A Partially Reduced-Bias Class of Value-at-Risk Estimators

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July 26, 2015

Abstract

For any level q, $0 < q < 1$, and on the basis of a sample $(X_1, \ldots, X_n)$ of either independent, identically distributed or possibly weakly dependent and stationary random variables from an unknown model F with a heavy right-tail function, the value-at-risk at the level q, denoted by VaR_q, the size of the loss that occurred with a small probability q, is estimated by a recent semi-parametric procedure based on a partially reduced-bias extreme value index (EVI) class of estimators, a generalization of the classical Hill EVI-estimator, related to the mean-of-order-p of an adequate set of statistics. Such an estimator depends on two tuning parameters p and k, with $p \geq 0$ and $1 \leq k < n$ the number of top order statistics involved in the semi-parametric estimation, and outperforms previous estimation procedures. The adequate choice of k and p can be done through the use of either a computer-intensive double-bootstrap method or through reliable heuristic procedures. An application in the field of finance is also provided.

Keywords: extreme value theory; semi-parametric estimation; statistics of extremes; value-at-risk.

1 Introduction and scope of the article

Let $(X_1,\ldots,X_n)$ be a sample of independent, identically distributed or possibly weakly dependent and stationary *random variables* (RVs), from an underlying *cumulative distribution function* (CDF) F. Let us denote by $(X_{1:n}\le\cdots\le X_{n:n})$ the sample of associated ascending order statistics. If there exist sequences of real numbers, (a_n,b_n), with $a_n>0$ and $b_n\in\mathbb{R}$, such that the sequence of linearly normalized maxima, $\{(X_{n:n}-b_n)/a_n\}_{n\ge1}$, converges to a non-degenerate RV, then (Gnedenko, 1943) such a RV is of the type of a general *extreme value* (EV) CDF,

$$\mathrm{EV}_\xi(x)=\begin{cases}\exp(-(1+\xi x)^{-1/\xi}), 1+\xi x>0, & \text{if} \quad \xi\neq 0,\\ \exp(-\exp(-x)),\ x>0, & \text{if} \quad \xi=0.\end{cases} \tag{1.1}$$

We then say that F is in the max-domain of attraction of EV_ξ, use the notation $F\in\mathcal{D}_{\mathcal{M}}(\mathrm{EV}_\xi)$, (a_n,b_n) are the so-called attraction coefficients of F to the limiting law EV_ξ, and the parameter ξ is the *extreme value index* (EVI), one of the most relevant parameters in the field of statistics of extremes.

We shall here consider heavy right tails, i.e. $\xi>0$ in (1.1), and we are interested in dealing with the semi-parametric estimation of the *value-at-risk* (VaR_q) at the level q, the size of the loss that occurs with a small probability q. We are thus dealing with the high quantile

$$\chi_{1-q}\equiv\mathrm{VaR}_q:=F^{\leftarrow}(1-q),$$

of the unknown CDF F, with $F^{\leftarrow}(y)=\inf\{x:F(x)\ge y\}$ denoting the generalized inverse function of F. As usual, let us denote by $U(t)$ the *tail quantile function* (TQF), i.e. $U(t):=F^{\leftarrow}(1-1/t),\quad t\ge1$, the generalized inverse function of $1/(1-F)$. For small q, we thus want to estimate the parameter $\mathrm{VaR}_q=U(1/q),\quad q=q_n\to0,\quad nq_n\le1$, extrapolating beyond the sample, possibly working in the whole $\mathcal{D}_{\mathcal{M}}(\mathrm{EV}_{\xi>0})=:\mathcal{D}^+_{\mathcal{M}}$, assuming thus that

$U(t) \sim Ct^{\xi}$, as $t \to \infty$, where the notation $a(t) \sim b(t)$ means that $a(t)/b(t) \to 1$, as $t \to \infty$.

Weissman (1978) proposed the semi-parametric VaR_q-estimator,

$$\text{Q}^{(q)}_{\hat{\xi}}(k) := X_{n-k:n}\left(k/(nq)\right)^{\hat{\xi}}, \tag{1.2}$$

where $\hat{\xi}$ can be any consistent estimator for ξ and Q stands for quantile. For $\xi > 0$, the classical EVI-estimator, usually the one which is used in (1.2), for a semi-parametric quantile estimation, is the Hill estimator $\hat{\xi} = \hat{\xi}(k) =: \text{H}(k)$ (Hill, 1975),

$$\text{H}(k) := \frac{1}{k}\sum_{i=1}^{k} V_{ik}, \quad V_{ik} = \ln\frac{X_{n-i+1:n}}{X_{n-k:n}}, \ 1 \le i \le k. \tag{1.3}$$

If we plug in (1.2) the Hill estimator, $\text{H}(k)$, we get the so-called Weissman-Hill quantile or VaR_q-estimator, with the obvious notation, $\text{Q}^{(q)}_{\text{H}}(k)$.

Noticing that we can write

$$\text{H}(k) = \sum_{i=1}^{k}\ln\left(\frac{X_{n-i+1:n}}{X_{n-k:n}}\right)^{1/k} = \ln\left(\prod_{i=1}^{k}\frac{X_{n-i+1:n}}{X_{n-k:n}}\right)^{1/k}, \quad 1 \le i \le k < n,$$

the Hill estimator is thus the logarithm of the geometric mean (or mean-of-order-0) of

$$\underline{\mathbb{U}} := \{U_{ik} := X_{n-i+1:n}/X_{n-k:n}, \ 1 \le i \le k < n\}. \tag{1.4}$$

More generally, Brilhante *et al.* (2013) considered as basic statistics the *mean-of-order-p* (MOP) of $\underline{\mathbb{U}}$, in (1.4), with $p \ge 0$, and the associated class of EVI-estimators,

$$\text{H}_p(k) := \begin{cases} \frac{1}{p}\left(1 - \left(\frac{1}{k}\sum_{i=1}^{k} U_{ik}^{p}\right)^{-1}\right), & \text{if} \quad p > 0, \\ \\ \text{H}(k), & \text{if} \quad p = 0, \end{cases} \tag{1.5}$$

with $\text{H}_0(k) \equiv \text{H}(k)$, given in (1.3). The class of MOP EVI-estimators in (1.5) depends now on this tuning parameter $p \ge 0$, and was shown to be valid for $0 \le p < 1/\xi$, whenever $k = k_n$ is an intermediate sequence, i.e. a sequence of integers $k = k_n$, $1 \le k < n$, such that $k = k_n \to \infty$ and $k_n = o(n)$, as $n \to \infty$. If we plug in (1.2) the MOP EVI-estimator, $\text{H}_p(k)$, we get the so-called MOP quantile or VaR_q-estimator, with the obvious notation, $\text{Q}^{(q)}_{\text{H}_p}(k)$, studied asymptotically and for finite samples in Gomes *et al.* (2015b).

The MOP EVI-estimators in (1.5) can often have a high asymptotic bias, and bias reduction has recently been a vivid topic of research in the area of statistics of extremes. Working just for technical simplicity in the particular class of Hall-Welsh models in (Hall and Welsh, 1986), with a TQF $U(t) = Ct^{\xi}\left(1 + \xi\beta t^{\rho}/\rho + o(t^{\rho})\right)$, as $t \to \infty$, dependent on a vector (β, ρ) of unknown second-order parameters, the asymptotic distributional representation of the Hill EVI-estimator, given in (1.3), or equivalently, of $\mathrm{H}_p(k)$, given in (1.5), for $p = 0$, led Caeiro *et al.* (2005) to directly remove the dominant component of the bias of the Hill EVI-estimator, given by $\xi\beta(n/k)^{\rho}/(1-\rho)$, considering the *corrected-Hill* (CH) EVI-estimators,

$$\mathrm{CH}(k) \equiv \mathrm{CH}_{\hat\beta,\hat\rho}(k) := \mathrm{H}(k)\Big(1 - \frac{\hat\beta}{1-\hat\rho}\left(\frac{n}{k}\right)^{\hat\rho}\Big), \tag{1.6}$$

a *minimum-variance reduced-bias* (MVRB) class of EVI-estimators for suitable second-order parameter estimators, $(\hat\beta, \hat\rho)$. Estimators of ρ can be found in a large variety of articles, including Fraga Alves *et al.* (2003). Regarding the β-estimation, we refer to Gomes and Martins (2002), also among others. Gomes and Pestana (2007) have used the EVI-estimator in (1.6) to build classes of MVRB VaR_q-estimators, that we obviously denote by $\mathrm{Q}^{(q)}_{\mathrm{CH}}(k)$. Recent overviews including the topic of reduced-bias estimation can be seen in Beirlant *et al.* (2012) and Gomes and Guillou (2014).

Working with values of p such that the asymptotic normality of the estimators in (1.5) holds, i.e. more specifically with $0 \le p < 1/(2\xi)$, Brilhante *et al.* (2014) noticed that there is an optimal value $p \equiv p_{_M} = \varphi_\rho/\xi$, with

$$\varphi_\rho = 1 - \rho/2 - \sqrt{(1-\rho/2)^2 - 1/2}, \tag{1.7}$$

which maximises the asymptotic efficiency of the class of estimators in (1.5). Then, they considered the optimal RV $\mathrm{H}_{p_M}(k)$, with $\mathrm{H}_p(k)$ given in (1.5), deriving its asymptotic behaviour. Such a behaviour has led Gomes *et al.* (2015a) to introduce a *partially reduced-bias* (PRB) class of MOP EVI-estimators based on $\mathrm{H}_p(k)$, in (1.5), with the functional expression

$$\mathrm{PRB}_p(k; \hat\beta, \hat\rho) := \mathrm{H}_p(k)\Big(1 - \frac{\hat\beta(1-\varphi_{\hat\rho})}{1-\hat\rho-\varphi_{\hat\rho}}\left(\frac{n}{k}\right)^{\hat\rho}\Big), \tag{1.8}$$

still dependent on a tuning parameter p and with φ_ρ defined in (1.7). It is thus sensible to use the class of EVI-estimators given in (1.8), and to consider the associated VaR_q-estimators, that we obviously denote by $\mathrm{Q}^{(q)}_{\mathrm{PRB}_p}(k)$.

In this article, apart from the description of a small-scale Monte-Carlo simulation, in Section 2, to illustrate the comparative behavior of the different VaR-estimators under consideration, an application in the field of finance is provided in Section 3. Finally, Section 4 sketches some conclusions of this study.

2 A Monte-Carlo illustration

We have implemented multi-sample Monte-Carlo simulation experiments of size, 5000×20, essentially for the class of VaR-estimators, $Q^{(q)}_{\mathrm{PRB}_p}(k)$, and for a few values of n and p, in comparison with the H and CH VaR-estimators. Further details on multi-sample simulation can be found in Gomes and Oliveira (2001).

In Figure 1 an illustration of the obtained results is given for the VaR-estimators under consideration and for an $\mathrm{EV}_{0.1}$ parent. In this figure, we show, for $n = 1000$, $q = 1/n$, and on the basis of the first $N = 5000$ runs, the simulated patterns of mean value, $\mathrm{E}_Q[\cdot]$, and root mean squared error, $\mathrm{RMSE}_Q[\cdot]$, of the standardized PRB MOP VaR-estimators, for $p = p_\ell = \ell/(8\xi)$, $\ell = 1(1)7$, representing only the best two among the considered ℓ-values, the classical H VaR-estimators and the MVRB VaR-estimators.

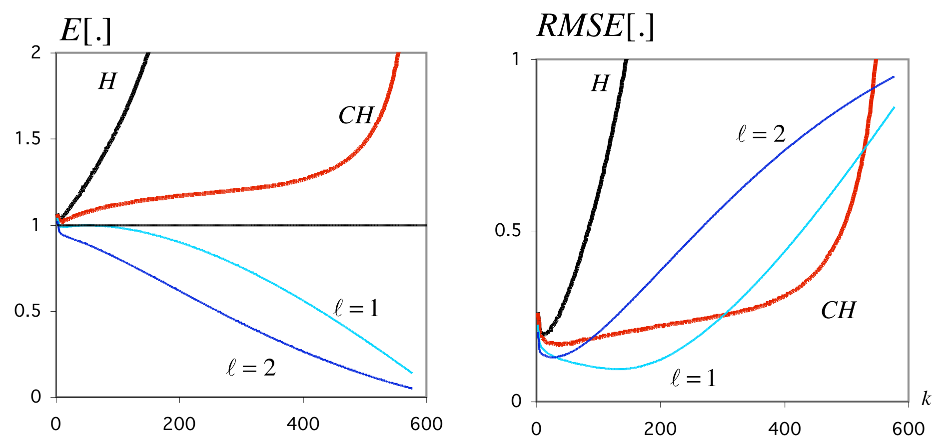

Figure 1: Mean values of $Q^{(1/n)}_{\bullet}(k)/\mathrm{VaR}_q$ (*left*) and RMSE of $Q^{(1/n)}_{\bullet}(k)/\mathrm{VaR}_q$ (*right*), for underlying EV parent with $\xi = 0.1$, for a sample size $n = 1000$

We have further computed the Weissman-Hill VaR-estimator $Q^{(q)}_{\mathrm{H}}(k)$ at the simulated value of $k^{(q)}_{0|\mathrm{H}} := \arg\min_k \mathrm{RMSE}\big(Q^{(q)}_{\mathrm{H}}(k)\big)$, the simulated optimal k in the sense of minimum RMSE. Such a value is not highly relevant in practice, but provides an indication of the best possible performance of the Weissman-Hill VaR-estimator. Such an estimator is denoted by $Q_{00} := Q_{H|0}$. We have also computed $Q_{0p} := Q_{PRB_p|0}$ at simulated optimal levels, for a few values of p, and the simulated indicators,

$$\text{REFF}_{0|p} := \text{RMSE}\,(Q_{00})/\text{RMSE}\,(Q_{p0}).$$

A similar REFF-indicator, $\text{REFF}_{CH|0}$ has also been computed for the MVRB VaR-estimator. For a visualisation of the obtained results, we represent Figure 2, again related to an $EV_{0.1}$ parent CDF.

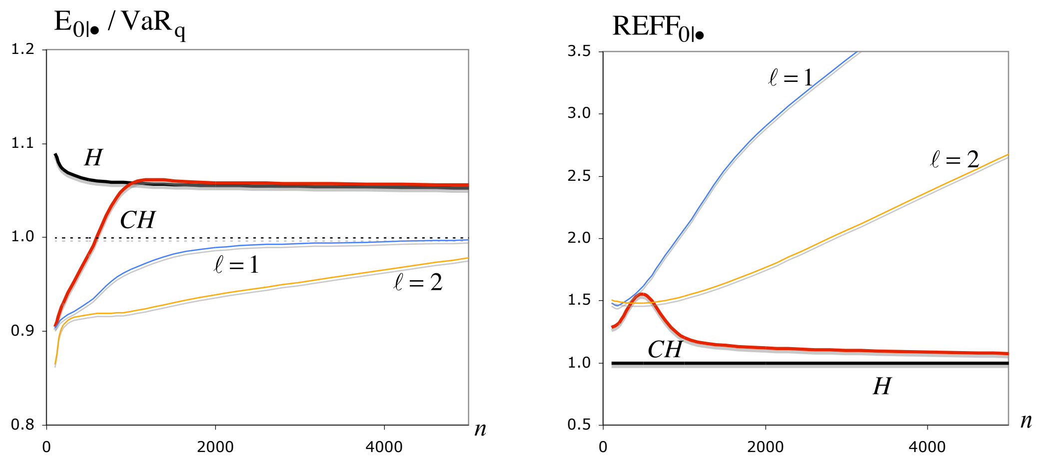

Figure 2: Normalized mean values (*left*) and REFF-indicators (*right*) of the VaR_q-estimators under study, at optimal levels, for $q = 1/n$, $EV_{0.1}$ parents and $100 \leq n \leq 5000$

3 A case-study in the field of finance

We shall here consider the performance of the above mentioned estimators in the analysis of Euro-UK Pound daily exchange rates from January 4, 1999 till December 14, 2004, the data already analyzed in Gomes and Pestana (2007). We have worked with the $n_0 = 725$ positive log-returns:

The sample paths of the VaR–estimators under study, for $q = 0.001$, are pictured in Figure 4, where PRB* represents the PRB_p VaR-estimator associated with an heuristic choice of p, performed in the lines of Gomes *et al.* (2013) and Neves *et al.* (2015).

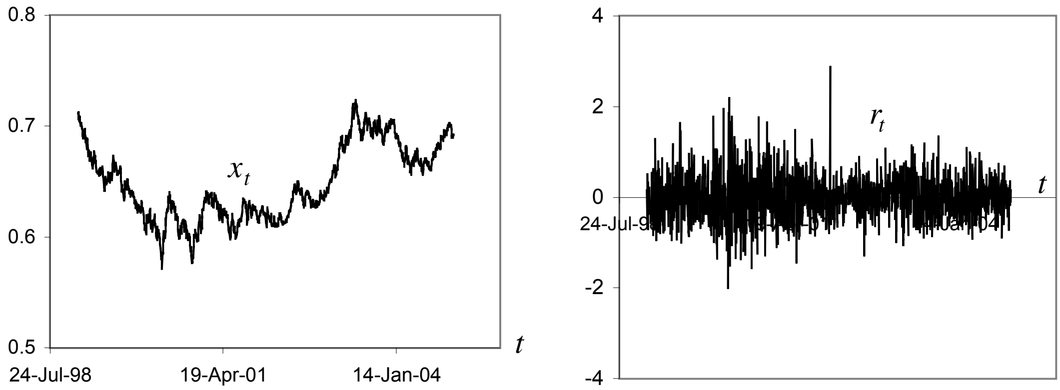

Figure 3: Euro-UK Pound daily exchange rates from January 4, 1999 till December 14, 2004 (*left*) and associated log-returns (*right*)

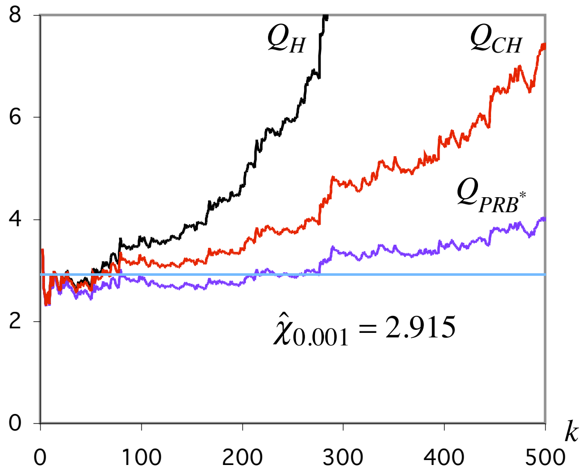

Figure 4: VaR_q-estimates provided through the different classes of VaR-estimators, for the Daily Log-Returns on the Euro-UK Pound and $q = 0.001$

For $q = 0.001$, any of the usual stability criterion for moderate values of k led us to the choice of the estimator Q_{PRB^*} and to the estimate 2.915 for $\text{VaR}_{0.001}$.

4 Concluding remarks

- It is clear that Weissman-Hill VaR-estimation leads to a strong over-estimation of VaR and the RB MOP, or even the MOP methodology can provide a more adequate VaR-estimation, being even able to beat the MVRB VaR-estimators in Gomes and Pestana (2007) in a large variety of situations.

- The obtained results lead us to strongly advise the use of the quantile estimator $\mathrm{Q}_{\mathrm{PRB}_p}(k)$, for a suitable choice of the tuning parameters p and k, provided by an algorithm like for instance the bootstrap algorithm of the type devised for an RB EVI-estimation in Gomes *et al.* (2012), among others, or heuristic algorithms of the type of the ones in Gomes *et al.* (2013) and Neves *et al.* (2015).

- For small values of $|\rho|$ the use of $\mathrm{Q}_{\mathrm{PRB}_p}$, with a suitable value of p, always enables a reduction in RMSE regarding the Weissman-Hill estimator and even the CH VaR_q-estimator. Moreover, the bias is also reduced comparatively with the bias of the Weissman-Hill VaR-estimator, resulting in estimates closer to the target value VaR_q, for small values of q comparatively to n.

Acknowledgements. Research partially supported by **FLAD**, Fundação Luso-Americana para o Desenvolvimento, COST Action IC1408, and National Funds through **FCT**—Fundação para a Ciência e a Tecnologia, projects PEst-OE/MAT/UI0006/2014 (CEA/UL) and PEst-OE/MAT/UI0297/2013, 2014 (CMA/UNL).

References

[1] Beirlant, J., Caeiro, F., & Gomes, M.I. (2012). An overview and open research topics in statistics of univariate extremes. *Revstat* **10**:1, 1–31.

[2] Brilhante, M.F., Gomes, M.I., & Pestana, D. (2013). A simple generalization of the Hill estimator. *Computational Statistics and Data Analysis* **57**:1, 518–535.

[3] Brilhante, M.F., Gomes, M.I., & Pestana, D. (2014). The mean-of-order p extreme value index estimator revisited. In A. Pacheco, R. Santos, M.R. Oliveira, and C.D.

Paulino (eds.), *New Advances in Statistical Modeling and Application*, Springer-Verlag, Berlin, Heidelberg, 163–175.

[4] Caeiro, F., Gomes, M.I., & Pestana, D. (2005). Direct reduction of bias of the classical Hill estimator. *Revstat* **3**:2, 113–136.

[5] Fraga Alves, M.I., Gomes, M.I., & de Haan, L. (2003). A new class of semi-parametric estimators of the second order parameter. *Portugaliae Mathematica* **60**:1, 193–213.

[6] Gnedenko, B.V. (1943). Sur la distribution limite du terme maximum d'une série aléatoire. *Annals of Mathematics* **44**:6, 423–453.

[7] Gomes, M.I., & Guillou, A. (2014). Extreme Value Theory and Statistics of Univariate Extremes: A Review. *International Statistical Review*, doi:10.1111/insr.12058.

[8] Gomes, M.I., & Martins, M.J. (2002). Asymptotically unbiased estimators of the tail index based on external estimation of the second order parameter. *Extremes*, **5**:1, 5–31.

[9] Gomes, M.I., & Oliveira, O. (2001). The bootstrap methodology in Statistics of Extremes—choice of the optimal sample fraction. *Extremes* **4**:4, 331–358.

[10] Gomes, M.I., & Pestana, D. (2007). A sturdy reduced-bias extreme quantile (VaR) estimator. *J. of the American Statistical Association* **102**:477, 280–292.

[11] Gomes, M.I., Figueiredo, F., & Neves, M.M. (2012). Adaptive estimation of heavy right tails: resampling-based methods in action. *Extremes* **15**, 463–48.

[12] Gomes, M.I., Henriques-Rodrigues, L., Fraga Alves, M.I., & Manjunath, B.G. (2013). Adaptive PORTMVRB estimation: an empirical comparison of two heuristic algorithms. *J. Statistical Computation and Simulation* **83**, 1129–1144.

[13] Gomes, M.I., Brilhante, M.F., Caeiro, F., & Pestana, D. (2015a). A new partially reduced-bias mean-of-order p class of extreme value index estimators. *Computational Statistics and Data Analysis* **82**, 223–237.

[14] Gomes, M.I., Brilhante, M.F., & Pestana, D. (2015b). A mean-of-order-p class of Value-at-Risk estimators. In C. Kitsos, T. Oliveira, A. Rigas and S. Gulati (eds.), *Theory*

and Practice of Risk Assessment, Springer Proceedings in Mathematics and Statistics, in press.

[15] Hall, P., & Welsh, A.W. (1985). Adaptive estimates of parameters of regular variation. *Annals of Statistics* **13**, 331–341.

[16] Hill, B.M. (1975). A simple general approach to inference about the tail of a distribution. *Annals of Statistics* **3**, 1163–1174.

[17] Neves, M.M., Gomes, M.I., Figueiredo, F., & Prata Gomes, D. (2015). Modeling extreme events: Sample fraction adaptive choice in parameter estimation. *J. of Statistical Theory and Practice* **9**, 184–199.

[18] Weissman, I. (1978). Estimation of parameters and large quantiles based on the k largest observations. *J. of the American Statistical Association* **73**, 812–815.