theorem 7.2, we need to work with a level k such that $\sqrt{k}A_q(n/k) \to \infty$, $\sqrt{k}A_0^2(n/k) \to \lambda_A$, $\sqrt{k}A_0(n/k)B_0(n/k) \to \lambda_B$, $\sqrt{k}/U_0(n/k) \to \lambda'$, all finite, and $\sqrt{k}A_0(n/k)\ln(n/k)(\widehat{\rho}^{(q)} - \rho_q) = o_p(1)$, as $n \to \infty$. This result is also true if we assume the validity of $C^\star_{U_2}$ for a level $k_1^{(q)} = O(n/\ln\ln n)$ and $\hat{\rho}^{(q)}_{U_2}$. Again, from Theorem 7.1, we can write,

$$\frac{\hat{\beta}_k^{(\eta_q,q)}(\hat{\rho}_k^{(\tau_q,q)})-\beta_q}{-\beta_q\ln(n/k)} \overset{d}{=} \frac{\sigma_{\rho_0,q}}{\sqrt{k}A_q(n/k)}W_k^{R_2}$$
$$+\begin{cases} \left(\dfrac{\chi_q f_{\rho_0}(\xi,\rho_0)}{A_0(n/k)U_0(n/k)}\right)(1+o_p(1)), & \text{in } \mathcal{R}_{21}, \\ \left(m_{\rho_0,\rho_0'}+\dfrac{\chi_q f_{\rho_0}(\xi,\rho_0)}{A_0(n/k)U_0(n/k)}\right)(1+o_p(1)), & \text{in } \mathcal{R}_{22}, \\ m_{\rho_0,\rho_0'}(1+o_p(1)), & \text{in } \mathcal{R}_{23}, \end{cases}$$

Consequently, in this case we need to have $\ln(n/k)/(\sqrt{k}A_q(n/k)) \to 0$, in order to guarantee that $\hat{\beta}^{(\eta_q,q)}(k;\hat{\rho}^{(\tau_q,q)}(k))$ is consistent for the estimation of β_q. If $\sqrt{k}A_0^2(n/k)\to\lambda_A$, $\sqrt{k}A_0(n/k)B_0(n/k)\to\lambda_B$ and $\sqrt{k}/U_0(n/k)\to\lambda'$, all finite, the remain of the Corollary follows.

iii) In $\mathcal{R}_3$, given (7.11) and with V^{R_3} given in (7.12),

$$\widehat{\beta}_k^{(\eta_q,q)}(\widehat{\rho}^{(q)}) = \beta_q + \frac{\beta_q\widetilde{\sigma}_{\rho_q}}{\sqrt{k}A_q(n/k)}W_k^{R_3} + \beta_q V^{R_3} - \beta_q(\widehat{\rho}^{(q)}-\rho_q)\ln(n/k)(1+o_p(1)),$$

provided that $(\widehat{\rho}^{(q)}-\rho_q)\ln(n/k) = o_p(1)$. In order to keep the same results of Theorem 7.2, we need to work with a level k such that $\sqrt{k}A_q(n/k)\to\infty$, $\sqrt{k}A_0^2(n/k)\to\lambda_A$, $\sqrt{k}A_0(n/k)B_0(n/k)\to\lambda_B$, $\sqrt{k}A_0(n/k)/U_0(n/k)\to\lambda_{AU}$, all finite, and $\sqrt{k}A_0(n/k)\ln(n/k)(\widehat{\rho}^{(q)}-\rho_q) = o_p(1)$, as $n\to\infty$. This result is also true if we assume the validity of $C^{\star}_{U_3}$ for a level $k_1^{(q)} = O(n/\ln\ln n)$ and $\hat{\rho}^{(q)}_{U_3}$. For this region, from Theorem 7.1, we can write,

$$\frac{\hat{\beta}_k^{(\eta_q,q)}(\hat{\rho}_k^{(\tau_q,q)})-\beta_q}{-\beta_q\ln(n/k)} \overset{d}{=} \frac{\widetilde{\sigma}_{\rho_0,q}}{\sqrt{k}A_q(n/k)}W_k^{R_3}$$
$$+\left(\widetilde{u}_{\rho_0}A_0(n/k)+\widetilde{v}_{\rho_0,\rho_0'}B_0(n/k)+\frac{\xi\chi_q\left(\widetilde{g}_{\rho_0}(\xi,\rho_0)+\widetilde{\lambda}\chi_q\widetilde{w}_{\rho_0}(\xi,\rho_0,\tau_q)\right)}{U_0(n/k)}\right)(1+o_p(1)),$$

Consequently, in this case we need to have $\ln(n/k)/(\sqrt{k}A_q(n/k))\to 0$, in order to guarantee that $\hat{\beta}^{(\eta_q,q)}(k;\hat{\rho}^{(\tau_q,q)}(k))$ is consistent for the estimation of β_q. If $\sqrt{k}A_0^2(n/k)\to\lambda_A$, $\sqrt{k}A_0(n/k)B_0(n/k)\to\lambda_B$ and $\sqrt{k}A_0(n/k)/U_0(n/k)\to\lambda_{AU}$, all finite, the remain of the Corollary follows.

$\square$

Remark 7.1. *If we consider* $\hat\beta^{(q)} \equiv \hat\beta^{(\eta_q,q)}(k_1^{(q)};\hat\rho^{(q)})$*, with* $\hat\rho^{(q)}$ *any of the estimators in* (3.1)*, computed at the high level* $k_1^{(q)}$*,* $\hat\beta^{(q)} - \beta_q$ *is thus, from* (7.13)*, of the order of* $(\hat\rho^{(q)} - \rho_q)\ln(n/k_1^{(q)})$*. Consequently, the validity of* (3.2)*, enable us to guarantee the consistency of* $\hat\beta^{(q)} = \hat\beta^{(\eta_q,q)}(k_1^{(q)};\hat\rho^{(q)})$*.*

8 Proof of the result in Section 4

We start by stating and proving a lemma related to condition $C^\star_{U_\bullet}$, $\bullet = 1,2,3$.

Lemma 8.1. *Under the third-order framework in* (1.7)*, if we further assume the validity of* $C^\star_{U_\bullet}$*,* $\bullet = 1,2,3$ *for a level* $k_1^{(q)} = O(n/\ln\ln n)$*, and for* k *values such that* $\lim_{n\to\infty}\sqrt{k}A_q^2(n/k)$ *is finite,*

$$\left(\sqrt{k}A_q(n/k)\right)\ln(n/k)\left(\hat\rho^{(q)}_{U_\bullet} - \rho_q\right) = o_p(1), \quad \text{as} \quad n\to\infty. \tag{8.1}$$

Proof. i) In $\mathcal{R}_1$, under the validity of $C^\star_{U_1}$, for a level $k_1^{(q)} = O(n/\ln\ln n)$, we may guarantee that

$$\hat\rho^{(q)}_{U_1} - \rho_q = O_p\left(1/\left(\sqrt{k_1^{(q)}}/U_0(n/k_1^{(q)})\right)\right) = O_p\left((\ln\ln n)^{(2\xi+1)/2}/\sqrt{n}\right).$$

Consequently, condition (3.2) holds with $\hat\rho^{(q)}$ replaced by $\hat\rho^{(U_1,q)}$. We can also write that $\left(\hat\rho^{(q)}_{U_1} - \rho_q\right)\ln(n/k) = o_p(1)$, as $n\to\infty$, and (8.1) holds whenever $\sqrt{k}/U_0(n/k) \to \lambda'$, finite. Next, if $\sqrt{k}/U_0(n/k) \to \infty$, k is of a larger order than $n^{2\xi/(2\xi+1)}$, and n/k is at most of the order of $n^{1/(2\xi+1)}$. Consequently, $\ln(n/k)U_0(n/k) \equiv \ln(n/k)/A_q(n/k)$ is at most of the order of $(\ln n)n^{\xi/(2\xi+1)}$, and $0 \le |\left(\hat\rho^{(q)} - \rho_q\right)\ln(n/k)/A_q(n/k)| < O\left((\ln\ln n)^{(2\xi+1)/2}\ln n/n^{1/(4\xi+2)}\right) \underset{n\to\infty}{\to} 0$. If $\lim_{n\to\infty}\sqrt{k}/U_0^2(n/k)$ is finite, i.e, if $\lim_{n\to\infty}\sqrt{k}A_q^2(n/k)$ is finite, then (8.1) follows for this type of k-levels.

ii) In $\mathcal{R}_2$, under the validity of $C^\star_{U_2}$, for a level $k_1^{(q)} = O(n/\ln\ln n)$, we may guarantee that

$$\hat\rho^{(q)}_{U_2} - \rho_q = O_p\left(1/\left(\sqrt{k_1^{(q)}}A_0(n/k_1^{(q)})\right)\right) = O_p\left((\ln\ln n)^{(1-2\rho)/2}/\sqrt{n}\right).$$

Consequently, condition (3.2) holds with $\hat\rho^{(q)}$ replaced by $\hat\rho^{(U_2,q)}$. We can also write that $\left(\hat\rho^{(q)}_{U_2} - \rho_q\right)\ln(n/k) = o_p(1)$, as $n\to\infty$, and (8.1) holds whenever $\sqrt{k}A_0(n/k) \to \lambda$, finite. Next, if $\sqrt{k}A_0(n/k) \to \infty$, k is of a larger order than $n^{-2\rho/(1-2\rho)}$, and n/k is at most of the order of $n^{1/(1-2\rho)}$. Consequently, $\ln(n/k)/A_0(n/k) \equiv \ln(n/k)/A_q(n/k)$ is at most of the order of $(\ln n)n^{-\rho/(1-2\rho)}$, and $0 \le |\left(\hat\rho^{(q)} - \rho_q\right)\ln(n/k)/A_q(n/k)| < O\left((\ln\ln n)^{(1-2\rho)/2}\ln n/n^{1/(2-4\rho)}\right) \underset{n\to\infty}{\to} 0$. If $\lim_{n\to\infty}\sqrt{k}A_0^2(n/k)$ is finite, i.e, if $\lim_{n\to\infty}\sqrt{k}A_q^2(n/k)$ is finite, then (8.1) follows for this type of k-levels.

iii) In $\mathcal{R}_3$, under the validity of $C^\star_{U_3}$, for a level $k_1^{(q)} = O(n/\ln\ln n)$, we may guarantee that

$$\hat\rho^{(q)}_{U_3} - \rho_q = O_p\left(1/\left(\sqrt{k_1^{(q)}}A_0(n/k_1^{(q)})\right)\right) = O_p\left((\ln\ln n)^{(1-2\rho)/2}/\sqrt{n}\right).$$

Consequently, condition (3.2) holds with $\hat\rho^{(q)}$ replaced by $\hat\rho^{(U_3,q)}$. We can also write that $\left(\hat\rho^{(q)}_{U_3} - \rho_q\right)\ln(n/k) = o_p(1)$, as $n \to \infty$, and (8.1) holds whenever $\sqrt{k}A_0(n/k) \to \lambda$, finite. Next, if $\sqrt{k}A_0(n/k) \to \infty$, k is of a larger order than $n^{-2\rho/(1-2\rho)}$, and n/k is at most of the order of $n^{1/(1-2\rho)}$. Consequently, $\ln(n/k)/A_0(n/k) \equiv \ln(n/k)/A_q(n/k)$ is at most of the order of $(\ln n)n^{-\rho/(1-2\rho)}$, and $0 \le |\left(\hat\rho^{(q)} - \rho_q\right)\ln(n/k)/A_q(n/k)| < O\left((\ln\ln n)^{(1-2\rho)/2}\ln n/n^{1/(2-4\rho)}\right) \underset{n\to\infty}{\to} 0$. If $\lim_{n\to\infty}\sqrt{k}A_0^2(n/k)$ is finite, i.e, if $\lim_{n\to\infty}\sqrt{k}A_q^2(n/k)$ is finite, then (8.1) follows for this type of k-levels.

□

Proof. [**Theorem 4.2**] If we estimate consistently (ρ_q, β_q) through the estimators $(\hat\rho^{(q)}, \hat\beta^{(q)})$, given in (3.1) and (3.3), respectively, and with $\overline{H}^{(q)}(k; \widehat\beta^{(q)}, \widehat\rho^{(q)})$ given in (1.21), delta's method enable us to write,

$$\overline{H}^{(q)}(k; \widehat\beta^{(q)}, \widehat\rho^{(q)}) = \overline{H}^{(q)}(k; \beta_q, \rho_q) + (\widehat\beta^{(q)} - \beta_q)\frac{\partial\overline{H}^{(q)}(k; \beta_q, \rho_q)}{\partial\beta_q}(1 + o_p(1)) + (\widehat\rho^{(q)} - \rho_q)\frac{\partial\overline{H}^{(q)}(k; \beta_q, \rho_q)}{\partial\rho_q}(1 + o_p(1)),$$

with

$$\frac{\partial\overline{H}^{(q)}(k; \beta_q, \rho_q)}{\partial\beta_q} \overset{p}{\sim} -\frac{A_q(n/k)}{\beta_q(1-\rho_q)},$$
$$\frac{\partial\overline{H}^{(q)}(k; \beta_q, \rho_q)}{\partial\rho_q} \overset{p}{\sim} -\frac{A_q(n/k)}{1-\rho_q}\left(\ln\left(\frac{n}{k}\right) + \frac{1}{1-\rho_q}\right).$$

Thence,

$$\overline{H}^{(q)}(k;\widehat{\beta}^{(q)},\widehat{\rho}^{(q)})-\overline{H}^{(q)}(k;\beta_q,\rho_q)\overset{p}{\sim}-\frac{A_q(n/k)}{1-\rho_q}\left[\frac{\widehat{\beta}^{(q)}-\beta_q}{\beta_q}+(\widehat{\rho}^{(q)}-\rho_q)\left(\ln\left(\frac{n}{k}\right)+\frac{1}{1-\rho_q}\right)\right].$$

Considering levels k such that $\sqrt{k}A_q(n/k)\to c$, finite, i.e., if $\sqrt{k}/U_0(n/k)\to\lambda'$, finite in $\mathcal{R}_1$, if $\sqrt{k}A_0(n/k)\to\lambda$, finite in $\mathcal{R}_2$, and if $\sqrt{k}/U_0(n/k)\to\lambda'$ or $\sqrt{k}A_0(n/k)\to\lambda$, finite in $\mathcal{R}_3$, then $\sqrt{k}\left(\overline{H}^{(q)}(k;\widehat{\beta}^{(q)},\widehat{\rho}^{(q)})-\overline{H}^{(q)}(k;\beta_q,\rho_q)\right)\underset{n\to\infty}{\overset{p}{\longrightarrow}}0$, and $\sqrt{k}(\overline{H}^{(q)}(k;\widehat{\beta}^{(q)},\widehat{\rho}^{(q)})-\xi)$ is asymptotically normal with null mean and variance ξ^2.

Next, if we consider levels k such that $\sqrt{k}A_q(n/k)\to\infty$, and assuming the validity of (7.13), i.e., $(\widehat{\beta}^{(q)}-\beta_q)/\beta_q\overset{p}{\sim}-\ln(n/k_1^{(q)})(\widehat{\rho}^{(q)}-\rho_q)$, we have

$$\overline{H}^{(q)}(k;\widehat{\beta}^{(q)},\widehat{\rho}^{(q)})-\overline{H}^{(q)}(k;\beta_q,\rho_q)\overset{p}{\sim}-\frac{A_q(n/k)}{1-\rho_q}(\widehat{\rho}^{(q)}-\rho_q)\left\{-\ln\left(\frac{k}{k_1^{(q)}}\right)+\frac{1}{1-\rho_q}\right\}$$
$$=:R^{(q)}_{k,k_1^{(q)}}. \quad (8.2)$$

i) In $\mathcal{R}_1$, for levels k such that $\sqrt{k}/U_0(n/k)\to\infty$, and with $k_1^{(q)}$ optimal for the estimation of ρ_q, i.e., $\hat{\rho}^{(q)}-\rho_q=O_p\left(1/\left(\sqrt{k_1^{(q)}}/U_0(n/k_1^{(q)})\right)\right)$,

$$\begin{aligned}\sqrt{k}R^{(q)}_{k,k_1^{(q)}} &= O_p\left(\frac{\sqrt{k}\;U_0(n/k_1^{(q)})}{\sqrt{k_1^{(q)}}\;U_0(n/k)}\ln\left(\frac{k}{k_1^{(q)}}\right)\right)\\ &= O_p\left(\left(\frac{k}{k_1^{(q)}}\right)^{\frac{1}{2}+\xi}\ln\left(\frac{k}{k_1^{(q)}}\right)\right)=o_p(1),\end{aligned}$$

if $k/k_1^{(q)}\to 0$, and the second part of the Theorem follows.

Finally, if we assume the validity of $C^{\star}_{U_1}$ for a level $k_1^{(q)}=O(n/\ln\ln n)$ and consider $\hat{\beta}^{(q)}_{U_1}=\hat{\beta}^{(q)}(k_1^{(q)},\hat{\rho}^{(q)}_{U_1})$, we guarantee, on the basis of Lemma 8.1, that $\sqrt{k}\left(\hat{\rho}^{(q)}_{U_1}-\rho_q\right)A_q(n/k)\ln(k/k_1^{(q)})=o_p(1)$. If we replace $(\hat{\beta}^{(q)},\hat{\rho}^{(q)})$ in (8.2) by $(\hat{\beta}^{(q)}_{U_1},\hat{\rho}^{(q)}_{U_1})$ or if we consider in (8.2) that $\hat{\rho}^{(q)}-\rho_q=o_p\left(\ln(n/k)/\left(\sqrt{k}A_q(n/k)\right)\right)$, the remain of the Theorem follows.

ii) In $\mathcal{R}_2$, for levels k such that $\sqrt{k}A_0(n/k) \to \infty$, and with $k_1^{(q)}$ optimal for the estimation of ρ_q, i.e., $\hat{\rho}^{(q)} - \rho_q = O_p\left(1/\left(\sqrt{k_1^{(q)}}A_0(n/k_1^{(q)})\right)\right)$,

$$\begin{aligned}\sqrt{k}R^{(q)}_{k,k_1^{(q)}} &= O_p\left(\frac{\sqrt{k}/A_0(n/k_1^{(q)})}{\sqrt{k_1^{(q)}}/A_0(n/k)}\ln\left(\frac{k}{k_1^{(q)}}\right)\right)\\ &= O_p\left(\left(\frac{k}{k_1^{(q)}}\right)^{\frac{1}{2}-\rho}\ln\left(\frac{k}{k_1^{(q)}}\right)\right) = o_p(1),\end{aligned}$$

if $k/k_1^{(q)} \to 0$, and the second part of the Theorem follows.

Finally, if we assume the validity of $C^\star_{U_2}$ for a level $k_1^{(q)} = O(n/\ln\ln n)$ and consider $\hat{\beta}^{(q)}_{U_2} = \hat{\beta}^{(q)}(k_1^{(q)}, \hat{\rho}^{(q)}_{U_2})$, we guarantee, on the basis of Lemma 8.1, that $\sqrt{k}\left(\hat{\rho}^{(q)}_{U_2} - \rho_q\right)A_q(n/k)\ln(k/k_1^{(q)}) = o_p(1)$. If we replace $(\hat{\beta}^{(q)}, \hat{\rho}^{(q)})$ in (8.2) by $(\hat{\beta}^{(q)}_{U_2}, \hat{\rho}^{(q)}_{U_2})$ or if we consider in (8.2) that $\hat{\rho}^{(q)} - \rho_q = o_p\left(\ln(n/k)/\left(\sqrt{k}A_q(n/k)\right)\right)$, the remain of the Theorem follows.

iii) In $\mathcal{R}_3$, for levels k such that $\sqrt{k}A_0(n/k) \to \infty$, and with $k_1^{(q)}$ optimal for the estimation of ρ_q, i.e., $\hat{\rho}^{(q)} - \rho_q = O_p\left(1/\left(\sqrt{k_1^{(q)}}A_0(n/k_1^{(q)})\right)\right)$,

$$\begin{aligned}\sqrt{k}R^{(q)}_{k,k_1^{(q)}} &= O_p\left(\frac{\sqrt{k}/A_0(n/k_1^{(q)})}{\sqrt{k_1^{(q)}}/A_0(n/k)}\ln\left(\frac{k}{k_1^{(q)}}\right)\right)\\ &= O_p\left(\left(\frac{k}{k_1^{(q)}}\right)^{\frac{1}{2}-\rho}\ln\left(\frac{k}{k_1^{(q)}}\right)\right) = o_p(1),\end{aligned}$$

if $k/k_1^{(q)} \to 0$, and the second part of the Theorem follows.

Finally, if we assume the validity of $C^\star_{U_3}$ for a level $k_1^{(q)} = O(n/\ln\ln n)$ and consider $\hat{\beta}^{(q)}_{U_3} = \hat{\beta}^{(q)}(k_1^{(q)}, \hat{\rho}^{(q)}_{U_3})$, we guarantee, on the basis of Lemma 8.1, that $\sqrt{k}\left(\hat{\rho}^{(q)}_{U_3} - \rho_q\right)A_q(n/k)\ln(k/k_1^{(q)}) = o_p(1)$. If we replace $(\hat{\beta}^{(q)}, \hat{\rho}^{(q)})$ in (8.2) by $(\hat{\beta}^{(q)}_{U_3}, \hat{\rho}^{(q)}_{U_3})$ or if we consider in (8.2) that $\hat{\rho}^{(q)} - \rho_q = o_p\left(\ln(n/k)/\left(\sqrt{k}A_q(n/k)\right)\right)$, the remain of the Theorem follows.

□

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