Value-at-Risk Estimation and the PORT Mean-of-order-p Methodology*

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Abstract

In finance, insurance and statistical quality control, among many other areas of application, a typical requirement is the estimation of the *value-at-risk* (VaR) at a small level q, i.e. a high quantile of probability $1-q$, a value, high enough, so that the chance of an exceedance of that value is equal to q, small. The semi-parametric estimation of high quantiles depends strongly on the estimation of the *extreme value index* (EVI), the primary

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parameter of extreme events. And most semi-parametric VaR-estimators do not enjoy the adequate behaviour, in the sense that they do not suffer the appropriate linear shift in the presence of linear transformations of the data. For heavy tails, i.e. for a positive EVI, new VaR-estimators with such a behaviour, the so-called PORT VaR-estimators, with PORT standing for *peaks over a random threshold*, were recently introduced in the literature. Regarding EVI-estimation, new classes of PORT-EVI estimators, based on a powerful generalization of Hill EVI-estimators related to adequate *mean-of-order-p* (MO_p) EVI-estimators, were even more recently introduced. In this article, also for heavy tails, we introduce a new class of PORT-MO_p VaR-estimators with the above mentioned behaviour, using the PORT-MO_p class of EVI-estimators. Under convenient but soft restrictions on the underlying model, these estimators are consistent and asymptotically normal. The behaviour of the PORT-MO_p VaR-estimators is studied for finite samples through Monte-Carlo simulation experiments.

Keywords. Asymptotic behaviour; Heavy tails; High quantiles; Mean-of-order-p estimation; Monte-Carlo simulation; PORT methodology; Semi-parametric methods; Statistics of extremes; Value-at-risk.

1 Introduction and scope of the article

In the field of *extreme value theory* (EVT) it is usually said that a *cumulative distribution function* (CDF) F has a heavy right-tail whenever the right tail function, given by $\overline{F} := 1 - F$, is a regularly varying function with a negative index of regular variation $\alpha = -1/\xi$, i.e. for every $x > 0$, $\lim_{t\to\infty} \overline{F}(tx)/\overline{F}(t) = x^{-1/\xi}$, $\xi > 0$. Then we are in the domain of attraction for maxima of an *extreme value* (EV) CDF,

$$\mathrm{EV}_\xi(x) = \exp(-(1+\xi x)^{-1/\xi}), \qquad x > -1/\xi, \quad \xi > 0, \tag{1.1}$$

and we write $F \in \mathcal{D}_\mathcal{M}(\mathrm{EV}_{\xi>0})$. More generally, we can have $\xi \in \mathbb{R}$, i.e. the CDF $\mathrm{EV}_\xi(x) = \exp(-(1+\xi x)^{-1/\xi})$, $1+\xi x > 0$, if $\xi \neq 0$, and by continuity the so-called Gumbel CDF, $\mathrm{EV}_0(x) =$

$\exp(-\exp(-x))$, $x \in \mathbb{R}$, for $\xi = 0$. The parameter ξ is the *extreme value index* (EVI), one of the primary parameters both in probabilistic and statistical EVT.

For heavy tails, and with the notation $U(t) := F^{\leftarrow}(1-1/t)$, $t \geq 1$, $F^{\leftarrow}(y) := \inf\{x : F(x) \geq y\}$ the generalized inverse function of the underlying model F, the positive EVI appears, for every $x > 0$, as the limiting value, as $t \to \infty$, of the quotient $(\ln U(tx) - \ln U(t))/\ln x$ (de Haan, 1970). Indeed, with the usual notation $\mathcal{R}_{\mathrm{a}}$ for the class of regularly varying functions with an index of regular variation a, we can further say that

$$F \in \mathcal{D}_{\mathcal{M}}^{+} := \mathcal{D}_{\mathcal{M}}(\mathrm{EV}_{\xi>0}) \iff \overline{F} = 1 - F \in \mathcal{R}_{-1/\xi} \iff U \in \mathcal{R}_{\xi}, \tag{1.2}$$

with the first necessary and sufficient condition given in Gnedenko (1943) and the second one in de Haan (1984). Heavy-tailed distributions have been accepted as realistic models for various phenomena in the most diverse areas of application, among which we mention bibliometrics, biometry, economics, ecology, finance, insurance, molecular biology and statistical quality control.

For small values of a level q, and as usual in the area of statistical EVT, we want to extrapolate beyond the sample, estimating the *value-at-risk* (VaR) at a level q, denoted by VaR_q, or equivalently, a high quantile χ_{1-q}, i.e. a value such that $F(\chi_{1-q}) = U(1/q) = 1 - q$, i.e.

$$\mathrm{VaR}_q \equiv \chi_{1-q} := U(1/q), \quad q = q_n \to 0, \quad \text{as } n \to \infty. \tag{1.3}$$

We further often assume that $nq_n \to K$ as $n \to \infty$, $K \in [0,\ 1]$, and base inference on the $k+1$ upper *order statistics* (OSs). As usual in semi-parametric estimation of parameters of extreme events, we assume that k is an *intermediate* sequence of integers in $[1, n[$, i.e.

$$k = k_n \to \infty \ \text{ and } \ k/n \to 0, \quad \text{as} \quad n \to \infty. \tag{1.4}$$

To derive the asymptotic non-degenerate behaviour of estimators of parameters of extreme events under a semi-parametric framework, it is further convenient to assume a bit more than the first-order condition, $U \in \mathcal{R}_\xi$, provided in (1.2). A common condition for heavy tails, also assumed now, is the second-order condition that guarantees that

$$\lim_{t\to\infty} \frac{\ln U(tx) - \ln U(t) - \xi \ln x}{A(t)} = \begin{cases} \frac{x^\rho - 1}{\rho} & \text{if} \quad \rho < 0, \\ \ln x & \text{if} \quad \rho = 0, \end{cases} \tag{1.5}$$

being ρ (≤ 0). Note that the limit function in (1.5) is necessarily of the given form and $|A| \in \mathcal{R}_\rho$ (Geluk and de Haan, 1987). Sometimes, for sake of simplicity and for technical reasons, we assume to be working in a sub-class of Hall-Welsh class of models (Hall and Welsh, 1985), where there exist $\xi > 0$, $\rho < 0$, $C > 0$ and $\beta \neq 0$, such that, as $t \to \infty$,

$$U(t) = C\ t^\xi \Big(1 + A(t)\big(1 + o(1)\big)/\rho\Big), \quad \text{with} \quad A(t) = \xi\beta t^\rho. \tag{1.6}$$

The parameters β and ρ are the so-called generalized scale and shape second-order parameters, respectively. Typical heavy-tailed models, like the $\text{EV}_{\xi>0}$ in (1.1) ($\rho = -\xi$), the Fréchet CDF, $\Phi_\alpha(x) = \exp(-x^{-\alpha})$, $x \geq 0$, $\alpha > 0$ ($\xi = 1/\alpha$, $\rho = -1$), the generalized Pareto, $\text{GP}_{\xi>0}(x) = 1 + \ln \text{EV}_\xi(x)$, $x \geq 0$ ($\rho = -\xi$), and the well-known Student-t_ν ($\xi = 1/\nu$, $\rho = -2/\nu$) belong to such a class. Then, the second-order condition in equation (1.5) holds, with $A(t) = \xi\beta t^\rho$, $\beta \neq 0$, $\rho < 0$, as given in (1.6). Further details on these semi-parametric frameworks can be seen in Beirlant *et al.* (2004), de Haan and Ferreira (2006) and Fraga Alves *et al.* (2007), among others.

Under the validity of condition (1.6), and using the notation $a(t) \sim b(t)$ if and only if $\lim_{t\to\infty} a(t)/b(t) = 1$, we can guarantee that $U(t) \sim Ct^\xi$, as $t \to \infty$, and from (1.3), we have

$$\text{VaR}_q = U(1/q) \ \sim \ Cq^{-\xi}, \quad \text{as} \quad q \to 0.$$

An obvious estimator of VaR_q is thus $\widehat{C}q^{-\hat{\xi}}$, with $\widehat{C}$ and $\hat{\xi}$ any consistent estimators of C and ξ, respectively. Given a sample $\underline{\mathbf{X}}_n := (X_1, \ldots, X_n)$, let us denote $(X_{1:n} \leq \cdots \leq X_{n:n})$ the set of associated ascending OSs. A common estimator of C, proposed in Hall (1982), is

$$\widehat{C} \equiv C_{k,n,\hat{\xi}} := X_{n-k:n}(k/n)^{\hat{\xi}} \quad \text{and} \quad Q_{k,q,\hat{\xi}} = \widehat{C}\ q^{-\hat{\xi}} = X_{n-k:n}\big(k/(nq)\big)^{\hat{\xi}} \tag{1.7}$$

is the straightforward VaR-estimator at the level q (Weissman, 1978). In classical approaches, we often consider for $\hat{\xi}$ the Hill (H) estimator (Hill, 1975), the average of the log-excesses, i.e.

$$\text{H}_k \equiv \text{H}_k(\underline{\mathbf{X}}_n) := \frac{1}{k}\sum_{i=1}^{k} \left(\ln X_{n-i+1:n} - \ln X_{n-k:n}\right). \tag{1.8}$$

But the Hill EVI-estimator is the logarithm of the *geometric mean* (or *mean of order* 0) of

$$U_{ik} := X_{n-i+1:n}/X_{n-k:n}, \quad 1 \leq i \leq k < n. \tag{1.9}$$

It is thus sensible to consider the *mean-of-order-p* (MO_p) of U_{ik}, $1 \le i \le k$, as done in Brilhante *et al.* (2013), for $p \ge 0$, and in Gomes and Caeiro (2014) for any $p \in \mathbb{R}$. See also, Paulauskas and Vaičiulis (2013, 2015), Beran *et al.* (2014), Gomes *et al.* (2015a, 2016a) and Caeiro *et al.* (2016a). We then more generally get the class of MO_p EVI-estimators,

$$\mathrm{H}_k(p) = \mathrm{H}_k(p; \underline{\mathbf{X}}_n) := \begin{cases} \frac{1}{p}\left(1 - k/\sum_{i=1}^{k} U_{ik}^p\right) & \text{if} \quad p < 1/\xi,\ p \neq 0, \\ \mathrm{H}_k & \text{if} \quad p = 0, \end{cases} \tag{1.10}$$

with $\mathrm{H}_k(0) \equiv \mathrm{H}_k$, given in (1.8), and U_{ik} given in (1.9), $1 \le i \le k < n$. Associated MO_p VaR-estimators, studied asymptotically and for finite samples in Gomes *et al.* (2015b), are thus a sensible generalization of the Weissman-Hill VaR-estimators.

The MO_p EVI-estimators, in (1.10), depend now on this *tuning* parameter $p \in \mathbb{R}$, are highly flexible, but, as often desirable, they are not location-invariant, depending strongly on possible shifts in the underlying data model. Also, most of the semi-parametric VaR-estimators in the literature, like the ones in Beirlant *et al.* (2008), Caeiro and Gomes (2008), the MO_p VaR-estimators in Gomes *et al.* (2015b), as well as in other papers on semi-parametric quantile estimation prior to 2008 (see also, the functional equation in (1.7), Beirlant *et al.*, 2004, and de Haan and Ferreira, 2006), do not enjoy the adequate behaviour in the presence of linear transformations of the data, a behaviour related to the fact that for any high-quantile, VaR_q, we have

$$\mathrm{VaR}_q(\lambda + \delta X) = \lambda + \delta \mathrm{VaR}_q(X) \tag{1.11}$$

for any model X, real λ and positive δ. Recently, and for $\xi > 0$, Araújo Santos *et al.* (2006) provided VaR-estimators with the linear property in (1.11), based on a *sample of excesses* over a random threshold $X_{n_s:n}$, $n_s := \lfloor ns \rfloor + 1$, $0 \le s < 1$, where $\lfloor x \rfloor$ denotes the integer part of x, being s possibly null only when the underlying parent has a finite left endpoint (see Gomes *et al.*, 2008b, for further details on this subject). Those VaR-estimators are based on the sample of size $n^{(s)} = n - n_s$, defined by

$$\underline{\mathbf{X}}_n^{(s)} := \big(X_{n:n} - X_{n_s:n}, \ldots, X_{n_s+1:n} - X_{n_s:n}\big). \tag{1.12}$$

Such estimators were named PORT-VaR estimators, with PORT standing for *peaks over a random threshold*, and were based on the PORT-Hill, $\mathrm{H}_k(\underline{\mathbf{X}}_n^{(s)})$, $k < n - n_s$, with $\mathrm{H}_k(\underline{\mathbf{X}}_n)$ provided in (1.8). Now, we further suggest for an adequate VaR-estimation, the use of the PORT-MO$_p$ EVI-estimators,

$$\mathrm{H}_k(p, s) := \mathrm{H}_k\big(p; \underline{\mathbf{X}}_n^{(s)}\big), \quad k < n - n_s, \tag{1.13}$$

introduced and studied both theoretically and for finite samples in Gomes *et al.* (2016c), with $\mathrm{H}_k(p; \underline{\mathbf{X}}_n)$ and $\underline{\mathbf{X}}_n^{(s)}$ respectively provided in (1.10) and (1.12). Such PORT-MO$_p$ VaR-estimators are given by

$$\widehat{\mathrm{VaR}}_q(k; p, s) := (X_{n-k:n} - X_{n_s:n}) \left(\frac{k}{nq}\right)^{\mathrm{H}_k(p,s)} + X_{n_s:n}. \tag{1.14}$$

Under convenient restrictions on the underlying model, this class of VaR-estimators is consistent and asymptotically normal for adequate k.

In Section 2 of this paper, and following closely Henriques-Rodrigues and Gomes (2009), Gomes and Henriques-Rodrigues (2016) and Gomes *et al.* (2016c), we present a few introductory technical details and asymptotic results associated with the PORT methodology. A few comments on the asymptotic behaviour of the PORT-classes of VaR-estimators under study will be provided in Section 3. In Section 4, through the use of Monte-Carlo simulation techniques, we shall exhibit the performance of the PORT-MO$_p$ VaR-estimators in (1.14), comparatively to the classical Weissman-Hill, MO$_p$ and a PORT version of the most simple *reduced-bias* (RB) VaR-estimators in Gomes and Pestana (2007). In Section 5, we refer possible methods for the adaptive choice of the tuning parameters (k, p, s), either based on the bootstrap or on heuristic methodologies, and provide some concluding remarks.

2 The PORT methodology: technical details

First note that if there is a shift $\lambda \in \mathbb{R}$ in the model, i.e. if the CDF $F(x) = F_\lambda(x) = F_0(x - \lambda)$, the EVI does not change with λ. Indeed, if a shift λ is induced in data associated with a *random variable* (RV) X, i.e. if we consider $Y = X + \lambda$, $U_\lambda(t) \equiv U_Y(t) = U_X(t) + \lambda$. Consequently, and due to the fact that $F \in \mathcal{D}_{\mathcal{M}}(\mathrm{EV}_\xi)$ if and only if there exists a function $a(\cdot)$ such that

$$\frac{U(tx) - U(t)}{a(t)} \underset{t\to\infty}{\longrightarrow} \frac{x^\xi - 1}{\xi} \quad \text{(de Haan, 1984)},$$

the EVI, ξ, does not depend on any shift λ. However, the same does not happen to the second-order parameters. Indeed, condition (1.5) can be rewritten as

$$\lim_{t\to\infty} \frac{\ln U_\lambda(tx) - \ln U_\lambda(t) - \xi \ln x}{A_\lambda(t)} = \frac{x^{\rho_\lambda} - 1}{\rho_\lambda}, \tag{2.1}$$

for all $x > 0$, with $|A_\lambda| \in \mathcal{R}_{\rho_\lambda}$, and for $\lambda \neq 0$,

$$\rho_\lambda = \begin{cases} \rho_0 & \text{if} \quad \rho_0 > -\xi, \\ -\xi & \text{if} \quad \rho_0 \leq -\xi. \end{cases}$$

Furthermore, and again for $\lambda \neq 0$, the function $A_\lambda(t)$ in (2.1) can be chosen as

$$A_\lambda(t) := \begin{cases} -\frac{\xi\ \lambda}{U_0(t)} & \text{if} \quad \rho_0 < -\xi, \\ A_0(t) - \frac{\xi\ \lambda}{U_0(t)} & \text{if} \quad \rho_0 = -\xi, \\ A_0(t) & \text{if} \quad \rho_0 > -\xi. \end{cases} \tag{2.2}$$

Under the validity of (1.6), we can thus consider the parameterization $A_\lambda(t) = \xi\beta_\lambda t^{\rho_\lambda}$. Further details on the influence of such a shift in $\big(\beta_0, \rho_0, A_0(\cdot)\big)$ and on the estimation of generalized shape and scale second-order parameters can be found in Henriques-Rodrigues *et al.* (2014, 2015).

2.1 Asymptotic behaviour of the PORT EVI-estimators

In this section we present, under the validity of the second-order condition in (1.5), the asymptotic distributional representations of the PORT-MO$_p$ EVI-estimators, $\mathrm{H}_k(p, s)$, in (1.13). Generalizing the results of Theorem 2.1 in Araújo Santos *et al.* (2006), and on the basis of the asymptotic behaviour of the MO$_p$ EVI-estimators derived in Brilhante *et al.* (2013), Gomes *et al.* (2016c), proved the following theorem:

Theorem 2.1 (Gomes *et al.*, 2016c)**.** *If the second order condition* (1.5) *holds,* $k = k_n$ *is an intermediate sequence of positive integers, i.e.* (1.4) *holds, for any real* s*,* $0 \leq s < 1$*, with*

$\chi_s := F^{\leftarrow}(s)$, finite, we have for $\mathrm{H}_k(p,s)$, *in* (1.13), *an asymptotic distributional representation of the type,*

$$\mathrm{H}_k(p,s) \stackrel{d}{=} \xi + \frac{\sigma_{_{\mathrm{H}(p)}}\ P_k^{\mathrm{H}(p)}}{\sqrt{k}} + \left(b_{_{\mathrm{H}(p)}} A_0(n/k) + \frac{c_{_{\mathrm{H}(p)}}\chi_s}{U_0(n/k)}\right)(1+o_p(1)), \tag{2.3}$$

where $P_k^{\mathrm{H}(p)}$ is a sequence of asymptotically standard normal RVs,

$$\sigma_{_{\mathrm{H}(p)}} := \frac{\xi(1-p\xi)}{\sqrt{1-2p\xi}}, \quad b_{_{\mathrm{H}(p)}} := \frac{1-p\xi}{1-p\xi-\rho}, \quad c_{_{\mathrm{H}(p)}} := \frac{\xi(1-p\xi)}{1-(p-1)\xi}. \tag{2.4}$$

3 Asymptotic behavior of the PORT VaR-estimators

Assuming that we are working with data from $F_\lambda(x) = F_0(x-\lambda)$, i.e. an underlying model with location parameter $\lambda \in \mathbb{R}$, we first present the following result on the asymptotic behaviour of intermediate OSs, proved in Ferreira *et al.* (2003).

Proposition 3.1 (Ferreira *et al.*, 2003)**.** *Under the second-order framework in* (2.1) *and for intermediate sequences of positive integers k, i.e. if* (1.4) *holds,*

$$X_{n-k:n} \stackrel{d}{=} U_\lambda(n/k)\Big(1 + \frac{\xi\ B_k}{\sqrt{k}} + o_p(A_\lambda(n/k))\Big)$$

with $U_\lambda(t) = \lambda + U_0(t)$, $A_\lambda(t)$ given in (2.2)*, and where B_k is asymptotically standard normal. Moreover, for $i<j$, $\mathbb{C}ov(B_i, B_j) = \sqrt{i\ j}\ (1-j/n)/(j-1)$.*

Straightforward generalizations of Theorem 3.1 in Araújo Santos *et al.* (2006) and Theorem 4.1 in Henriques-Rodrigues and Gomes (2009), enable us to state the following theorem.

Theorem 3.1. *Let us assume that the second-order condition in* (2.1) *holds, with $A_\lambda(t) = \xi\beta_\lambda t^{\rho_\lambda}$, that k is an intermediate sequence of integers, i.e.* (1.4) *holds, and that $\ln(nq)/\sqrt{k} \to 0$, as $n\to\infty$, with $q = q_n$ given in* (1.3)*. Let us further use the notation $r_n := k/(nq)$. Then, for any* *real s, $0 \le s < 1$, $\chi_s = F^{\leftarrow}(s)$, finite, and the* PORT-*quantile estimator in* (1.14),

$$\frac{\sqrt{k}}{\ln r_n}\left(\frac{\widehat{\mathrm{VaR}}_q(k;p,s)}{\mathrm{VaR}_q}-1\right) \stackrel{d}{=} \sigma_{_{\mathrm{H}(p)}} P_k^{\mathrm{H}(p)} + \sqrt{k}\big(b_{_{\mathrm{H}(p)}} A_0(n/k) + c_{_{\mathrm{H}(p)}}\ \xi\chi_s/U_0(n/k)\big)(1+o_p(1)), \quad (3.1)$$

with $\big(\sigma_{_{\mathrm{H}(p)}}, b_{_{\mathrm{H}(p)}}, c_{_{\mathrm{H}(p)}}\big)$ *given in* (2.4), *and where* $P_k^{\mathrm{H}(p)}$ *is asymptotically standard normal.*

Proof. The PORT-quantile estimator in (1.14) can be written as

$$\widehat{\mathrm{VaR}}_q(k;p,s) := X_{n-k:n}\left\{\left(1-\frac{X_{n_s:n}}{X_{n-k:n}}\right) r_n^{\mathrm{H}_k(p,s)} + \frac{X_{n_s:n}}{X_{n-k:n}}\right\},$$

with the notation $r_n := k/(nq)$. Therefore,

$$\frac{\widehat{\mathrm{VaR}}_q(k;p,s) - \mathrm{VaR}_q}{X_{n-k:n}} = \left(1-\frac{X_{n_s:n}}{X_{n-k:n}}\right) r_n^{\mathrm{H}_k(p,s)} + \frac{X_{n_s:n}}{X_{n-k:n}} - \frac{\mathrm{VaR}_q}{X_{n-k:n}}.$$

The use of the delta method enables us to write

$$r_n^{\mathrm{H}_k(p,s)} \stackrel{d}{=} r_n^{\xi}\Big(1+\ln r_n\big(\mathrm{H}_k(p,s)-\xi\big)\big(1+o_p(1)\big)\Big).$$

Since $\mathrm{VaR}_q = U_\lambda(1/q)$, the second-order condition in (2.1) and the result in Proposition 2.1 enable us to write

$$\frac{\mathrm{VaR}_q}{X_{n-k:n}} = \frac{U_\lambda\left(\frac{n}{k}\ r_n\right)}{U_\lambda\left(\frac{n}{k}\right)} \times \frac{U_\lambda\left(\frac{n}{k}\right)}{X_{n-k:n}} \stackrel{d}{=} r_n^{\xi}\Big(1 - \frac{\xi\ B_k}{\sqrt{k}} - \frac{A_\lambda(n/k)}{\rho_\lambda}(1 + o_p(1))\Big),$$

Therefore, as $X_{n_s:n}/X_{n-k:n} = o_p(1)$,

$$\frac{\sqrt{k}}{\ln r_n}\left(\frac{\widehat{\mathrm{VaR}}_q(k;p,s)-\mathrm{VaR}_q}{\mathrm{VaR}_q}\right) = \sqrt{k}\big(\mathrm{H}_k(p,s)-\xi\big) + \frac{\xi B_k}{\ln r_n} + O_p\left(\frac{\sqrt{k}A_\lambda(n/k)}{\ln r_n}\right).$$

From (2.3), the result in (3.1) follows. □

Corollary 3.1. *Under the conditions of* Theorem 3.1, *with* $\mathcal{N}(\mu,\sigma^2)$ *denoting a normal* RV *with mean value* μ *and variance* σ^2, $\big(\sigma_{_{\mathrm{H}(p)}}, b_{_{\mathrm{H}(p)}}, c_{_{\mathrm{H}(p)}}\big)$ *given in* (2.4), *and* $P_k^{\mathrm{H}(p)}$ *an asymptotically standard normal* RV, *the following results hold:*

- *For values of* $\xi+\rho_0<0$ *and* $\chi_s\neq 0$,

$$\sqrt{k}\Big(\widehat{\mathrm{VaR}}_q(k;p,s)-\mathrm{VaR}_q\Big)/\big(\ln r_n \mathrm{VaR}_q\big) \;\stackrel{d}{=}\; \sigma_{_{\mathrm{H}(p)}} P_k^{\mathrm{H}(p)}+\sqrt{k}\Big(c_{_{\mathrm{H}(p)}}\frac{\chi_s}{U_0(n/k)}\Big)\;(1+o_p(1)).$$

If $\sqrt{k}/U_0(n/k)\to\lambda_{_U}$ *finite, then*

$$\sqrt{k}\Big(\widehat{\mathrm{VaR}}_q(k;p,s)-\mathrm{VaR}_q\Big)/\big(\ln r_n \mathrm{VaR}_q\big) \;\underset{n\to\infty}{\stackrel{d}{\longrightarrow}}\; \mathcal{N}(\lambda_{_U} c_{_{\mathrm{H}(p)}}\chi_s,\sigma^2_{_{\mathrm{H}(p)}}).$$

- *For values of* $\xi+\rho_0>0$ *or* $\xi+\rho_0\leq 0$ *and* $\chi_s=0$,

$$\sqrt{k}\Big(\widehat{\mathrm{VaR}}_q(k;p,s)-\mathrm{VaR}_q\Big)/\big(\ln r_n \mathrm{VaR}_q\big) \;\stackrel{d}{=}\; \sigma_{_{\mathrm{H}(p)}} P_k^{\mathrm{H}(p)}+\sqrt{k}\,\Big(b_{_{\mathrm{H}(p)}}A_0(n/k)\Big)\,(1+o_p(1)).$$

If $\sqrt{k}A_0(n/k)\to\lambda_{_A}$ *finite, then*

$$\sqrt{k}\Big(\widehat{\mathrm{VaR}}_q(k;p,s)-\mathrm{VaR}_q\Big)/\big(\ln r_n \mathrm{VaR}_q\big) \;\underset{n\to\infty}{\stackrel{d}{\longrightarrow}}\; \mathcal{N}(\lambda_{_A} b_{_{\mathrm{H}(p)}},\sigma^2_{_{\mathrm{H}(p)}}).$$

- *For values of* $\xi+\rho_0=0$ *and* $\chi_s\neq 0$,

$$\sqrt{k}\Big(\widehat{\mathrm{VaR}}_q(k;p,s)-\mathrm{VaR}_q\Big)/\big(\ln r_n \mathrm{VaR}_q\big)$$
$$\stackrel{d}{=}\; \sigma_{_{\mathrm{H}(p)}} P_k^{\mathrm{H}(p)}+\sqrt{k}\Big(b_{_{\mathrm{H}(p)}}A_0(n/k)+c_{_{\mathrm{H}(p)}}\frac{\chi_s}{U_0(n/k)}\Big)(1+o_p(1)).$$

If $\sqrt{k}/U_0(n/k)\to\lambda_{_U}$ *and* $\sqrt{k}A_0(n/k)\to\lambda_{_A}$, *with* $\lambda_{_U}$ *and* $\lambda_{_A}$ *both finite, then*

$$\sqrt{k}\Big(\widehat{\mathrm{VaR}}_q(k;p,s)-\mathrm{VaR}_q\Big)/\big(\ln r_n \mathrm{VaR}_q\big) \;\underset{n\to\infty}{\stackrel{d}{\longrightarrow}}\; \mathcal{N}(\lambda_{_U} c_{_{\mathrm{H}(p)}}\chi_s+\lambda_{_A} b_{_{\mathrm{H}(p)}},\sigma^2_{_{\mathrm{H}(p)}}).$$

4 A Monte-Carlo simulation study

Monte-Carlo multi-sample simulation experiments, of size 5000×20, have been implemented for the classes of MO_p and PORT-MO_p VaR-estimators associated with $p=p_\ell=2\ell/(5\xi)$, $\ell=0,\ 1,\ 2$. Apart from the MO_p and PORT-MO_p VaR-estimators, we have further considered in the VaR-estimator in (1.7), the replacement of the estimator $\hat\xi\equiv\hat\xi_k$ by one of the most simple classes of *corrected-Hill* (CH) EVI-estimators in Caeiro *et al.* (2005). Such a class is defined as

$$\mathrm{CH}_k\equiv\mathrm{CH}_k(\hat\beta,\hat\rho):=\mathrm{H}_k\Big(1-\hat\beta(n/k)^{\hat\rho}/(1-\hat\rho)\Big). \qquad (4.1)$$

The estimators in (4.1) can be second-order *minimum-variance reduced-bias* (MVRB) EVI-estimators, for adequate levels k and an adequate external estimation of the vector of second-order parameters, (β, ρ), introduced in (1.6), i.e. the use of CH_k can enable us to eliminate the dominant component of bias of the Hill estimator, H_k, keeping its asymptotic variance. Indeed, from the results in Caeiro *et al.* (2005), we know that it is possible to adequately estimate the second-order parameters β and ρ, so that we get

$$\sqrt{k}\,(\mathrm{CH}_k - \xi) \;\stackrel{d}{=}\; \mathcal{N}\left(0, \xi^2\right) + o_p\big(\sqrt{k}(n/k)^{\rho}\big),$$

i.e. CH_k overpasses H_k for all k. Overviews on reduced-bias estimation can be found in Chapter 6 of Reiss and Thomas, 2007, Gomes *et al.* (2008a), Beirlant *et al.* (2012) and Gomes and Guillou (2015). For the estimation of the vector of second-order parameters (β, ρ), and just as in the aforementioned review articles, we propose an algorithm of the type of the ones presented in Gomes and Pestana (2007), where the authors used the β-estimator in Gomes and Martins (2002) and the simplest ρ-estimator in Fraga Alves *et al.* (2003), both computed at a level $k_1 = \lfloor n^{0.999} \rfloor$. For updated references of recent β and ρ estimators, see Caeiro *et al.* (2016a).

It is well-known that the PORT methodology works efficiently only when the left endpoint of the underlying parent is negative, and $q = 0$ does not work when the left endpoint is infinite, like happens with the Student model (see Araújo Santos *et al.*, 2006, Gomes *et al.*, 2008b, 2011, 2016c, Caeiro *et al.*, 2016b, and Gomes and Henriques-Rodrigues, 2016, for further details related to the topic of PORT estimation). Consequently, only models with this characteristic have been considered, the EV_ξ, in (1.1) and the Student-t_ν $(\xi = 1/\nu, \rho = -2/\nu)$. The values $s = 0$ (for the EV_ξ parents), the value of s associated with the best performance of the PORT methodology for these models, and $s = 0.1$ (for the Student parents) were the ones used for illustration of the results. Sample sizes from $n = 100(100)500$ and $n = 1000(1000)5000$ were simulated from the aforementioned underlying models, for different values of ξ.

4.1 Mean values and mean square error patterns as k-functionals

For each value of n and for each of the aforementioned models, we have first simulated, on the basis of the initial 5000 runs, the mean value (E) and the *root mean square error* (RMSE) of the scale normalized VaR-estimators, i.e. the Var-estimators over VaR_q, as functions of k. For the EVI-estimation, apart from H_p, in (1.10), $p = 0$ $\big(\mathrm{H}_0 \equiv \mathrm{H}\big)$ and $p = p_\ell = 2\ell/(5\xi)$, $\ell = 1$ (for which asymptotic normality holds), and $\ell = 2$ (where only consistency was proved), and the MVRB (CH) EVI-estimators, in (4.1), we have also included their PORT versions, for the above mentioned values of s, using the notation $\bullet|s$, where $\bullet$ refers to the acronymous of the EVI-estimator.

The results are illustrated in Figure 1, for samples of size $n = 1000$ from an EV_ξ underlying parent, with $\xi = 0.1$ and $s = 0$. In this case, and for all k, there is a clear reduction in RMSE, as well as in bias, with the obtention of estimates closer to the target value ξ, particularly when we consider the PORT-version associated with H_{p_1}. Further note that, at optimal levels, in the sense of minimal RMSE, even the H_{p_2} beat the PORT-MVRB VaR-estimators.

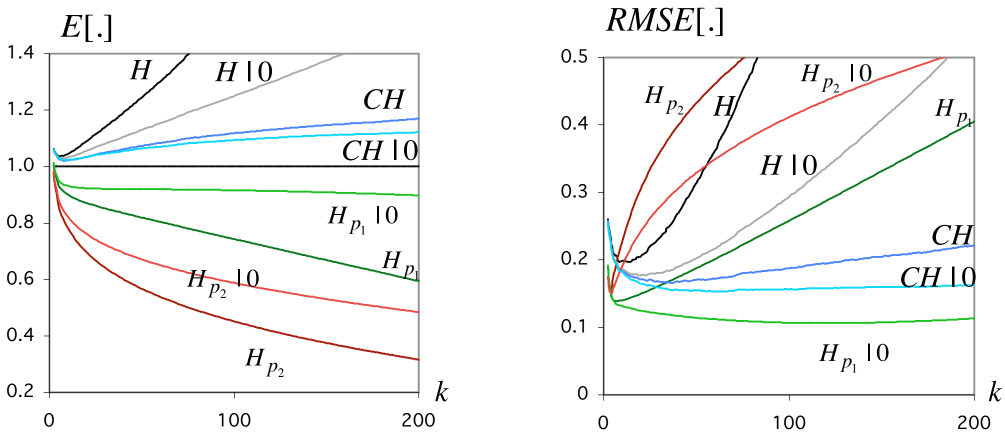

Figure 1: Mean values (*left*) and RMSEs (*right*) of the normalized H, CH, and H_p, $p = p_\ell = 2\ell/(5\xi)$, $\ell = 1, 2$ VaR-estimators for $q = 1/n$, together with their PORT versions, associated with $s = 0$ and generally denoted $\bullet|0$, for $\mathrm{EV}_{0.1}$ underlying parents and sample size $n = 1000$

Similar patterns were obtained for all other simulated models.

4.2 Mean values at optimal levels

Table 1 is also related to the EV_ξ model, with $\xi = 0.1$. We there present, for different sample sizes n, the simulated mean values at optimal levels (levels where RMSEs are minima as functions of k) of some of the normalized VaR-estimators, under consideration in this study. Information on 95% confidence intervals are also given. Among the estimators considered, and distinguishing 2 regions, a first one with (H, CH, H_{p_1}, H_{p_2}), and a second one with the associated PORT versions, (H|0, CH|0, $H_{p_1}|0, H_{p_2}|0$), the one providing the smallest squared bias is written in **bold** whenever there is an out-performance of the behaviour achieved in the previous regions.

Table 1: Simulated mean values of normalized VaR-estimators at their optimal levels for $EV_{0.1}$ parents.

	$n = 100$	$n = 200$	$n = 500$	$n = 1000$	$n = 5000$
H	**1.089** $\pm$ 0.0048	1.073 $\pm$ 0.0042	1.061 $\pm$ 0.0031	1.058 $\pm$ 0.0030	1.053 $\pm$ 0.0018
CH	0.905 $\pm$ 0.0081	**0.930** $\pm$ 0.0049	**0.983** $\pm$ 0.0073	**1.056** $\pm$ 0.0035	**1.052** $\pm$ 0.0025
H_{p_1}	0.885 $\pm$ 0.0014	0.901 $\pm$ 0.0056	0.910 $\pm$ 0.0029	0.915 $\pm$ 0.0022	0.918 $\pm$ 0.0006
H_{p_2}	0.865 $\pm$ 0.0014	0.889 $\pm$ 0.0012	0.912 $\pm$ 0.0006	0.924 $\pm$ 0.0008	0.926 $\pm$ 0.0065
H|0	**1.078** $\pm$ 0.0037	1.069 $\pm$ 0.0033	1.063 $\pm$ 0.0037	1.060 $\pm$ 0.0032	1.057 $\pm$ 0.0027
CH|0	0.922 $\pm$ 0.0036	**0.945** $\pm$ 0.0038	1.025 $\pm$ 0.0006	1.116 $\pm$ 0.0005	1.060 $\pm$ 0.0021
$H_{p_1}\|0$	0.887 $\pm$ 0.0037	0.898 $\pm$ 0.0031	0.893 $\pm$ 0.0009	0.915 $\pm$ 0.0005	**0.998** $\pm$ 0.0002
$H_{p_2}\|0$	0.889 $\pm$ 0.0014	0.909 $\pm$ 0.0012	0.920 $\pm$ 0.0070	0.926 $\pm$ 0.0050	0.928 $\pm$ 0.0006

Tables 2, 3, 4 and 5 are similar to Table 1, but respectively associated with $EV_{0.25}$, $EV_{0.5}$, Student-t_4 and t_2 underlying parents.

Note that contrarily to what happens with the non-PORT and PORT EVI-estimation, where the values associated with p_2 have a minimum squared bias smaller than the ones associated with p_1, things work the other way round for the VaR-estimation.

4.3 RMSEs and relative efficiency indicators at optimal levels

We have further computed the Weissman-Hill VaR-estimator, i.e. the VaR-estimator $Q_{k,q,\hat{\xi}}$, in (1.7), with $\hat{\xi}$ replaced by the H EVI-estimator, in (1.8), at the simulated optimal k in the sense

Table 2: Simulated mean values of normalized VaR-estimators at their optimal levels for $\mathrm{EV}_{0.25}$ parents.

	$n = 100$	$n = 200$	$n = 500$	$n = 1000$	$n = 5000$
H	1.143 ± 0.0068	1.125 ± 0.0070	1.108 ± 0.0048	1.106 ± 0.0052	1.094 ± 0.0034
CH	0.848 ± 0.0092	0.874 ± 0.0041	0.925 ± 0.0027	**1.036** ± 0.0041	1.094 ± 0.0036
H_{p_1}	**0.862** ± 0.0023	**0.912** ± 0.0014	**0.993** ± 0.0013	1.083 ± 0.0038	**1.049** ± 0.0014
H_{p_2}	0.854 ± 0.0046	0.848 ± 0.0014	0.868 ± 0.0043	0.869 ± 0.0035	0.881 ± 0.0024
H|0	1.133 ± 0.0059	1.109 ± 0.0052	1.104 ± 0.0047	1.101 ± 0.0048	1.088 ± 0.0012
CH|0	0.878 ± 0.0004	0.906 ± 0.0031	0.941 ± 0.0020	**0.965** ± 0.0018	1.063 ± 0.0004
$\mathrm{H}_{p_1}\|0$	**0.983** ± 0.0017	**1.060** ± 0.0022	1.048 ± 0.0021	1.055 ± 0.0022	1.064 ± 0.0017
$\mathrm{H}_{p_2}\|0$	0.848 ± 0.0050	0.859 ± 0.0034	0.867 ± 0.0025	0.872 ± 0.0023	0.851 ± 0.0009

Table 3: Simulated mean values of semi-parametric normalized VaR-estimators at their optimal levels for underlying $\mathrm{EV}_{0.5}$ parents.

	$n = 100$	$n = 200$	$n = 500$	$n = 1000$	$n = 5000$
H	1.298 ± 0.0156	1.245 ± 0.0100	1.211 ± 0.0088	1.189 ± 0.0077	1.157 ± 0.0055
CH	**0.905** ± 0.1764	0.800 ± 0.0108	0.842 ± 0.0040	0.874 ± 0.0026	**0.997** ± 0.0022
H_{p_1}	1.117 ± 0.0069	**1.077** ± 0.0059	**1.086** ± 0.0069	**1.102** ± 0.0045	1.131 ± 0.0037
$\mathrm{H}_{p_2}\|0$	0.780 ± 0.0030	0.771 ± 0.0015	0.784 ± 0.0008	0.812 ± 0.0010	0.898 ± 0.0007
H|0	1.233 ± 0.0109	1.203 ± 0.0075	1.171 ± 0.0062	1.157 ± 0.0062	1.118 ± 0.0042
CH|0	0.789 ± 0.0051	0.825 ± 0.0041	0.865 ± 0.0033	0.892 ± 0.0030	**0.944** ± 0.0016
$\mathrm{H}_{p_1}\|0$	**1.084** ± 0.0088	**1.092** ± 0.0054	**1.110** ± 0.0052	**1.124** ± 0.0041	1.116 ± 0.0032
H_{p_2}	0.778 ± 0.0042	0.783 ± 0.0033	0.788 ± 0.0027	0.772 ± 0.0013	0.817 ± 0.0007

of minimum RMSE. Such an estimator is denoted by Q_{00}. For any of the VaR-estimators under study, generally denoted Q_k, we have also computed Q_0, the estimator Q_k computed at the simulated value of $k_{0|Q} := \arg\min_k \mathrm{RMSE}\big(\mathrm{Q}_k\big)$. The simulated indicators are

$$\mathrm{REFF}_{\mathrm{Q}|0} := \frac{\mathrm{RMSE}\,(\mathrm{Q}_{00})}{\mathrm{RMSE}\,(\mathrm{Q}_0)}. \tag{4.2}$$

Remark 4.1. *Note that, as usual, an indicator higher than one means a better performance than the Weissman-Hill VaR-estimator. Consequently, the higher the indicators in* (4.2) *are, the better the associated VaR-estimators perform, comparatively to* Q_{00}.

Again as an illustration of the obtained results, we present Tables 6–10. In the first row,

Table 4: Simulated mean values of normalized VaR-estimators at their optimal levels for Student t_4 parents ($\xi = 0.25$)

	$n = 100$	$n = 200$	$n = 500$	$n = 1000$	$n = 5000$
H	1.114 ± 0.0056	1.099 ± 0.0043	1.089 ± 0.0037	1.085 ± 0.0037	1.077 ± 0.0037
CH	0.903 ± 0.0292	0.903 ± 0.0053	0.922 ± 0.0030	0.978 ± 0.0028	1.056 ± 0.0015
H_{p_1}	$\mathbf{0.932} \pm 0.0014$	$\mathbf{1.009} \pm 0.0019$	$\mathbf{1.035} \pm 0.0023$	$\mathbf{1.032} \pm 0.0019$	$\mathbf{1.054} \pm 0.0017$
H_{p_2}	0.875 ± 0.0062	0.882 ± 0.0029	0.886 ± 0.0022	0.889 ± 0.0020	0.877 ± 0.0005
H $\|0.1$	1.095 ± 0.0063	1.081 ± 0.0027	1.070 ± 0.0027	1.061 ± 0.0020	1.035 ± 0.0015
CH $\|0.1$	0.890 ± 0.0030	0.950 ± 0.0031	$\mathbf{0.980} \pm 0.0020$	$\mathbf{0.990} \pm 0.0012$	$\mathbf{0.998} \pm 0.0006$
$\mathrm{H}_{p_1}\|0.1$	$\mathbf{1.056} \pm 0.0027$	1.055 ± 0.0023	1.057 ± 0.0019	1.056 ± 0.0023	1.041 ± 0.0012
$\mathrm{H}_{p_2}\|0.1$	0.876 ± 0.0011	0.904 ± 0.0008	0.953 ± 0.0005	0.982 ± 0.0005	0.998 ± 0.0002

Table 5: Simulated mean values of normalized VaR-estimators at their optimal levels for Student t_2 parents ($\xi = 0.5$)

	$n = 100$	$n = 200$	$n = 500$	$n = 1000$	$n = 5000$
H	1.236 ± 0.0090	1.198 ± 0.0107	1.168 ± 0.0043	1.145 ± 0.0038	1.106 ± 0.0038
CH	1.115 ± 0.1919	0.809 ± 0.0072	0.825 ± 0.0053	0.848 ± 0.0030	0.848 ± 0.0043
H_{p_1}	$\mathbf{1.094} \pm 0.0073$	$\mathbf{1.082} \pm 0.0048$	$\mathbf{1.084} \pm 0.0031$	$\mathbf{1.080} \pm 0.0040$	$\mathbf{1.062} \pm 0.0021$
H_{p_2}	0.803 ± 0.0036	0.796 ± 0.0026	0.795 ± 0.0012	0.813 ± 0.0008	0.873 ± 0.0005
H $\|0.1$	1.163 ± 0.0056	1.121 ± 0.0048	1.077 ± 0.0030	1.049 ± 0.0027	1.007 ± 0.0021
CH $\|0.1$	0.793 ± 0.0053	0.813 ± 0.0048	0.828 ± 0.0036	0.840 ± 0.0038	0.864 ± 0.0028
$\mathrm{H}_{p_1}\|0.1$	1.098 ± 0.0058	1.087 ± 0.0034	$\mathbf{1.072} \pm 0.0033$	$\mathbf{1.051} \pm 0.0023$	$\mathbf{1.010} \pm 0.0017$
$\mathrm{H}_{p_2}\|0.1$	0.836 ± 0.0014	0.868 ± 0.0013	0.915 ± 0.0008	0.949 ± 0.0007	1.065 ± 0.0004

we provide RMSE_0, the RMSE of Q_{00}, so that we can easily recover the RMSE of all other estimators. The following rows provide the REFF-indicators for the different VaR-estimators under study. A similar mark (**bold**) is used for the highest REFF indicator, again considering the aforementioned two regions and $q = 1/n$.

For a better visualization of the results presented in some of the tables above, we further present Figure 2, associated with an $\mathrm{EV}_{0.1}$ underlying parent.

Table 6: Simulated RMSE_0 (first row) and of $\text{REFF}_{\bullet|0}$ indicators, for $\text{EV}_{0.1}$ parents.

	$n = 100$	$n = 200$	$n = 500$	$n = 1000$	$n = 5000$
RMSE_0	0.329 ± 0.1224	0.273 ± 0.1209	0.225 ± 0.1059	0.200 ± 0.0754	0.157 ± 0.0324
CH	1.287 ± 0.0154	1.323 ± 0.0147	1.252 ± 0.0123	1.202 ± 0.0083	1.073 ± 0.0041
H_{p_1}	$\mathbf{1.566} \pm 0.0174$	$\mathbf{1.505} \pm 0.0129$	$\mathbf{1.460} \pm 0.0103$	$\mathbf{1.440} \pm 0.0093$	$\mathbf{1.545} \pm 0.0113$
H_{p_2}	1.450 ± 0.0177	1.379 ± 0.0117	1.316 ± 0.0084	1.279 ± 0.0086	1.189 ± 0.0063
H|0	1.132 ± 0.0093	1.121 ± 0.0060	1.118 ± 0.0049	1.122 ± 0.0049	1.136 ± 0.0057
CH|0	1.659 ± 0.0196	1.833 ± 0.0179	1.548 ± 0.0202	1.373 ± 0.0110	1.202 ± 0.0077
$\text{H}_{p_1}\|0$	$\mathbf{1.695} \pm 0.0190$	$\mathbf{1.626} \pm 0.0149$	$\mathbf{1.614} \pm 0.0128$	$\mathbf{1.874} \pm 0.0160$	$\mathbf{4.988} \pm 0.0340$
$\text{H}_{p_2}\|0$	1.529 ± 0.0184	1.440 ± 0.0113	1.359 ± 0.0082	1.323 ± 0.0097	1.240 ± 0.0066

Table 7: Simulated RMSE_0 (first row) and of $\text{REFF}_{\bullet|0}$ indicators, for $\text{EV}_{0.25}$ parents.

	$n = 100$	$n = 200$	$n = 500$	$n = 1000$	$n = 5000$
RMSE_0	0.469 ± 0.1207	0.394 ± 0.1350	0.329 ± 0.1453	0.294 ± 0.1498	0.231 ± 0.1538
CH	1.393 ± 0.0144	1.431 ± 0.0155	1.681 ± 0.0215	1.908 ± 0.0257	1.197 ± 0.0045
H_{p_1}	$\mathbf{2.132} \pm 0.0218$	$\mathbf{2.522} \pm 0.0233$	$\mathbf{3.802} \pm 0.0333$	$\mathbf{3.866} \pm 0.0248$	$\mathbf{3.108} \pm 0.0229$
H_{p_2}	1.771 ± 0.0148	1.658 ± 0.0154	1.540 ± 0.0118	1.464 ± 0.0118	1.283 ± 0.095
H|0	1.178 ± 0.0081	1.174 ± 0.0101	1.185 ± 0.0053	1.206 ± 0.0060	1.245 ± 0.0043
CH|0	1.837 ± 0.0164	1.907 ± 0.0206	2.215 ± 0.0222	2.678 ± 0.0251	2.681 ± 0.0180
$\text{H}_{p_1}\|0$	$\mathbf{3.527} \pm 0.0300$	$\mathbf{2.754} \pm 0.0221$	1.703 ± 0.0135	1.584 ± 0.0128	1.443 ± 0.0102
$\text{H}_{p_2}\|0$	1.896 ± 0.0176	1.757 ± 0.0162	1.614 ± 0.0144	1.526 ± 0.0131	1.338 ± 0.0110

Table 8: Simulated RMSE_0 (first row) and of $\text{REFF}_{\bullet|0}$ indicators, for underlying $\text{EV}_{0.5}$ parents.

	$n = 100$	$n = 200$	$n = 500$	$n = 1000$	$n = 5000$
RMSE_0	0.811 ± 0.1588	0.664 ± 0.1728	0.539 ± 0.1793	0.467 ± 0.1810	0.341 ± 0.1782
CH	1.376 ± 0.2291	1.564 ± 0.0183	$\mathbf{1.702} \pm 0.0217$	$\mathbf{1.871} \pm 0.0167$	$\mathbf{3.047} \pm 0.0223$
H_{p_1}	$\mathbf{2.179} \pm 0.0278$	$\mathbf{1.650} \pm 0.0185$	1.343 ± 0.0160	1.201 ± 0.0204	1.044 ± 0.0135
H_{p_2}	2.439 ± 0.0361	2.137 ± 0.0277	1.843 ± 0.0194	1.677 ± 0.0153	1.560 ± 0.0133
H|0	1.262 ± 0.0106	1.255 ± 0.0119	1.289 ± 0.0119	1.319 ± 0.0093	1.387 ± 0.0068
CH|0	$\mathbf{2.192} \pm 0.0308$	$\mathbf{2.082} \pm 0.0248$	$\mathbf{2.093} \pm 0.0206$	$\mathbf{2.166} \pm 0.0173$	$\mathbf{2.560} \pm \pm 0.0226$
$\text{H}_{p_1}\|0$	1.890 ± 0.0258	1.586 ± 0.0224	1.404 ± 0.0261	1.340 ± 0.0317	1.344 ± 0.0150
$\text{H}_{p_2}\|0$	$\mathbf{2.595} \pm 0.0418$	$\mathbf{2.259} \pm 0.0309$	2.081 ± 0.0218	2.063 ± 0.0186	2.389 ± 0.0203

Table 9: Simulated RMSE_0 (first row) and of $\text{REFF}_{\bullet|0}$ indicators, for Student t_4 parents ($\xi = 0.25$)

	$n = 100$	$n = 200$	$n = 500$	$n = 1000$	$n = 5000$
RMSE_0	0.378 ± 0.1445	0.320 ± 0.1507	0.270 ± 0.1556	0.240 ± 0.1572	0.185 ± 0.1554
CH	1.217 ± 0.1176	1.310 ± 0.0129	1.480 ± 0.0134	$\mathbf{1.881} \pm 0.0114$	$\mathbf{1.531} \pm 0.0095$
H_{p_1}	$\mathbf{2.143} \pm 0.0187$	$\mathbf{2.483} \pm 0.0209$	$\mathbf{1.821} \pm 0.0148$	1.422 ± 0.0100	1.151 ± 0.0088
H_{p_2}	1.713 ± 0.0167	1.631 ± 0.0152	1.518 ± 0.0122	1.427 ± 0.0066	1.270 ± 0.0069
H $\|0.1$	1.243 ± 0.0105	1.273 ± 0.0081	1.359 ± 0.0066	1.457 ± 0.0064	1.808 ± 0.0069
CH $\|0.1$	1.773 ± 0.0160	2.038 ± 0.0181	$\mathbf{2.599} \pm 0.0252$	$\mathbf{3.082} \pm 0.0198$	$\mathbf{4.431} \pm 0.0269$
$\text{H}_{p_1}\|0.1$	1.640 ± 0.0119	1.516 ± 0.0138	1.463 ± 0.0161	1.477 ± 0.0206	1.664 ± 0.0240
$\text{H}_{p_2}\|0.1$	2.080 ± 0.0205	2.288 ± 0.0238	$\mathbf{3.045} \pm 0.0248$	$\mathbf{4.026} \pm 0.0291$	$\mathbf{6.345} \pm 0.0502$

Table 10: Simulated RMSE_0 (first row) and of $\text{REFF}_{\bullet|0}$ indicators, for underlying Student t_2 parents ($\xi = 0.5$)

	$n = 100$	$n = 200$	$n = 500$	$n = 1000$	$n = 5000$
RMSE_0	0.675 ± 0.1735	0.559 ± 0.1793	0.449 ± 0.1804	0.379 ± 0.1789	0.255 ± 0.1684
CH	0.684 ± 0.3593	1.359 ± 0.0145	$\mathbf{1.388} \pm 0.0099$	$\mathbf{1.371} \pm 0.0080$	$\mathbf{1.449} \pm 0.0343$
H_{p_1}	$\mathbf{1.728} \pm 0.0148$	$\mathbf{1.468} \pm 0.0116$	1.308 ± 0.0085	1.240 ± 0.0047	1.209 ± 0.051
H_{p_2}	2.271 ± 0.0214	1.992 ± 0.0179	1.766 ± 0.0126	1.665 ± 0.0151	1.691 ± 0.0124
H $\|0.1$	1.318 ± 0.0097	1.382 ± 0.0099	1.532 ± 0.0117	$\mathbf{1.667} \pm 0.0079$	$\mathbf{2.110} \pm 0.0132$
CH $\|0.1$	$\mathbf{1.969} \pm 0.0160$	$\mathbf{1.786} \pm 0.0150$	$\mathbf{1.573} \pm 0.0103$	1.419 ± 0.0091	1.123 ± 0.0065
$\text{H}_{p_1}\|0.1$	1.609 ± 0.0149	1.516 ± 0.0116	1.560 ± 0.0109	1.647 ± 0.0072	2.037 ± 0.0108
$\text{H}_{p_2}\|0.1$	$\mathbf{2.810} \pm 0.0276$	$\mathbf{2.821} \pm 0.0287$	$\mathbf{3.116} \pm 0.0227$	$\mathbf{3.547} \pm 0.0280$	$\mathbf{2.848} \pm 0.0169$

5 CONCLUSIONS

The new PORT-MO_p VaR-estimators, defined in (1.14), generalize the Weissman-Hill PORT-quantile estimator studied in Araújo Santos *et al.* (2006). Consequently, both asymptotically and for finite sample sizes, we were expecting a much better behaviour of this new VaR-estimator. The gain in efficiency of the PORT-MO_p VaR-estimators is, in most cases, greater than the one obtained with the MVRB and PORT-MVRB VaR-estimators. The simulated mean values of the normalized PORT-MO_p VaR-estimators are always better, for moderate to large values of n, in the Student-t_ν parents. For the EV_ξ-parents, we have different behaviours accordingly to the

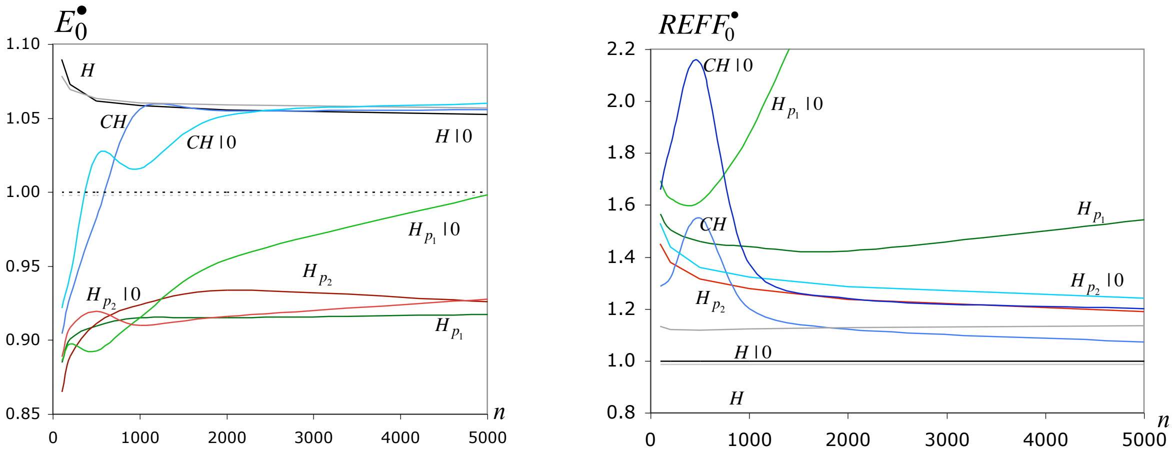

Figure 2: Mean values (*left*) and REFF-indicators (*right*) at optimal levels of the different normalized VaR-estimators under study, for $q = 1/n$, an underlying $\mathrm{EV}_{0.1}$ parent and sample sizes $n = 100(100)500(500)5000$

size of the sample but there is a general out-performance of the PORT-MO$_p$ VaR-estimators. And indeed, for an adequate choice of k, p and s, the PORT-MO$_p$ VaR–estimators are able to outperform the MVRB and even the PORT-MVRB VaR-estimators, in most cases. The choice of (k, p, s) can be done through heuristic sample-path stability algorithms, like the ones in Gomes *et al.* (2013) or through a bootstrap algorithm of the type of the ones presented in Caeiro and Gomes (2015a) and in Gomes *et al.* (2016b), where R-scripts are provided.

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