New reduced-bias estimators of a positive extreme value index

M. Ivette Gomes

CEAUL and DEIO, FCUL, Universidade de Lisboa, e-mail: ivette.gomes@fc.ul.pt

M. Fátima Brilhante

CEAUL and DM, Universidade dos Açores, e-mail: fbrilhante@uac.pt

Dinis Pestana

CEAUL and DEIO, FCUL, Universidade de Lisboa, e-mail: dinis.pestana@fc.ul.pt

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Abstract

Reduced-bias versions of a very simple generalization of the classical Hill estimator of a positive *extreme value index* (EVI) are introduced. The Hill estimator can be regarded as the logarithm of the mean-of-order-0 of a certain set of statistics. Instead of such a geometric mean, we can more generally consider the mean-of-order-p (MOP) of those statistics, with $p \geq 0$. The asymptotic behaviour of the class of MOP EVI-estimators for $p < 1/\gamma$ is reviewed and associated reduced-bias MOP (RBMOP) and optimal RBMOP classes of EVI-estimators are suggested and studied both asymptotically and for finite samples, through a large-scale simulation study.

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1 Introduction

Let us consider a sample of size n of independent and identically distributed (i.i.d.) random variables (r.v.'s), $X_1, \ldots, X_n$, with a common distribution function (d.f.) F. Let us denote by $X_{1:n} \leq \cdots \leq X_{n:n}$ the associated ascending order statistics. Let us further assume that there exist sequences of real constants $\{a_n > 0\}$ and $\{b_n \in \mathbb{R}\}$ such that the maximum, linearly normalized, i.e. $(X_{n:n} - b_n)/a_n$, converges in distribution to a non-degenerate random variable. Then (Gnedenko, 1943), the limit distribution is necessarily of the type of the general *extreme value* (EV) d.f., given by the functional expression,

$$\mathrm{EV}_\gamma(x) = \begin{cases} \exp(-(1+\gamma x)^{-1/\gamma}), \; 1+\gamma x > 0, & \text{if} \quad \gamma \neq 0, \\ \\ \exp(-\exp(-x)), \; x \in \mathbb{R}, & \text{if} \quad \gamma = 0. \end{cases} \tag{1.1}$$

The d.f. F is then said to belong to the *max-domain of attraction* of EV_γ, and we write $F \in \mathcal{D}_{\mathcal{M}}(\mathrm{EV}_\gamma)$. The parameter γ is the *extreme value index* (EVI), the primary parameter of extreme events. This index measures the heaviness of the *right-tail function* $\overline{F}(x) := 1 - F(x)$, and the heavier the right-tail, the larger γ is. We shall work here with *heavy-tailed* models, i.e., *Pareto-type* underlying d.f.'s, with a positive EVI. These heavy-tailed models are quite common in the most diverse areas of application, like bibliometrics, biostatistics, computer science, finance, insurance, social sciences and telecommunications, among others.

For heavy-tailed models, the classical EVI-estimators are the Hill estimators (Hill, 1975), which are the averages of the log-excesses, $V_{ik} := \ln X_{n-i+1:n} - \ln X_{n-k:n}, \; 1 \leq i \leq k < n$, i.e.

$$\mathrm{H}(k) := \frac{1}{k} \sum_{i=1}^{k} V_{ik}, \qquad 1 \leq k < n. \tag{1.2}$$

Since

$$\mathrm{H}(k) = \sum_{i=1}^{k} \ln \left(\frac{X_{n-i+1:n}}{X_{n-k:n}} \right)^{1/k} = \ln \left(\prod_{i=1}^{k} \frac{X_{n-i+1:n}}{X_{n-k:n}} \right)^{1/k},$$

the Hill estimator is the logarithm of the *geometric mean* (or *mean-of-order*-0) of the statistics $U_{ik} := X_{n-i+1:n}/X_{n-k:n}, \; 1 \leq i \leq k < n$. More generally, we can consider as basic statistics the *mean-of-order-p* (MOP) of U_{ik}, with $p \geq 0$, i.e. the class of statistics

$$A_p(k) = \begin{cases} \left(\sum_{i=1}^{k} U_{ik}^p/k\right)^{1/p}, & \text{if} \quad p > 0, \\ \\ \left(\prod_{i=1}^{k} U_{ik}\right)^{1/k}, & \text{if} \quad p = 0, \end{cases} \tag{1.3}$$

and the associated class of MOP EVI-estimators, introduced and studied in Brilhante *et al.* (2013a), dependent now on a *tuning* parameter $p \geq 0$, and with the functional expression,

$$\mathrm{H}_p(k) := \begin{cases} \left(1 - A_p^{-p}(k)\right)/p, & \text{if} \quad 0 < p < 1/\gamma, \\ \\ \ln A_0(k) = \mathrm{H}(k), & \text{if} \quad p = 0, \end{cases} \tag{1.4}$$

with $\mathrm{H}_0(k) \equiv \mathrm{H}(k)$ given in (1.2) and $A_p(k)$ given in (1.3). The class of MOP EVI-estimators in (1.4) is highly flexible, but it is not asymptotically unbiased for large and even for moderate k-values, the ones that lead to minimum *mean square error* (MSE) in a classical EVI-estimation done through $\mathrm{H}_p(k)$.

In this paper, after the introduction, in Section 2, of a few technical details in the field of *extreme value theory* (EVT), we provide a brief reference to one of the most simple minimum-variance reduced-bias (MVRB) EVI-estimators, the corrected-Hill (CH) estimator introduced and studied in Caeiro *et al.* (2005). In Section 3, we introduce a class of reduced-bias MOP (RBMOP) EVI-estimators, still dependent on a tuning parameter p, and an optimal RBMOP (ORBMOP) EVI-estimator for the p-values where we can guarantee the asymptotic normality of the statistics in (1.4), i.e. for $0 \leq p < 1/(2\gamma)$. We also deal with the derivation of the asymptotic behaviour of such classes and the adequate estimation of second-order parameters. Section 4 is dedicated to the study of the finite sample properties of the new classes of estimators comparatively to the behaviour of the aforementioned MOP and CH EVI-estimators, done through a large-scale Monte-Carlo simulation study.

2 Technical details in the field of EVT

In the area of *statistics of extremes* and whenever working with large values, i.e. with the right tail of the model F underlying the data, a model F is usually said to be *heavy-tailed* whenever the right tail-function $\overline{F} = 1 - F$ is a regularly varying function with a negative index of regular variation equal to $-1/\gamma$, $\gamma > 0$, and we use the notation $\overline{F} \in \mathrm{RV}_{-1/\gamma}$. Note that a regular varying function with an index of regular variation equal to $a \in \mathbb{R}$, i.e. an element of RV_a, is a positive measurable function $g(\cdot)$ such that for all $x > 0$, $g(tx)/g(t) \to x^a$, as $t \to \infty$ (see Bingham *et al.*, 1987). Heavy-tailed models are thus such that $\overline{F}(x) = x^{-1/\gamma}L(x)$, $\gamma > 0$, with $L \in \mathrm{RV}_0$, a *regularly varying* function with an *index of regular variation* equal to zero, also called a *slowly varying* function at infinity. Equivalently, with $F^{\leftarrow}(x) := \inf\{y : F(y) \geq x\}$, the *reciprocal quantile function* $U(t) := F^{\leftarrow}(1 - 1/t),\ t \geq 1$, is of regular variation with index γ (de Haan, 1984), i.e. $U \in \mathrm{RV}_\gamma$.

2.1 A brief review of first and second-order conditions for a heavy right-tail

We assume the validity of any of the equivalent first-order conditions,

$$F \in \mathcal{D}^+_{\mathcal{M}} := \mathcal{D}_{\mathcal{M}}\left(EV_\gamma\right)_{\gamma>0} \iff \lim_{t\to\infty} \frac{\overline{F}(tx)}{\overline{F}(t)} = x^{-1/\gamma},\ \forall x > 0\ \left(\overline{F} \in RV_{-1/\gamma}\right)$$
$$\iff \lim_{t\to\infty} \frac{U(tx)}{U(t)} = x^{\gamma},\ \forall x > 0\ \left(U \in RV_\gamma\right). \quad (2.1)$$

The second-order parameter ρ (≤ 0) measures the rate of convergence in the first-order conditions, in (2.1), and can be defined as the non-positive parameter in the limiting relation,

$$\lim_{t\to\infty} \frac{\ln U(tx) - \ln U(t) - \gamma \ln x}{A(t)} = \begin{cases} \frac{x^\rho - 1}{\rho}, & \text{if} \quad \rho < 0, \\ \ln x, & \text{if} \quad \rho = 0, \end{cases} \quad (2.2)$$

$x > 0$, and where $|A|$ must be of regular variation with an index ρ (Geluk and de Haan, 1987). This condition has been widely accepted as an appropriate condition to specify the right tail of a Pareto-type distribution in a semi-parametric way and enables easily the derivation of the non-degenerate bias of EVI-estimators, under a semi-parametric framework. Further developments of the topic can be found in de Haan and Ferreira (2006).

In order to have consistency of EVI-estimators in all $\mathcal{D}_{\mathcal{M}}^{+}$, we need to work with intermediate values of k, i.e., a sequence of integers $k = k_n$, $1 \le k < n$, such that

$$k = k_n \to \infty \quad \text{and} \quad k_n = o(n), \text{ as } n \to \infty. \tag{2.3}$$

Under the second-order framework in (2.2), the asymptotic distributional representation

$$\mathrm{H}(k) - \gamma \equiv H_0(k) - \gamma \overset{d}{=} \frac{\gamma Z_k^{(0)}}{\sqrt{k}} + \frac{A(n/k)}{1-\rho}(1 + o_p(1)) \tag{2.4}$$

holds (de Haan and Peng, 1998), where, with $\{E_i\}$ a sequence of i.i.d. standard exponential r.v.'s, $Z_k^{(0)} = \sqrt{k}\big(\sum_{i=1}^{k} E_i/k - 1\big)$ is an asymptotically standard normal random variable.

Thinking on the MOP functionals in (1.4), we can further state the following theorem:

Theorem 1 (Brilhante *et al.*, 2013a). *Under the validity of the first-order condition in* (2.1), *and for intermediate sequences* $k = k_n$, *i.e. whenever* (2.3) *holds, the class of estimators* $\mathrm{H}_p(k)$, *in* (1.4) *is consistent for the estimation of* γ *whenever* $0 \le p < 1/\gamma$.

If we further assume the validity of the second-order condition, in (2.2), *and with the notation*

$$\sigma_{\mathrm{H}_p} \equiv \sigma_{\mathrm{H}_p}(\gamma) := \frac{\gamma(1-p\gamma)}{\sqrt{1-2p\gamma}}, \quad b_{\mathrm{H}_p} \equiv b_{\mathrm{H}_p}(\gamma,\rho) := \frac{1-p\gamma}{1-p\gamma-\rho}, \tag{2.5}$$

the asymptotic distributional representation

$$\mathrm{H}_p(k) \overset{d}{=} \gamma + \frac{\sigma_{\mathrm{H}_p} Z_k^{(p)}}{\sqrt{k}} + b_{\mathrm{H}_p} A(n/k) + o_p\big(A(n/k)\big), \tag{2.6}$$

holds for $0 \le p < 1/(2\gamma)$, *with* $Z_k^{(p)}$ *asymptotically standard normal.*

2.2 The class of CH EVI-estimators

If we look at the asymptotic distributional representation in (2.4), or more generally to the one in (2.6), we see that these estimators reveal usually a high asymptotic bias, i.e., as $n \to \infty$, for intermediate k and under the validity of the general second-order condition, in (2.2), $\sqrt{k}\,(\mathrm{H}_p(k) - \gamma)$

is asymptotically normal with variance $\sigma^2_{\mathrm{H}_p}$ and a non-null mean value, equal to λb_{H_p}, whenever $\sqrt{k}\ A(n/k) \ \rightarrow\ \lambda \neq 0$, finite, the type of k-values which lead to minimum MSE of $\mathrm{H}_p(k)$. More precisely, it follows from the results of de Haan and Peng (1998) and Brilhante *et al.* (2013a) that under the general second-order condition, in (2.2), and with the notation $\mathcal{N}_{\mu,\sigma^2}$ for a normal r.v. with mean value μ and variance σ^2,

$$\sqrt{k}\,(\mathrm{H}_p(k)-\gamma) \ \stackrel{d}{=}\ \mathcal{N}_{0,\sigma^2_{\mathrm{H}_p}} + b_{\mathrm{H}_p}\sqrt{k}A(n/k) + o_p\big(\sqrt{k}A(n/k)\big),$$

where the bias $b_{\mathrm{H}_p}\sqrt{k}A(n/k)$ can be very large, moderate or small (i.e. go to infinity, constant or zero) as $n \rightarrow \infty$. It is thus sensible to try reducing the dominant component of bias, a topic extensively addressed in the more recent literature in the area of statistics of extremes.

Whenever dealing with bias reduction in the field of extremes, it is usual to consider a slightly more restrict class than $\mathcal{D}^+_{\mathcal{M}}$, the class of models

$$U(t) = C\ t^{\gamma}\Big(1 + A(t)/\rho + o(t^{\rho})\Big), \qquad A(t) := \gamma\beta t^{\rho}, \tag{2.7}$$

as $t \rightarrow \infty$, where $C > 0$, $\gamma > 0$, $\rho < 0$ and $\beta \neq 0$ (Hall and Welsh, 1985). This means that the slowly varying function $L(t)$ in $U(t) = t^{\gamma}L(t)$ is assumed to behave asymptotically as a constant. To assume (2.7) is equivalent to assume the possibility of choosing $A(t) = \gamma\beta t^{\rho}$, $\rho < 0$, in the more general second-order condition in (2.2). Models like the log-gamma and the log-Pareto ($\rho = 0$) are thus excluded from this class. The standard Pareto is also excluded. But most heavy-tailed models used in applications, like the EV_{γ}, in (1.1) and the Student's t d.f.'s, among others, belong to Hall-Welsh class.

The simplest class of CH EVI-estimators, introduced in Caeiro *et al.* (2005), is now considered. Such a class is defined as

$$\mathrm{CH}(k) \equiv \mathrm{CH}(k;\hat\beta,\hat\rho) := \mathrm{H}(k)\Big(1 - \hat\beta(n/k)^{\hat\rho}/(1-\hat\rho)\Big). \tag{2.8}$$

The estimators in (2.8) can be second-order minimum-variance reduced-bias (MVRB) EVI-estimators, for adequate levels k and an adequate external estimation of the vector of second-order parameters, (β,ρ), defined in (2.7), i.e. the use of $\mathrm{CH}(k)$ can enable us to eliminate the dominant component of bias of the Hill estimator, $\mathrm{H}(k)$, keeping its asymptotic variance. Indeed, from the results in Caeiro *et al.* (2005), we know that it is possible to adequately estimate the second-order parameters β and ρ, so that we get

$$\sqrt{k}\,(\mathrm{CH}(k)-\gamma) \stackrel{d}{=} \mathcal{N}_{0,\gamma^2} + o_p\big(\sqrt{k}A(n/k)\big),$$

i.e. $\mathrm{CH}(k)$ overpasses $\mathrm{H}(k)$ for all k.

For details on algorithms for the (β,ρ)-estimation, see Gomes and Pestana (2007a,b) and Gomes *et al.* (2008b). We have so far suggested the use of the ρ-estimators in Fraga Alves *et al.* (2003) and the β-estimators in Gomes and Martins (2002), described later on, in *Algorithm* 3.1. Note however that recent classes of β-estimators (Caeiro and Gomes, 2006, 2008; Gomes *et al.*, 2010) and ρ-estimators (Goegebeur *et al.*, 2008, 2010; Ciuperca and Mercadier, 2010; Caeiro and Gomes, 2012a,b), among others, are potential candidates for the (β,ρ)-estimation. Overviews on reduced-bias estimation can be found in Chapter 6 of Reiss and Thomas (2007), Gomes *et al.* (2008a) and Beirlant *et al.* (2012).

3 The new classes of RBMOP and ORBMOP EVI-estimators

If we look at the dominant component of bias, $b_{\mathrm{H}_p} := (1-p\gamma)A(n/k)/(1-p\gamma-\rho)$, provided in Theorem 1, we see that for $0 \le p < 1/(2\gamma)$ it is sensible to consider the class of RBMOP EVI-estimators,

$$\mathrm{CH}_p(k) \equiv \mathrm{RBMOP}(k) \equiv \mathrm{CH}_p(k;\hat\beta,\hat\rho) := \mathrm{H}_p(k)\Big(1-\frac{\hat\beta(1-p\mathrm{H}_p(k))}{1-\hat\rho-p\mathrm{H}_p(k)}\Big(\frac{n}{k}\Big)^{\hat\rho}\Big). \tag{3.1}$$

This class is similar in spirit to the MVRB EVI-estimators, in (2.8), and the main reasons for such a consideration are also similar to the ones presented in the aforementioned article, and already sketched above.

Working in the class of models in (2.7) for technical simplicity, and working with values of p such that asymptotic normality of the functionals in (1.4) holds, i.e. $0 \le p < 1/(2\gamma)$, Brilhante *et al.* (2013b) noticed that there is an optimal value $p \equiv p_{\mathrm{M}} = \varphi(\rho)/\gamma$, with $\varphi(\rho) =$

$1-\rho/2-\sqrt{\rho^2-4\rho+2}/2$, which maximizes the efficiency of the class of estimators in (1.4). Then, under the second-oder framework, in (2.2), and with the notation H^* for the MOP EVI-estimator associated to $p \equiv p_{_\mathrm{M}}$, and with the notation

$$\sigma_{\mathrm{H}^*} \equiv \sigma_{\mathrm{H}^*}(\gamma,\rho) := \frac{\gamma(1-\varphi(\rho))}{\sqrt{1-2\varphi(\rho)}}, \qquad b_{\mathrm{H}^*} \equiv b_{\mathrm{H}^*}(\rho) := \frac{1-\varphi(\rho)}{1-\varphi(\rho)-\rho} \tag{3.2}$$

they get the validity of the asymptotic distributional representation

$$\mathrm{H}^*(k) := \mathrm{H}_{p_{_\mathrm{M}}}(k) \stackrel{d}{=} \gamma + \frac{\sigma_{\mathrm{H}^*} Z_k^{(p_\mathrm{M})}}{\sqrt{k}} + b_{\mathrm{H}^*} A(n/k) + o_p\big(A(n/k)\big), \tag{3.3}$$

which immediately suggests the class of ORBMOP EVI-estimators, with a functional expression

$$\mathrm{CH}^*(k) \equiv \mathrm{ORBMOP}(k) \equiv \mathrm{CH}^*(k;\hat\beta,\hat\rho) := \mathrm{H}^*(k)\Big(1 - \frac{\hat\beta(1-\varphi(\hat\rho))}{1-\hat\rho-\varphi(\hat\rho)}\left(\frac{n}{k}\right)^{\hat\rho}\Big). \tag{3.4}$$

3.1 Asymptotic behaviour of the RBMOP and ORBMOP EVI-estimators

We next state the main theorem in this paper.

Theorem 2. *Under the second-order framework in* (2.7), $Z_k^{(p)}$ *and* $Z_k^* := Z_k^{(p_\mathrm{M})}$ *asymptotically standard normal r.v.'s, and for intermediate* k, *i.e., if* (2.3) *holds, we can write*

$$\mathrm{CH}_p(k;\beta,\rho) \stackrel{d}{=} \gamma + \frac{\sigma_{\mathrm{H}_p} Z_k^{(p)}}{\sqrt{k}} + o_p\big(A(n/k)\big)$$

and

$$\mathrm{CH}^*(k;\beta,\rho) \stackrel{d}{=} \gamma + \frac{\sigma_{\mathrm{H}^*}\ Z_k^*}{\sqrt{k}} + o_p\big(A(n/k)\big),$$

with σ_{H_p} *and* σ_{H^*} *defined in* (2.5) *and* (3.2), *respectively. Consequently, if* $\sqrt{k}\ A(n/k) \to \lambda$, *finite, and denoting by* UH *either* CH_p *or* CH^*

$$\sqrt{k}\ \ (\mathrm{UH}(k;\beta,\rho)-\gamma) \underset{n\to\infty}{\overset{d}{\longrightarrow}} N_{0,\sigma^2_{\mathrm{UH}}}. \tag{3.5}$$

Moreover, if we consistently estimate the vector (β,ρ) *of second-order parameters through* $(\hat\beta,\hat\rho)$, *with* $\hat\rho-\rho = o_p(ln(n/k))$, (3.5) *still holds, with* $\mathrm{UH}(k;\beta,\rho)$ *replaced by* $\mathrm{UH}(k;\hat\beta,\hat\rho)$.

Proof. We can write

$$\mathrm{CH}_p(k;\beta,\rho) = \mathrm{H}_p(k)\Big(1 - \frac{\beta(1-p\mathrm{H}_p(k))}{1-\rho-p\mathrm{H}_p(k)}\left(\frac{n}{k}\right)^{\rho}\Big) = \mathrm{H}_p(k)\Big(1 - \frac{(1-p\mathrm{H}_p(k))A(n/k)}{\gamma(1-\rho-p\mathrm{H}_p(k))}\Big).$$

But

$$\frac{(1-p\mathrm{H}_p(k))A(n/k)}{\gamma(1-\rho-p\mathrm{H}_p(k))} \stackrel{d}{=} \frac{(1-p\gamma)A(n/k)}{\gamma(1-\rho-p\gamma)} + o_p(A(n/k)). \tag{3.6}$$

If we look at (2.6) and (3.6), we immediately get

$$\mathrm{CH}_p(k;\beta,\rho) \stackrel{d}{=} \gamma + \frac{\gamma(1-p\gamma)Z_k^{(p)}}{\sqrt{k}\sqrt{1-2p\gamma}} + o_p(A(n/k)),$$

and (3.5) follows for $\mathrm{UH} = \mathrm{CH}_p$.

If we think about $\mathrm{CH}^*(k;\beta,\rho)$, with $\mathrm{CH}^*(k;\hat\beta,\hat\rho)$, defined in (3.4), and on the asymptotic distributional representation of $\mathrm{H}^*(k)$, provided in (3.3), we get the asymptotic distributional representation,

$$\begin{aligned}\mathrm{CH}^*(k;\beta,\rho) = \mathrm{H}^*(k)\Big(1 - \frac{\beta(1-\varphi(\rho))}{1-\rho-\varphi(\rho)}\left(\frac{n}{k}\right)^{\rho}\Big) &= \mathrm{H}^*(k)\Big(1 - \frac{(1-\varphi(\rho))A(n/k)}{\gamma(1-\rho-\varphi(\rho))}\Big)\\ &\stackrel{d}{=} \gamma + \frac{\gamma(1-\varphi(\rho))Z_k^{(p_\mathrm{M})}}{\sqrt{k}\sqrt{1-2\varphi(\rho)}} + o_p\big(A(n/k)\big).\end{aligned}$$

Consequently, (3.5) follows for $\mathrm{UH} = \mathrm{CH}^*$.

Finally, and in the same lines of Caeiro *et al.* (2009), if we estimate consistently β and ρ through the estimators $\hat\beta$ and $\hat\rho$, we can use Cramer's delta-method, to guarantee that there exist values $a_{_\mathrm{UH}}$ and $b_{_\mathrm{UH}}$, such that

$$\mathrm{UH}(k;\hat\beta,\hat\rho) - \mathrm{UH}(k;\beta,\rho) \stackrel{p}{\sim} a_{_\mathrm{UH}}A(n/k)\Big\{\Big(\frac{\hat\beta-\beta}{\beta}\Big) + (\hat\rho-\rho)\left[\ln(n/k) - b_{_\mathrm{UH}}\right]\Big\}. \tag{3.7}$$

An asymptotic normal behaviour for $\mathrm{UH}(k;\hat\beta,\hat\rho)$, of the type of the one in (3.5), follows thus straightforwardly from (3.7). ■

On the basis of Theorem 2, we can say that whereas

$$\sqrt{k}\,(\mathrm{H}^*(k)-\gamma) \stackrel{d}{=} \mathcal{N}_{0,\sigma^2_{\mathrm{H}^*}} + O_p\big(\sqrt{k}A(n/k)\big),$$

$$\sqrt{k}\,(\mathrm{CH}^*(k)-\gamma) \stackrel{d}{=} \mathcal{N}_{0,\sigma^2_{\mathrm{H}^*}} + o_p\big(\sqrt{k}A(n/k)\big),$$

i.e. $\mathrm{CH}^*(k)$, in (3.4), outperforms $\mathrm{H}^*(k)$, in (3.3), for all k, just as $\mathrm{CH}(k)$, in (2.8), outperforms $\mathrm{H}(k)$, in (1.2), also for all k. Also, $\mathrm{CH}_p(k)$, in (3.1), outperforms $\mathrm{H}_p(k)$, in (1.4), for all k.

Remark 1. *We cannot forget that* $p_{\mathrm{M}} = \varphi(\rho)/\gamma$ *depends on* γ *and* ρ*, being thus sensible to develop an algorithm for the adequate choice of* p_{M}*, based either on sample path stability and similar to the one in* Gomes *et al.* (2012b) *or on the bootstrap methodology and similar to the ones in* Gomes *et al.* (2012a) *and* Brilhante *et al.* (2013a)*. However, note that it is also possible to consider the* RBMOP *classes of* EVI*-estimators in* (3.1)*, dependent on a tuning parameter* p *and similar algorithms for the choice of* p*. Further note that we can also work with* $\mathrm{CH}^*(k)$ *computed at an adequate estimate of* $k_{0|\mathrm{H}^*} := \arg\min_k \mathrm{MSE}(\mathrm{H}^*(k))$*, like*

$$\hat{k}_{0|\mathrm{H}^*} := \min\left(n-1, \left\lfloor \left((1-\varphi(\hat\rho)-\hat\rho)^2 n^{-2\hat\rho}/\big(-2\hat\rho\hat\beta^2(1-2\varphi(\hat\rho))\big)\right)^{1/(1-2\hat\rho)} \right\rfloor + 1\right),$$

where $\lfloor x \rfloor$ *denotes, as usual, the integer part of* x*. We are sure that* $\mathrm{CH}^*\big(\hat{k}_{0|\mathrm{H}^*}\big)$ *outperforms* $\mathrm{H}^*\big(\hat{k}_{0|\mathrm{H}^*}\big)$*, which on its turn overpasses* $\mathrm{H}\big(\hat{k}_{0|\mathrm{H}}\big)$*, with*

$$\hat{k}_{0|\mathrm{H}} := \min\left(n-1, \left\lfloor \left((1-\hat\rho)^2 n^{-2\hat\rho}/\big(-2\hat\rho\hat\beta^2\big)\right)^{1/(1-2\hat\rho)} \right\rfloor + 1\right),$$

the k*-estimate of* $k_{0|\mathrm{H}} := \arg\min_k \mathrm{MSE}(\mathrm{H}(k))$ *suggested in* Hall (1982)*.*

3.2 Estimation of second-order parameters

For the estimation of the vector of second-order parameters (β, ρ), we propose an algorithm of the type of the ones presented in Gomes and Pestana (2007a,b):

Algorithm 3.1 (Second-order parameters estimation).

Step 1 *Given an observed sample* $(x_1, \ldots, x_n)$*, compute for the tuning parameters* $\tau = 0$ *and* $\tau = 1$*, the observed values of* $\hat\rho_\tau(k)$*, the most simple class of estimators in* Fraga Alves *et al.* (2003)*. Such estimators have the functional form*

$$\hat\rho_\tau(k) := -\big|3(W_{k,n}^{(\tau)} - 1)/(W_{k,n}^{(\tau)} - 3)\big|,$$

dependent on the statistics

$$W_{k,n}^{(0)} := \frac{\ln\left(M_{k,n}^{(1)}\right) - \frac{1}{2}\ln\left(M_{k,n}^{(2)}/2\right)}{\frac{1}{2}\ln\left(M_{k,n}^{(2)}/2\right) - \frac{1}{3}\ln\left(M_{k,n}^{(3)}/6\right)}, \quad W_{k,n}^{(1)} := \frac{M_{k,n}^{(1)} - \left(M_{k,n}^{(2)}/2\right)^{1/2}}{\left(M_{k,n}^{(2)}/2\right)^{1/2} - \left(M_{k,n}^{(3)}/6\right)^{1/3}},$$

where

$$M_{k,n}^{(j)} := \frac{1}{k}\sum_{i=1}^{k}\Big(\ln X_{n-i+1:n} - \ln X_{n-k:n}\Big)^j, \; j = 1, 2, 3.$$

Step 2 *Consider* $\mathcal{K} = \left(\lfloor n^{0.995}\rfloor,\ \lfloor n^{0.999}\rfloor\right)$. *Compute the median of* $\{\hat\rho_\tau(k)\}_{k\in\mathcal{K}}$, *denoted* χ_τ, *and compute* $I_\tau := \sum_{k\in\mathcal{K}}\left(\hat\rho_\tau(k) - \chi_\tau\right)^2$, $\tau = 0, 1$. *Next choose the tuning parameter* $\tau^* = 0$ *if* $I_0 \le I_1$; *otherwise, choose* $\tau^* = 1$.

Step 3 *Work with* $\hat\rho \equiv \hat\rho_{\tau^*} = \hat\rho_{\tau^*}(k_1)$ *and* $\hat\beta \equiv \hat\beta_{\tau^*} := \hat\beta_{\hat\rho_{\tau^*}}(k_1)$, *with* $k_1 = \lfloor n^{0.999}\rfloor$, *being* $\hat\beta_{\hat\rho}(k)$ *the estimator in* Gomes and Martins (2002), *given by*

$$\hat\beta_{\hat\rho}(k) := \left(\frac{k}{n}\right)^{\hat\rho}\frac{d_k(\hat\rho)\ D_k(0) - D_k(\hat\rho)}{d_k(\hat\rho)\ D_k(\hat\rho) - D_k(2\hat\rho)},$$

dependent on the estimator $\hat\rho = \hat\rho_{\tau^*}(k_1)$, *and where, for any* $\alpha \le 0$,

$$d_k(\alpha) := \frac{1}{k}\sum_{i=1}^{k}(i/k)^{-\alpha} \quad \text{and} \quad D_k(\alpha) := \frac{1}{k}\sum_{i=1}^{k}(i/k)^{-\alpha}\,U_i,$$

with $U_i = i\ \left(\ln X_{n-i+1:n} - \ln X_{n-i:n}\right),\ 1 \le i \le k < n$, *the scaled log-spacings.*

4 Finite sample properties of the EVI-estimators

We have implemented large-scale multi-sample Monte-Carlo simulation experiments of size 5000×20 for the new classes of RBMOP and ORBMOP EVI-estimators, in (3.3) and (3.4), respectively, and for sample sizes $n = 100, 200, 500, 1000, 2000$ and 5000, $\gamma = 0.1, 0.25, 0.5$ and 1, from the following models:

1. Fréchet model, with d.f. $F(x) = \exp(-x^{-1/\gamma}),\ x \ge 0$;

2. Extreme value model, with d.f. $F(x) = \mathrm{EV}_\gamma(x)$, in (1.1);

3. $\mathrm{Burr}_{\gamma,\rho}$ model, with d.f. $F(x) = 1 - (1 + x^{-\rho/\gamma})^{1/\rho}$, $x \geq 0$, for the aforementioned values of γ and for $\rho = -0.1, -0.25, -0.5$ and -1;

4. Generalised Pareto model, with d.f. $F(x) = \mathrm{GP}_\gamma(x) = 1 + \ln \mathrm{EV}_\gamma(x) = 1 - \left(1 + \gamma x\right)^{-1/\gamma}$, $x \geq 0$.

We have further considered

5. Student-t_ν underlying parents, with $\nu = 2, 4$ ($\gamma = 1/\nu;\ \rho = -2/\nu$).

For details on multi-sample simulation, see Gomes and Oliveira (2001).

4.1 Mean values and MSE patterns

For each value of n and for each of the aforementioned models, we have first simulated the mean values (E) and root MSEs (RMSEs) of $\mathrm{H}(k)$, $\mathrm{CH}(k)$, $\mathrm{H}^*(k)$ and $\mathrm{CH}^*(k)$ EVI-estimators, as functions of the number of top order statistics k involved in the estimation, and on the basis of the first run of size 5000. As an illustration, we present Figures 1, 2 and 3, respectively associated to $\mathrm{EV}_{0.1}$, Student-t_4 and $\mathrm{GP}_{0.5}$ parents.

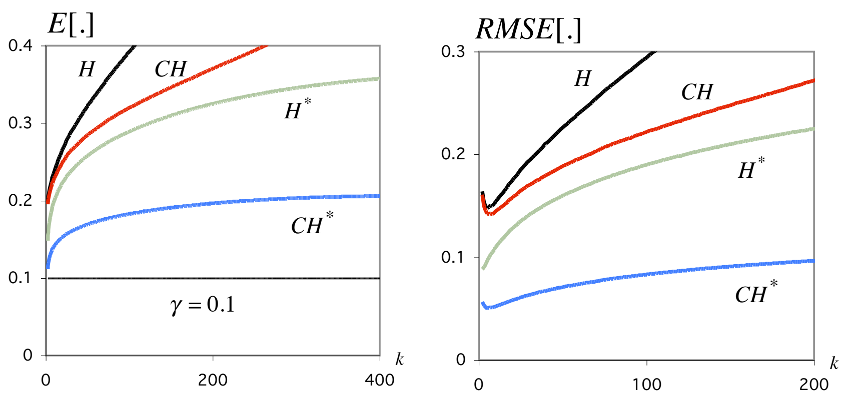

Figure 1: Mean values (*left*) and RMSEs (*right*) of $\mathrm{H}(k)$, $\mathrm{CH}(k)$, $\mathrm{H}^*(k)$ and $\mathrm{CH}(k) \equiv$ORBMOP(k), for an $\mathrm{EV}_{0.1}$ underlying parent

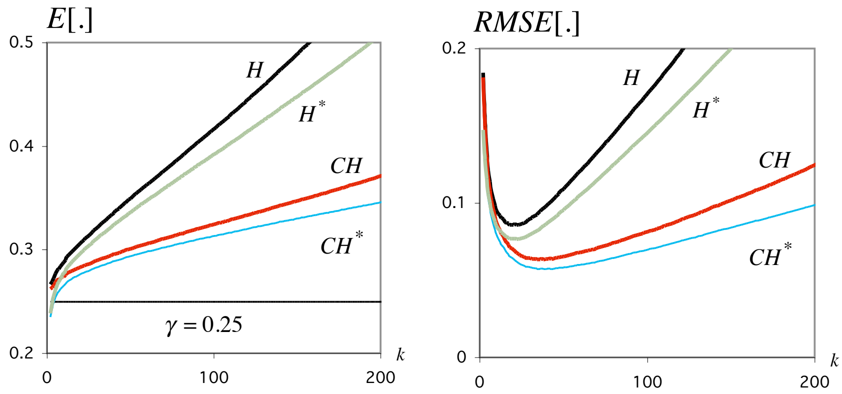

Figure 2: Mean values (*left*) and RMSEs (*right*) of H(k), CH(k), H*(k) and CH(k) ≡ORBMOP(k), for an Student-t_4 underlying parent ($\gamma = 1/4 = 0.25$)

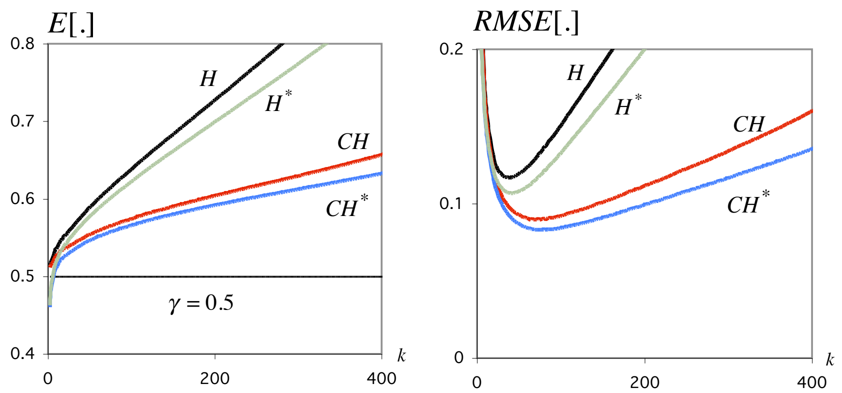

Figure 3: Mean values (*left*) and RMSEs (*right*) of H(k), CH(k), H*(k) and CH(k) ≡ORBMOP(k), for a GP$_{0.5}$ underlying parent

A few comments:

- There is always a reduction in RMSE, as well as in bias, with the obtention of estimates closer to the target value γ.
- Such a reduction is particularly high for values of ρ close to zero, even when we work with models out of the scope of Therem 2, like the the log-gamma and the log-Pareto.

4.1.1 Mean values of the EVI-estimators at optimal levels

As an illustration of the bias reduction achieved with the RBMOP EVI-estimators in (3.1) at optimal levels (levels where RMSE are minimal as functions of k), see Tables 1 and 2, respectively related to Fréchet and EV_1 models. We there present, for $n = 100,\ 200,\ 500,\ 1000,\ 2000$ and 5000, the simulated mean values at optimal levels of the $H(k)$, $CH(k)$ and $CH_p(k)$ EVI-estimators, in (1.2), (2.8) and (3.1), respectively, for $p = j/(10\gamma)$, $j = 1, 2, 3, 4$. We further present the same characteristics for the new EVI-estimators $H^*(k)$, in (3.3), and $CH^*(k) \equiv \mathrm{ORBMOP}(k)$, in (3.4). Information on 95% confidence intervals, computed on the basis of the 20 replicates with 5000 runs each, is also provided. Among the estimators considered, the one providing the smallest squared bias is underlined, and written in **bold**. Generally denoting T any of the aforementioned EVI-estimators, note that for Fréchet underlying parents, T/γ does not depend on γ.

Table 1: Simulated mean values, at optimal levels, of $H(k)/\gamma$, $CH(k)/\gamma$, $H_p(k)/\gamma$, $CH_p(k)/\gamma$ (independent on γ), with $p_j = j/(10\gamma)$, $j = 1, 2, 3, 4$, H^* and $CH^* \equiv$ ORBMOP, for *Fréchet* parents, together with 95% confidence intervals.

Fréchet parent, γ						
n	100	200	500	1000	2000	5000
H	1.109 ± 0.0027	1.085 ± 0.0028	1.063 ± 0.0013	1.049 ± 0.0014	1.039 ± 0.0009	1.029 ± 0.0006
CH	0.982 ± 0.0030	0.986 ± 0.0395	0.995 ± 0.0016	0.999 ± 0.0008	1.000 ± 0.0005	1.000 ± 0.0003
H_{p_1}	1.092 ± 0.0030	1.072 ± 0.0025	1.054 ± 0.0010	1.045 ± 0.0009	1.035 ± 0.0010	1.026 ± 0.0005
CH_{p_1}	1.002 ± 0.0029	1.006 ± 0.0377	1.010 ± 0.0014	1.010 ± 0.0009	1.010 ± 0.0006	1.009 ± 0.0003
H_{p_2}	1.096 ± 0.0022	1.078 ± 0.0026	1.058 ± 0.0011	1.047 ± 0.0010	1.038 ± 0.0008	1.028 ± 0.0005
CH_{p_2}	1.002 ± 0.0028	1.002 ± 0.0186	1.004 ± 0.0008	1.004 ± 0.0007	1.002 ± 0.0006	1.001 ± 0.0002
H_{p_3}	1.088 ± 0.0024	1.073 ± 0.0016	1.056 ± 0.0014	1.046 ± 0.0010	1.038 ± 0.0007	1.028 ± 0.0005
CH_{p_3}	1.011 ± 0.0028	1.010 ± 0.0162	1.009 ± 0.0008	1.006 ± 0.0006	1.005 ± 0.0005	1.002 ± 0.0002
H_{p_4}	1.078 ± 0.0018	1.069 ± 0.0018	1.054 ± 0.0010	1.045 ± 0.0010	1.037 ± 0.0007	1.0283 ± 0.0005
CH_{p_4}	1.011 ± 0.0028	1.010 ± 0.0162	1.009 ± 0.0008	1.006 ± 0.0006	1.005 ± 0.0005	1.002 ± 0.0002
H^*	1.095 ± 0.0024	1.075 ± 0.0026	1.057 ± 0.0010	1.047 ± 0.0009	1.037 ± 0.0011	1.028 ± 0.0006
CH^*	**0.983** ± 0.0034	**0.992** ± 0.0329	**0.998** ± 0.0015	**1.000** ± 0.0008	**1.000** ± 0.0005	**1.000** ± 0.0002

In Tables 3, 4, 5 and 6, we present, for the same values of n as in Tables 1 and 2, the simulated mean values at optimal levels of $H(k)$, in (1.2), $CH(k)$, in (2.8), $H^*(k)$, in (3.3), and $CH^*(k) \equiv \mathrm{ORBMOP}(k)$, in (3.4), for all other simulated models. Information on 95% confidence intervals, computed on the basis of the 20 replicates with 5000 runs each, is again

Table 2: Simulated mean values, at optimal levels, of $\mathrm{H}(k)$, $\mathrm{CH}(k)$, $\mathrm{H}_p(k)$, $\mathrm{CH}_p(k)$, with $p_j = j/(10\gamma)$, $j = 1, 2, 3, 4$, H^* and $\mathrm{CH}^* \equiv$ ORBMOP, for an EV_1 underlying parent, together with 95% confidence intervals

n	100	200	500	1000	2000	5000
			EV_γ **parent,** $\gamma = 1$ $(\rho = -1)$			
H	1.159 ± 0.0049	1.124 ± 0.0032	1.091 ± 0.0030	1.072 ± 0.0020	1.058 ± 0.0014	1.042 ± 0.0009
CH	0.894 ± 0.0099	0.975 ± 0.0046	$\underline{1.003} \pm 0.0024$	$\underline{1.004} \pm 0.0013$	$\underline{1.003} \pm 0.0007$	$\underline{1.001} \pm 0.0004$
H_{p_1}	1.1468 ± 0.0039	1.1167 ± 0.0031	1.0869 ± 0.0021	1.0690 ± 0.0020	1.0545 ± 0.0015	1.0408 ± 0.0008
CH_{p_1}	0.903 ± 0.0106	0.987 ± 0.0043	1.008 ± 0.0025	1.007 ± 0.0012	1.004 ± 0.0007	1.001 ± 0.0003
H_{p_2}	1.131 ± 0.0044	1.109 ± 0.0032	1.083 ± 0.0018	1.067 ± 0.0014	1.054 ± 0.0013	1.040 ± 0.0011
CH_{p_2}	0.918 ± 0.0129	0.996 ± 0.0034	1.012 ± 0.0018	1.010 ± 0.0009	1.006 ± 0.0006	1.002 ± 0.0003
H_{p_3}	1.1162 ± 0.0040	1.1008 ± 0.0024	1.0776 ± 0.0018	1.0653 ± 0.0015	1.0531 ± 0.0011	1.0399 ± 0.0006
CH_{p_3}	0.925 ± 0.0157	1.005 ± 0.0037	1.017 ± 0.0014	1.012 ± 0.0010	1.008 ± 0.0007	1.003 ± 0.0004
H_{p_4}	1.098 ± 0.0031	1.085 ± 0.0019	1.070 ± 0.0017	1.060 ± 0.0015	1.050 ± 0.0010	1.039 ± 0.0010
CH_{p_4}	$\underline{0.935} \pm 0.0211$	$\underline{1.013} \pm 0.0034$	1.021 ± 0.0017	1.016 ± 0.0009	1.010 ± 0.0007	1.004 ± 0.0004
CH	0.894 ± 0.0099	0.975 ± 0.0046	1.003 ± 0.0024	1.004 ± 0.0013	1.003 ± 0.0007	1.001 ± 0.0004
CH^*	$\underline{\mathbf{0.895}} \pm 0.0105$	$\underline{\mathbf{0.974}} \pm 0.0040$	$\underline{\mathbf{1.003}} \pm 0.0018$	$\underline{\mathbf{1.007}} \pm 0.0011$	$\underline{\mathbf{1.003}} \pm 0.0007$	$\underline{\mathbf{1.002}} \pm 0.0003$

provided. Among the estimators considered, the one providing the smallest squared bias is again underlined, and written in **bold**. Further note the following facts:

- Just as happens with Fréchet underlying parents, for $\mathrm{Burr}_{\gamma,\rho}$ models T/γ does not depend on γ, again with T denoting any of the aforementioned EVI-estimators.
- For $\gamma + \rho = 0$, the results associated to $\mathrm{Burr}_{\gamma,\rho}$ parents are equal to the ones associated to GP_γ parents.
- Regarding bias, the ORBMOP EVI–estimators outperform the MVRB EVI-estimators, unless ρ is small, say $|\rho| \geq 1$ in a GP model (see Table 5), for all n, and in a Student-t_2 model, for small values of n.

4.1.2 RMSEs and relative efficiency indicators at optimal levels

We have computed the Hill estimator at the simulated value of $k_{0|\mathrm{H}} := \arg\min_k \mathrm{RMSE}\big(\mathrm{H}(k)\big)$, the simulated optimal k in the sense of minimum RMSE, not relevant in practice, but providing an indication of the best possible performance of Hill's estimator. Such an estimator is denoted

Table 3: Simulated mean values, at optimal levels, of H(k), H*(k), CH(k) and CH*(k), for EV underlying parents, together with 95% confidence intervals.

n	100	200	500	1000	2000	5000
			EV$_\gamma$ **parent,** $\gamma = 0.1$ ($\rho = -0.1$)			
H	0.334 ± 0.0009	0.284 ± 0.0007	0.243 ± 0.0005	0.223 ± 0.0016	0.209 ± 0.0014	0.195 ± 0.0011
H*	0.260 ± 0.0005	0.233 ± 0.0005	0.207 ± 0.0004	0.192 ± 0.0005	0.180 ± 0.0004	0.167 ± 0.0004
CH	0.276 ± 0.0016	0.258 ± 0.0014	0.234 ± 0.0012	0.221 ± 0.0013	0.208 ± 0.0015	0.1945 ± 0.0012
CH*	**0.214** ± 0.0010	**0.220** ± 0.0010	**0.199** ± 0.0004	**0.188** ± 0.0005	**0.178** ± 0.0004	**0.166** ± 0.0005
			EV$_\gamma$ **parent,** $\gamma = 0.25$ ($\rho = -0.25$)			
H	0.427 ± 0.0012	0.391 ± 0.0026	0.365 ± 0.0019	0.348 ± 0.0012	0.335 ± 0.0013	0.321 ± 0.0010
H*	0.372 ± 0.0009	0.338 ± 0.0008	0.330 ± 0.0017	0.323 ± 0.0015	0.317 ± 0.0008	0.308 ± 0.0009
CH	0.382 ± 0.0027	0.372 ± 0.0021	0.353 ± 0.0014	0.342 ± 0.0017	0.330 ± 0.0008	0.317 ± 0.0008
CH*	**0.312** ± 0.0011	**0.307** ± 0.0012	**0.301** ± 0.0009	**0.296** ± 0.0007	**0.291** ± 0.0006	**0.286** ± 0.0005
			EV$_\gamma$ **parent,** $\gamma = 0.5$ ($\rho = -0.5$)			
H	0.654 ± 0.0032	0.624 ± 0.0033	0.596 ± 0.0011	0.579 ± 0.0016	0.565 ± 0.0010	0.551 ± 0.0010
H*	0.620 ± 0.0030	0.604 ± 0.0020	0.582 ± 0.0016	0.570 ± 0.0013	0.560 ± 0.0012	0.547 ± 0.0009
CH	0.554 ± 0.0053	0.573 ± 0.0016	0.564 ± 0.0014	0.558 ± 0.0010	0.550 ± 0.0009	0.541 ± 0.0006
CH*	**0.539** ± 0.0048	**0.564** ± 0.0016	**0.559** ± 0.0012	**0.553** ± 0.0011	**0.546** ± 0.0007	**0.539** ± 0.0005

Table 4: Simulated mean values, at optimal levels, of H(k)/γ, H*(k)/γ, CH(k)/γ and CH*(k)/γ, for Burr underlying parents, together with 95% confidence intervals.

n	100	200	500	1000	2000	5000
			Burr$_{\gamma,\rho}$ **parent,** $\rho = -0.1$			
H	3.256 ± 0.0088	2.814 ± 0.0069	2.424 ± 0.0096	2.241 ± 0.0147	2.100 ± 0.0137	1.947 ± 0.0101
H*	2.335 ± 0.0046	2.134 ± 0.0040	1.928 ± 0.0050	1.802 ± 0.0054	1.701 ± 0.0041	1.584 ± 0.0034
CH	3.011 ± 0.0082	2.689 ± 0.0065	2.378 ± 0.0135	2.217 ± 0.0155	2.097 ± 0.0149	1.943 ± 0.0114
CH*	**1.572** ± 0.0031	**1.485** ± 0.0028	**1.392** ± 0.0036	**1.351** ± 0.0094	**1.342** ± 0.0072	**1.328** ± 0.0067
			Burr$_{\gamma,\rho}$ **parent,** $\rho = -0.25$			
H	1.675 ± 0.0098	1.558 ± 0.0112	1.458 ± 0.0070	1.390 ± 0.0064	1.340 ± 0.0050	1.281 ± 0.0046
H*	1.463 ± 0.0036	1.354 ± 0.0103	1.316 ± 0.0086	1.292 ± 0.0064	1.270 ± 0.0057	1.233 ± 0.0040
CH	1.625 ± 0.0119	1.528 ± 0.0068	1.441 ± 0.0069	1.380 ± 0.0072	1.333 ± 0.0051	1.275 ± 0.0035
CH*	**1.277** ± 0.0063	**1.257** ± 0.0061	**1.221** ± 0.0040	**1.201** ± 0.0030	**1.180** ± 0.0022	**1.155** ± 0.0024
			Burr$_{\gamma,\rho}$ **parent,** $\rho = -0.5$			
H	1.295 ± 0.0086	1.242 ± 0.0048	1.186 ± 0.0039	1.155 ± 0.0024	1.131 ± 0.0027	1.102 ± 0.0022
H*	1.235 ± 0.0065	1.201 ± 0.0044	1.165 ± 0.0035	1.1390 ± 0.0030	1.118 ± 0.0023	1.095 ± 0.0017
CH	1.226 ± 0.0047	1.190 ± 0.0040	1.153 ± 0.0027	1.130 ± 0.0017	1.110 ± 0.0019	1.087 ± 0.0015
CH*	**1.194** ± 0.0031	**1.171** ± 0.0036	**1.139** ± 0.0026	**1.118** ± 0.0015	**1.102** ± 0.0017	**1.083** ± 0.0011

Table 5: Simulated mean values, at optimal levels, of H(k), H*(k), CH(k) and CH*(k), for GP underlying parents, together with 95% confidence intervals.

n	100	200	500	1000	2000	5000
			GP_γ **parent,** $\gamma = 0.1$ $(\rho = -0.1)$			
H	0.326 ± 0.0009	0.281 ± 0.0007	0.242 ± 0.0010	0.224 ± 0.0015	0.210 ± 0.0014	0.195 ± 0.0010
H*	0.233 ± 0.0005	0.213 ± 0.0004	0.193 ± 0.0005	0.180 ± 0.0005	0.170 ± 0.0004	0.158 ± 0.0003
CH	0.303 ± 0.0008	0.270 ± 0.0007	0.238 ± 0.0014	0.222 ± 0.0016	0.210 ± 0.0015	0.194 ± 0.0011
CH*	$\underline{\mathbf{0.157}} \pm 0.0003$	$\underline{\mathbf{0.149}} \pm 0.0003$	$\underline{\mathbf{0.139}} \pm 0.0004$	$\underline{\mathbf{0.135}} \pm 0.0009$	$\underline{\mathbf{0.134}} \pm 0.0007$	$\underline{\mathbf{0.133}} \pm 0.0007$
			GP_γ **parent,** $\gamma = 0.25$ $(\rho = -0.25)$			
H	0.419 ± 0.0024	0.390 ± 0.0028	0.365 ± 0.0018	0.347 ± 0.0016	0.335 ± 0.0012	0.320 ± 0.0011
H*	0.366 ± 0.0009	0.338 ± 0.0026	0.329 ± 0.0021	0.323 ± 0.0016	0.317 ± 0.0014	0.308 ± 0.0010
CH	0.406 ± 0.0030	0.382 ± 0.0017	0.360 ± 0.0017	0.345 ± 0.0018	0.333 ± 0.0013	0.319 ± 0.0009
CH*	$\underline{\mathbf{0.319}} \pm 0.0016$	$\underline{\mathbf{0.314}} \pm 0.0015$	$\underline{\mathbf{0.305}} \pm 0.0010$	$\underline{\mathbf{0.300}} \pm 0.0008$	$\underline{\mathbf{0.295}} \pm 0.0005$	$\underline{\mathbf{0.289}} \pm 0.0006$
			GP_γ **parent,** $\gamma = 0.5$ $(\rho = -0.5)$			
H	0.647 ± 0.0043	0.621 ± 0.0024	0.593 ± 0.0020	0.578 ± 0.0012	0.565 ± 0.0014	0.551 ± 0.0011
H*	0.617 ± 0.0032	0.601 ± 0.0022	0.583 ± 0.0018	0.569 ± 0.0015	0.559 ± 0.0012	0.548 ± 0.0009
CH	0.613 ± 0.0024	0.595 ± 0.0020	0.577 ± 0.0014	0.565 ± 0.0009	0.555 ± 0.0010	0.544 ± 0.0007
CH*	$\underline{\mathbf{0.597}} \pm 0.0016$	$\underline{\mathbf{0.586}} \pm 0.0018$	$\underline{\mathbf{0.570}} \pm 0.0013$	$\underline{\mathbf{0.559}} \pm 0.0008$	$\underline{\mathbf{0.551}} \pm 0.0009$	$\underline{\mathbf{0.542}} \pm 0.0006$
			GP_γ **parent,** $\gamma = 1$ $(\rho = -1)$ **and** $Burr_{\gamma,\rho}$ **parent,** $\rho = -1$			
H	1.138 ± 0.0042	1.110 ± 0.0033	1.079 ± 0.0022	1.064 ± 0.0011	1.050 ± 0.0010	1.037 ± 0.0009
H*	1.118 ± 0.0036	1.098 ± 0.0024	1.072 ± 0.0015	1.060 ± 0.0012	1.048 ± 0.0007	1.035 ± 0.0006
CH	$\underline{\mathbf{1.011}} \pm 0.0025$	$\underline{\mathbf{1.006}} \pm 0.0018$	$\underline{\mathbf{1.003}} \pm 0.0013$	$\underline{\mathbf{1.002}} \pm 0.0009$	$\underline{\mathbf{1.001}} \pm 0.0004$	$\underline{\mathbf{1.000}} \pm 0.0002$
CH*	1.016 ± 0.0023	1.010 ± 0.0015	1.005 ± 0.0007	1.003 ± 0.0005	1.001 ± 0.0004	1.001 ± 0.0003

Table 6: Simulated mean values, at optimal levels, of H(k), H*(k), CH(k) and CH*(k), for Student underlying parents, together with 95% confidence intervals.

n	100	200	500	1000	2000	5000
			Student t_4 **parent** $(\gamma = 0.25, \rho = -0.5)$			
H	0.361 ± 0.0009	0.339 ± 0.0026	0.317 ± 0.0016	0.305 ± 0.0013	0.296 ± 0.0009	0.286 ± 0.0007
H*	0.329 ± 0.0007	0.317 ± 0.0019	0.304 ± 0.0014	0.297 ± 0.0011	0.290 ± 0.0007	0.282 ± 0.0007
CH	0.311 ± 0.0023	0.310 ± 0.0009	0.300 ± 0.0013	0.294 ± 0.0008	0.288 ± 0.0006	0.281 ± 0.0004
CH*	$\underline{\mathbf{0.295}} \pm 0.0037$	$\underline{\mathbf{0.300}} \pm 0.0010$	$\underline{\mathbf{0.294}} \pm 0.0008$	$\underline{\mathbf{0.289}} \pm 0.0006$	$\underline{\mathbf{0.284}} \pm 0.0005$	$\underline{\mathbf{0.278}} \pm 0.0004$
			Student t_2 **parent** $(\gamma = 0.5, \rho = -1)$			
H	0.601 ± 0.0039	0.577 ± 0.0027	0.556 ± 0.0011	0.544 ± 0.0008	0.535 ± 0.0010	0.526 ± 0.0005
H*	0.580 ± 0.0028	0.566 ± 0.0021	0.550 ± 0.0013	0.540 ± 0.0012	0.532 ± 0.0005	0.523 ± 0.0005
CH	$\underline{\mathbf{0.464}} \pm 0.0123$	0.506 ± 0.0020	0.512 ± 0.0011	0.507 ± 0.0006	0.504 ± 0.0006	0.502 ± 0.0003
CH*	0.457 ± 0.0131	$\underline{\mathbf{0.504}} \pm 0.0019$	$\underline{\mathbf{0.512}} \pm 0.0009$	$\underline{\mathbf{0.507}} \pm 0.0006$	$\underline{\mathbf{0.505}} \pm 0.0004$	$\underline{\mathbf{0.502}} \pm 0.0003$

by H_{00}. We have also computed CH^*_{00}, i.e. the EVI-estimator $\mathrm{CH}^*(k) = \mathrm{ORBMOP}(k)$ computed at the simulated value of $k_{0|\mathrm{CH}^*} := \arg\min_k \mathrm{MSE}\big(\mathrm{CH}^*(k)\big)$. The simulated indicators are

$$\mathrm{REFF}_{\mathrm{CH}^*|\mathrm{H}} := \frac{\mathrm{RMSE}\,(\mathrm{H}_{00})}{\mathrm{RMSE}\,(\mathrm{CH}^*_{00})} = \sqrt{\frac{\mathrm{MSE}\,(\mathrm{H}_{00})}{\mathrm{MSE}\,(\mathrm{CH}^*_{00})}}. \tag{4.1}$$

Similar REFF-indicators, $\mathrm{REFF}_{\mathrm{H}^*|\mathrm{H}}$ and $\mathrm{REFF}_{\mathrm{CH}|\mathrm{H}}$, have also been computed for the H^* and CH EVI-estimators.

Remark 2. *An indicator higher than one means a better performance than the* Hill *estimator. Consequently, the higher these indicators are, the better the associated* EVI-*estimators perform, comparatively to* H_{00}.

Again as an illustration of the results obtained for $\mathrm{CH}_p(k)$, in (3.1), we present Tables 7 and 8. In the first row, we provide the RMSE of H_{00}, so that we can easily recover the RMSE of all other estimators. The following rows provide the REFF-indicators of CH and CH_p. We further present the same characteristics for H^* and $\mathrm{CH}^* \equiv$ ORBMOP. A similar mark (underlined and **bold**) is used for the highest REFF indicator.

Table 7: Simulated RMSE of H_{00}/γ (first row) and REFF-indicators of CH, H_{p_j}, with $p_j = j/(10\gamma)$, $j = 1,2,3,4$, H^* and $\mathrm{CH}^* \equiv$ ORBMOP (independent on γ), for *Fréchet* parents, together with 95% confidence intervals.

Fréchet parent, γ						
n	100	200	500	1000	2000	5000
$\mathrm{RMSE}(\mathrm{H}_{00}/\gamma)$	0.212 ± 0.3959	0.163 ± 0.3544	0.117 ± 0.3147	0.091 ± 0.2922	0.071 ± 0.2739	0.052 ± 0.2545
CH	1.257 ± 0.0072	1.237 ± 0.1591	1.337 ± 0.0080	1.460 ± 0.0123	1.574 ± 0.0123	1.795 ± 0.0097
H_{p_1}	1.038 ± 0.0012	1.031 ± 0.0011	1.026 ± 0.0010	1.023 ± 0.0009	1.020 ± 0.0010	1.019 ± 0.0010
CH_{p_1}	1.269 ± 0.0071	1.251 ± 0.1554	1.352 ± 0.0079	1.471 ± 0.0079	1.585 ± 0.0117	1.804 ± 0.0095
H_{p_2}	1.077 ± 0.0027	1.059 ± 0.0028	1.046 ± 0.0020	1.039 ± 0.0020	*1.032* ± 0.0024	*1.030* ± 0.0019
CH_{p_2}	1.268 ± 0.0072	1.249 ± 0.1611	1.343 ± 0.0081	1.455 ± 0.0073	1.565 ± 0.0105	1.778 ± 0.0094
H_{p_3}	1.120 ± 0.0046	1.084 ± 0.0057	1.055 ± 0.0032	*1.041* ± 0.0037	1.028 ± 0.0040	1.022 ± 0.0033
CH_{p_3}	1.261 ± 0.0078	1.232 ± 0.1847	1.307 ± 0.0082	1.404 ± 0.0078	1.505 ± 0.0102	1.702 ± 0.0089
H_{p_4}	1.185 ± 0.0055	1.120 ± 0.0081	1.060 ± 0.0047	1.027 ± 0.0055	0.999 ± 0.0069	0.982 ± 0.0053
CH_{p_4}	1.265 ± 0.0083	1.214 ± 0.2133	1.251 ± 0.0085	1.319 ± 0.0090	1.394 ± 0.0118	1.556 ± 0.0095
H^*	1.081 ± 0.0018	1.065 ± 0.0018	1.050 ± 0.0014	1.041 ± 0.0014	1.033 ± 0.0017	1.029 ± 0.0015
CH^*	<u>**1.276**</u> ± 0.0076	<u>**1.257**</u> ± 0.1738	<u>**1.358**</u> ± 0.0079	<u>**1.483**</u> ± 0.0078	<u>**1.599**</u> ± 0.0119	<u>**1.822**</u> ± 0.0097

Table 8: Simulated RMSE of H_{00} (first row) and REFF-indicators of CH, H_{p_j}, with $p_j = j/(10\gamma)$, $j = 1, 2, 3, 4$, H^* and $CH^* \equiv$ ORBMOP, for an EV_1 underlying parent, together with 95% confidence intervals

EV_γ **parent,** $\gamma = 1$ $(\rho = -1)$						
RMSE(H_{00})	0.314 ± 0.4431	0.239 ± 0.3776	0.170 ± 0.3249	0.132 ± 0.2981	0.104 ± 0.2775	0.076 ± 0.2565
CH	0.814 ± 0.1168	1.182 ± 0.0230	1.410 ± 0.0212	1.679 ± 0.0192	2.005 ± 0.0192	2.500 ± 0.0218
H_{p_1}	1.0543 ± 0.0013	1.0426 ± 0.0015	1.0329 ± 0.0010	1.0279 ± 0.0012	1.0242 ± 0.0010	1.0217 ± 0.0006
CH_{p_1}	0.816 ± 0.1199	1.191 ± 0.0223	1.408 ± 0.0199	1.66 ± 0.0188	1.972 ± 0.0182	2.445 ± 0.0197
H_{p_2}	1.117 ± 0.0032	1.088 ± 0.0032	1.064 ± 0.0018	1.052 ± 0.0025	1.042 ± 0.0019	1.036 ± 0.0015
CH_{p_2}	0.822 ± 0.1224	1.196 ± 0.0212	1.391 ± 0.0183	1.622 ± 0.0173	1.910 ± 0.0164	2.357 ± 0.0188
H_{p_3}	1.1976 ± 0.0046	1.1433 ± 0.0053	1.0937 ± 0.0031	1.0681 ± 0.0040	1.0485 ± 0.0033	1.0358 ± 0.0028
CH_{p_3}	0.841 ± 0.1223	1.201 ± 0.0199	1.360 ± 0.0163	1.557 ± 0.0165	1.809 ± 0.0152	2.214 ± 0.0172
H_{p_4}	1.318 ± 0.0055	1.225 ± 0.0068	1.134 ± 0.0040	1.084 ± 0.0062	1.045 ± 0.0055	1.014 ± 0.0046
CH_{p_4}	0.885 ± 0.1188	1.222 ± 0.0189	1.330 ± 0.0144	1.474 ± 0.0156	1.665 ± 0.0142	1.991 ± 0.0153
H^*	1.140 ± 0.0035	1.102 ± 0.0025	1.071 ± 0.0014	1.054 ± 0.0019	1.043 ± 0.0017	1.036 ± 0.0012
CH^*	**1.519** ± 0.0124	**1.454** ± 0.0084	**1.431** ± 0.0219	**1.709** ± 0.0211	**2.044** ± 0.0203	**2.568** ± 0.0221

We also present Tables 9, 10, 11 and 12, with the REFF indicators of $H^*|H$, $CH|H$ and $CH^*|H$. A similar mark (underlined and **bold**) is used for the highest REFF indicator. Again, 95% confidence intervals are provided.

Remark 3. *We now provide a few comments related to the* REFF*-indicators:*

- *For* Fréchet *and* $\text{Burr}_{\gamma,\rho}$ *underlying parents, the* REFF*-indicators and* $\text{RMSE}(H_{00})/\gamma$ *do not depend of* γ.
- *Just as for mean values at optimal levels, and again if we restrict ourselves to the region of* p*-values where we can guarantee the asymptotic normality of* H_p*, the best results were obtained for* $p = 4/(10\gamma)$ *for all simulated models but the* Fréchet *(independently of* γ*) and some of the simulated models with* $\gamma = 1$.
- *Regarding* RMSE*, the consistent* MOP EVI*-estimators, at optimal levels, can always beat the* MVRB EVI*-estimators, also computed at optimal levels, being these ones always beaten by the* ORBMOP EVI*-estimators, but for* $n = 100$ *and underlying parents with a support partially containing negative values.*

Table 9: Simulated RMSE of H_{00} (first row) and REFF-indicators of H^*, CH and $CH^* \equiv$ ORBMOP, for EV underlying parents, together with 95% confidence intervals.

n	100	200	500	1000	2000	5000
			EV_γ **parent,** $\gamma = 0.1$			
RMSE(H_{00})	0.268 ± 0.4059	0.216 ± 0.3270	0.174 ± 0.2601	0.151 ± 0.2227	0.133 ± 0.1920	0.114 ± 0.1600
H^*	1.801 ± 0.0036	1.679 ± 0.0024	1.580 ± 0.0027	1.530 ± 0.0028	1.481 ± 0.0041	1.421 ± 0.0047
CH	1.245 ± 0.0050	1.140 ± 0.0027	1.070 ± 0.0019	1.045 ± 0.0015	1.029 ± 0.0011	1.019 ± 0.0008
CH^*	$\underline{\mathbf{4.088}} \pm 0.0114$	$\underline{\mathbf{3.605}} \pm 0.0062$	$\underline{\mathbf{3.164}} \pm 0.0052$	$\underline{\mathbf{2.913}} \pm 0.0054$	$\underline{\mathbf{2.713}} \pm 0.0065$	$\underline{\mathbf{2.528}} \pm 0.0062$
			EV_γ **parent,** $\gamma = 0.25$			
RMSE(H_{00})	0.246 ± 0.3905	0.200 ± 0.3126	0.157 ± 0.2504	0.133 ± 0.2150	0.113 ± 0.1865	0.092 ± 0.1557
H^*	1.402 ± 0.0027	1.338 ± 0.0052	1.249 ± 0.0032	1.202 ± 0.0039	1.165 ± 0.0028	1.133 ± 0.0022
CH	1.328 ± 0.0108	1.237 ± 0.0056	1.171 ± 0.0042	1.130 ± 0.0031	1.101 ± 0.0021	1.072 ± 0.0020
CH^*	$\underline{\mathbf{2.402}} \pm 0.0122$	$\underline{\mathbf{2.224}} \pm 0.0097$	$\underline{\mathbf{2.076}} \pm 0.0069$	$\underline{\mathbf{2.003}} \pm 0.0067$	$\underline{\mathbf{1.949}} \pm 0.0064$	$\underline{\mathbf{1.905}} \pm 0.0079$
			EV_γ **parent,** $\gamma = 0.5$			
RMSE(H_{00})	0.256 ± 0.3846	0.202 ± 0.3086	0.151 ± 0.2508	0.122 ± 0.2197	0.100 ± 0.1939	0.077 ± 0.1656
H^*	1.187 ± 0.0026	1.141 ± 0.0020	1.103 ± 0.0019	1.084 ± 0.0021	1.070 ± 0.0019	1.058 ± 0.0021
CH	1.492 ± 0.0258	1.501 ± 0.0097	1.476 ± 0.0059	1.452 ± 0.0057	1.417 ± 0.0057	1.359 ± 0.0052
CH^*	$\underline{\mathbf{1.605}} \pm 0.0358$	$\underline{\mathbf{1.655}} \pm 0.0097$	$\underline{\mathbf{1.607}} \pm 0.0065$	$\underline{\mathbf{1.566}} \pm 0.0059$	$\underline{\mathbf{1.513}} \pm 0.0057$	$\underline{\mathbf{1.439}} \pm 0.0053$

Table 10: Simulated RMSE of H_{00} (first row) and REFF-indicators of H^*, CH and $CH^* \equiv$ ORBMOP, for Burr underlying parents, together with 95% confidence intervals.

n	100	200	500	1000	2000	5000
			$Burr_{\gamma,\rho}$ **parent,** $\rho = -0.1$			
RMSE(H_{00})	2.592 ± 0.2258	2.133 ± 0.1771	1.726 ± 0.1748	1.504 ± 0.1596	1.326 ± 0.1351	1.134 ± 0.0968
H^*	1.776 ± 0.0039	1.667 ± 0.0022	1.576 ± 0.0031	1.526 ± 0.0035	1.479 ± 0.0053	1.420 ± 0.0042
CH	1.110 ± 0.0008	1.066 ± 0.0002	1.034 ± 0.0007	1.024 ± 0.0008	1.017 ± 0.0007	1.011 ± 0.0003
CH^*	$\underline{\mathbf{3.719}} \pm 0.0076$	$\underline{\mathbf{3.354}} \pm 0.0046$	$\underline{\mathbf{3.000}} \pm 0.0065$	$\underline{\mathbf{2.782}} \pm 0.0068$	$\underline{\mathbf{2.606}} \pm 0.0059$	$\underline{\mathbf{2.424}} \pm 0.0044$
			$Burr_{\gamma,\rho}$ **parent,** $\rho = -0.25$			
RMSE(H_{00})	0.478 ± 0.2684	0.381 ± 0.2339	0.289 ± 0.2022	0.236 ± 0.1847	0.193 ± 0.1709	0.150 ± 0.1567
H^*	2.130 ± 0.0073	2.009 ± 0.0054	1.902 ± 0.0055	1.842 ± 0.0064	1.798 ± 0.0076	1.751 ± 0.0070
CH	1.148 ± 0.0049	1.118 ± 0.0026	1.088 ± 0.0025	1.069 ± 0.0023	1.057 ± 0.0018	1.042 ± 0.0012
CH^*	$\underline{\mathbf{1.388}} \pm 0.0026$	$\underline{\mathbf{1.323}} \pm 0.0048$	$\underline{\mathbf{1.243}} \pm 0.0038$	$\underline{\mathbf{1.198}} \pm 0.0028$	$\underline{\mathbf{1.164}} \pm 0.0028$	$\underline{\mathbf{1.129}} \pm 0.0026$
			$Burr_{\gamma,\rho}$ **parent,** $\rho = -0.5$			
RMSE(H_{00})	0.949 ± 0.1828	0.783 ± 0.1476	0.621 ± 0.1188	0.525 ± 0.1008	0.449 ± 0.0883	0.367 ± 0.0755
H^*	1.161 ± 0.0023	1.127 ± 0.0019	1.096 ± 0.0022	1.080 ± 0.0017	1.067 ± 0.0019	1.056 ± 0.0017
CH	1.422 ± 0.0066	1.383 ± 0.0065	1.337 ± 0.0052	1.300 ± 0.0057	1.262 ± 0.0045	1.230 ± 0.0048
CH^*	$\underline{\mathbf{1.639}} \pm 0.0065$	$\underline{\mathbf{1.550}} \pm 0.0070$	$\underline{\mathbf{1.462}} \pm 0.0051$	$\underline{\mathbf{1.402}} \pm 0.0052$	$\underline{\mathbf{1.346}} \pm 0.0037$	$\underline{\mathbf{1.298}} \pm 0.0045$

Table 11: Simulated RMSE of H_{00} (first row) and REFF-indicators of H^*, CH and $CH^* \equiv$ ORBMOP, for the aforementioned GP underlying parents, together with 95% confidence intervals.

n	100	200	500	1000	2000	5000
GP_γ parent, $\gamma = 0.1$ ($\rho = -0.1$)						
RMSE(H_{00})	0.259 ± 0.2609	0.213 ± 0.2091	0.173 ± 0.1614	0.150 ± 0.1342	0.133 ± 0.1121	0.113 ± 0.0889
H^*	1.776 ± 0.0038	1.667 ± 0.0022	1.576 ± 0.0031	1.526 ± 0.0035	1.479 ± 0.0053	1.420 ± 0.0042
CH	1.099 ± 0.0007	1.061 ± 0.0002	1.032 ± 0.0006	1.023 ± 0.0008	1.016 ± 0.0007	1.010 ± 0.0005
CH^*	$\underline{\mathbf{3.710}} \pm 0.0076$	$\underline{\mathbf{3.349}} \pm 0.0047$	$\underline{\mathbf{2.998}} \pm 0.0063$	$\underline{\mathbf{2.780}} \pm 0.0068$	$\underline{\mathbf{2.605}} \pm 0.0060$	$\underline{\mathbf{2.423}} \pm 0.0043$
GP_γ parent, $\gamma = 0.25$ ($\rho = -0.25$)						
RMSE(H_{00})	0.237 ± 0.2639	0.196 ± 0.2142	0.155 ± 0.1672	0.131 ± 0.1397	0.112 ± 0.1172	0.092 ± 0.0931
H^*	1.388 ± 0.0026	1.323 ± 0.0048	1.243 ± 0.0038	1.198 ± 0.0028	1.164 ± 0.0028	1.129 ± 0.0026
CH	1.148 ± 0.0049	1.118 ± 0.0027	1.088 ± 0.0025	1.069 ± 0.0023	1.057 ± 0.0018	1.042 ± 0.0012
CH^*	$\underline{\mathbf{2.130}} \pm 0.0073$	$\underline{\mathbf{2.009}} \pm 0.0053$	$\underline{\mathbf{1.902}} \pm 0.0055$	$\underline{\mathbf{1.842}} \pm 0.0064$	$\underline{\mathbf{1.798}} \pm 0.0076$	$\underline{\mathbf{1.751}} \pm 0.0070$
GP_γ parent, $\gamma = 0.5$ ($\rho = -0.5$)						
RMSE(H_{00}))	0.239 ± 0.2832	0.191 ± 0.2379	0.145 ± 0.1948	0.118 ± 0.1690	0.097 ± 0.1473	0.075 ± 0.1232
H^*	1.161 ± 0.0023	1.127 ± 0.0019	1.096 ± 0.0021	1.080 ± 0.0017	1.067 ± 0.0019	1.056 ± 0.0017
CH	1.422 ± 0.0066	1.383 ± 0.0065	1.337 ± 0.0052	1.300 ± 0.0057	1.262 ± 0.0045	1.230 ± 0.0048
CH^*	$\underline{\mathbf{1.639}} \pm 0.0065$	$\underline{\mathbf{1.550}} \pm 0.0070$	$\underline{\mathbf{1.462}} \pm 0.0050$	$\underline{\mathbf{1.402}} \pm 0.0052$	$\underline{\mathbf{1.346}} \pm 0.0037$	$\underline{\mathbf{1.298}} \pm 0.0046$
GP_γ parent, $\gamma = 1$ ($\rho = -1$) or $Burr_{\gamma,\rho}$ parent, $\rho = -1$						
RMSE(H_{00})	0.266 ± 0.3757	0.205 ± 0.3370	0.147 ± 0.3005	0.115 ± 0.2798	0.090 ± 0.2629	0.066 ± 0.2451
H^*	1.094 ± 0.0026	1.073 ± 0.0023	1.053 ± 0.0022	1.045 ± 0.0024	1.038 ± 0.0020	1.031 ± 0.0020
CH	1.977 ± 0.0113	2.123 ± 0.0130	2.428 ± 0.0151	2.702 ± 0.0119	2.962 ± 0.0146	3.394 ± 0.0186
CH^*	$\underline{\mathbf{2.171}} \pm 0.0137$	$\underline{\mathbf{2.302}} \pm 0.0136$	$\underline{\mathbf{2.619}} \pm 0.0150$	$\underline{\mathbf{2.908}} \pm 0.0108$	$\underline{\mathbf{3.174}} \pm 0.0160$	$\underline{\mathbf{3.624}} \pm 0.0198$

Table 12: Simulated RMSE of H_{00} (first row) and REFF-indicators of H^*, CH and $CH^* \equiv$ ORBMOP, for Student underlying parents, together with 95% confidence intervals.

n	100	200	500	1000	2000	5000
Student t_4 parent ($\gamma = 0.25, \rho = -0.5$)						
RMSE(H_{00})	0.183 ± 0.5264	0.143 ± 0.4352	0.106 ± 0.3562	0.085 ± 0.3117	0.070 ± 0.2753	0.054 ± 0.2336
H^*	1.262 ± 0.0027	1.201 ± 0.0024	1.146 ± 0.0027	1.117 ± 0.0024	1.095 ± 0.0023	1.075 ± 0.0017
CH	1.435 ± 0.0468	1.398 ± 0.0084	1.361 ± 0.0053	1.322 ± 0.0056	1.283 ± 0.0057	1.236 ± 0.0048
CH^*	$\underline{\mathbf{1.680}} \pm 0.0714$	$\underline{\mathbf{1.629}} \pm 0.0097$	$\underline{\mathbf{1.539}} \pm 0.0063$	$\underline{\mathbf{1.467}} \pm 0.0054$	$\underline{\mathbf{1.400}} \pm 0.0052$	$\underline{\mathbf{1.325}} \pm 0.0045$
Student t_2 parent ($\gamma = 0.5, \rho = -1$)						
RMSE(H_{00})	0.203 ± 0.4920	0.153 ± 0.4029	0.108 ± 0.3264	0.083 ± 0.2827	0.065 ± 0.2467	0.047 ± 0.2091
H^*	$\underline{\mathbf{1.177}} \pm 0.0026$	1.126 ± 0.0028	1.091 ± 0.0018	1.077 ± 0.0014	1.074 ± 0.0012	1.072 ± 0.0022
CH	0.980 ± 0.1394	1.418 ± 0.0172	1.706 ± 0.0152	1.944 ± 0.0162	2.227 ± 0.0179	2.641 ± 0.0218
CH^*	0.995 ± 0.1453	$\underline{\mathbf{1.477}} \pm 0.0217$	$\underline{\mathbf{1.811}} \pm 0.0185$	$\underline{\mathbf{2.071}} \pm 0.0182$	$\underline{\mathbf{2.382}} \pm 0.0218$	$\underline{\mathbf{2.833}} \pm 0.0240$

5 Concluding remarks

- It is clear that Hill's estimation leads to a strong over-estimation of the EVI and the MOP provides a more adequate EVI-estimation, being even able to beat the MVRB EVI-estimators in a large variety of situations. The ORBMOP beat always the MOP and often beat the MVRB EVI-estimators.
- The patterns of the estimators' sample paths are always of the same type, in the sense that the ORBMOP$\equiv$ CH* often beats the CH EVI-estimators for all k. This is surely mainly due to the reduction of bias, but also to the small increase in the variance. Indeed, the function $(1-\varphi(\rho))/\sqrt{1-2\varphi(\rho)}$ is not a long way from 1 for all $\rho < 0$. Such a function indeed attains a maximum at $\rho = 0$, equal to $\sqrt{(1+\sqrt{2})/2} \approx 1.099$, as illustrated in Figure 4.

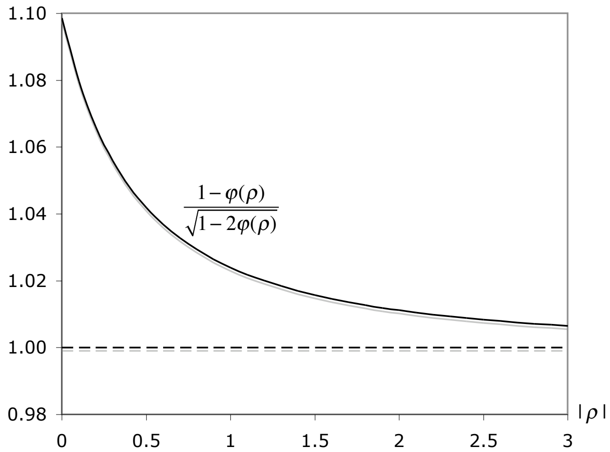

Figure 4: Comparative variance indicators of the CH and the ORBMOP EVI-estimators, as a function of $|\rho|$.

- Also, there is surely a high reduction of bias of the ORBMOP comparatively with the CH, a topic still under investigation, and not discussed here, due to the deep involvement of a third-order framework.

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