Diagnostics for pairwise region extremal dependence in random fields

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Abstract: A coefficient of tail dependence η and a slowly varying function $\mathcal{L}$ provide information about pairwise extremal dependence of some spatial processes. They enable to know whether the spatial process is asymptotically dependent, asymptotically independent or independent for any pair of locations $\mathbf{i}$ and $\mathbf{j}$. We propose a generalization of such diagnostic tools in order to describe the type and strength of the dependence for any pair of sets A and B of locations.

We apply the properties and use of such measures to a space modelling for duration of extremes.

Keywords: Random fields, extremal dependence coefficients, tail dependence

1 Introduction

Let $\mathbf{X} = \{X_{\mathbf{t}}\}_{\mathbf{t}\in\mathbb{Z}^2}$ be a stationary random field with continous univariate marginal distribution F. Quantifying dependence between extreme events occurring at several locations is essential. For max-stable processes an important measure of dependence is

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the extremal coefficient function that generalizes the extremal coefficient ϵ introduced by Tiago de Oliveira (1962/63). The extremal coefficient function, $\epsilon(\mathbf{i},\mathbf{j})$, is such that

$$P(X_{\mathbf{i}} \leq x, X_{\mathbf{j}} \leq x) = F^{\epsilon(\mathbf{i},\mathbf{j})}(x), \quad x \in \mathbb{R},\ \mathbf{i},\mathbf{j} \in \mathbb{Z}^2, \tag{1.1}$$

and satisfies $1 \leq \epsilon(\mathbf{i},\mathbf{j}) \leq 2$.

This function can be computed for several models in the literature (Schlather (2002)) and is related with other measures, namely the following tail dependence function

$$\begin{aligned}\lambda(\mathbf{i},\mathbf{j}) &= \lim_{x \to x^F} P(X_{\mathbf{i}} > x \mid X_{\mathbf{j}} > x) \\ &= 2 - \lim_{u \uparrow 1} \frac{\log P(F(X_{\mathbf{i}}) \leq u,\ F(X_{\mathbf{j}}) \leq u)}{\log P(F(X_{\mathbf{i}}) \leq u)},\end{aligned} \tag{1.2}$$

where x^F is the upper limit of the support of F. By (1.2) we have $\lambda(\mathbf{i},\mathbf{j}) = 2 - \epsilon(\mathbf{i},\mathbf{j})$.

When $\lambda(\mathbf{i},\mathbf{j}) = 0$ ($\epsilon(\mathbf{i},\mathbf{j}) = 2$) the variables $X_{\mathbf{i}}$ and $X_{\mathbf{j}}$ are said to be asymptotically independent, being exactly independent in the case of max-stable random fields. However, in this case, at finite levels x quite different degrees of dependence are attainable. This argument motivates the introduction of a dependence parameter $\eta(\mathbf{i},\mathbf{j})$ to quantify such degrees of dependence in Ledford and Tawn (1996).

They consider that the joint survivor function of an arbitrary random pair $(X_{\mathbf{i}}, X_{\mathbf{j}})$ satisfies the asymptotic condition

$$P(X_{\mathbf{i}} > x, X_{\mathbf{j}} > x) \sim \mathcal{L}_{\mathbf{i},\mathbf{j}}(x) P(X_{\mathbf{i}} > x)^{\frac{1}{\eta(\mathbf{i},\mathbf{j})}} \tag{1.3}$$

for large x, where $\mathcal{L}_{\mathbf{i},\mathbf{j}}(x)$ is a slowly varying function as $x \to \infty$ and $\eta(\mathbf{i},\mathbf{j})$ denotes a tail dependence coefficient that lies in $(0,1]$.

The tail behaviour given in (1.3) does not characterize distributions in the domain of attraction of a bivariate extreme value distribution (Schlather (2001)). Nevertheless, the parameter $\eta(\mathbf{i},\mathbf{j})$ and the function $\mathcal{L}_{\mathbf{i},\mathbf{j}}(x)$ are pivotal in characterizing the spatial extremal dependence as is presented in Ancona-Navarrete and Tawn (2002).

The above coefficients $\lambda(\mathbf{i},\mathbf{j})$ and $\eta(\mathbf{i},\mathbf{j})$ focus only on bivariate distributions.

This paper proposes a generalization of the above approach for the study of the pairwise dependence via the parameter η. We evaluate the dependence of the extremal events $\bigcap_{\mathbf{i} \in \mathbf{A}} \{X_{\mathbf{i}} > x\}$ and $\bigcap_{\mathbf{i} \in \mathbf{B}} \{X_{\mathbf{i}} > x\}$, x large, for any pair of sets A and B of locations.

2 Main definitions and properties

We will assume that the stationary random filed $\mathbf{X}$ is such that, for any set of locations C of $\mathbb{Z}^2$,

$$P\left(\bigcap_{\mathbf{i} \in \mathbf{C}} \{F(X_{\mathbf{i}}) > 1 - t\}\right) \sim t^{\frac{1}{\eta(\mathbf{C})}} \mathcal{L}_C(t^{-1}), \tag{2.1}$$

as $t \downarrow 0$, where $\mathcal{L}_C(x)$ is a slowly varying function as $x \to \infty$, and $\eta(C) \in (0,1]$.

We now present an interpretation for the range of the values of $\eta(A \cup B)$, when A and B satisfies the following particular case of (2.1): for each x,

$$P\left(\bigcap_{\mathbf{i}\in\mathbf{A}} \{X_{\mathbf{i}} > x\}\right) = P^{\overline{\epsilon}(A)}(X_{\mathbf{1}} > x) \text{ and } P\left(\bigcap_{\mathbf{i}\in\mathbf{B}} \{X_{\mathbf{i}} > x\}\right) = P^{\overline{\epsilon}(A)}(X_{\mathbf{1}} > x), \quad (2.2)$$

where $\overline{\epsilon}(A)$ and $\overline{\epsilon}(B)$ are positive constants.

The general case will be discussed later.

The aditional conditions (2.2) are satisfied if, for instance, the random vectors $\left(Y_{\mathbf{i}_1}, \ldots, Y_{\mathbf{i}_p}\right) = \left(-X_{\mathbf{i}_1}, \ldots, -X_{\mathbf{i}_p}\right)$ and $\left(Y_{\mathbf{j}_1}, \ldots, Y_{\mathbf{j}_q}\right) = \left(-X_{\mathbf{j}_1}, \ldots, -X_{\mathbf{j}_q}\right)$ have multivariate extreme value distributions, for some arrangements $(\mathbf{i_1}, \ldots, \mathbf{i_p})$ and $(\mathbf{j_1}, \ldots, \mathbf{j_q})$ of the elements in A and in B, respectively. In fact, the coefficients $\overline{\epsilon}$ are the multivariate extensions, for $\mathbf{Y} = -\mathbf{X}$, of the coefficients ϵ in (1.1) considered in Smith (1990). If $\left(Y_{\mathbf{i}_1}, \ldots, Y_{\mathbf{i}_p}\right)$ has multivariate extreme value distribution then its max-stability equation enables to conclude that there exists a constant $\epsilon(\mathbf{i_1}, \ldots, \mathbf{i_p}) \in [1, p]$ such that $P\left(Y_{\mathbf{i}_1} \leq y, \ldots, Y_{\mathbf{i}_p} \leq y\right) = P\left(Y_{\mathbf{1}} \leq y\right)^{\epsilon(\mathbf{i}_1, \ldots, \mathbf{i}_p)}$.

For the sake of simplicity we shall write $U_{\mathbf{i}} = F(X_{\mathbf{i}})$, $\mathbf{i} \in \mathbb{Z}^2$.

Proposition 2.1 *For any set of locations A and B satisfying (2.2) it holds:*

(1)

$$P\left(\bigcap_{\mathbf{i}\in\mathbf{A}\cup\mathbf{B}} \{U_{\mathbf{i}} > u\}\right) > P\left(\bigcap_{\mathbf{i}\in\mathbf{A}} \{U_{\mathbf{i}} > u\}\right) P\left(\cap_{\mathbf{i}\in\mathbf{B}} \{U_{\mathbf{i}} > u\}\right), \; u \geq \textit{some } u_0,$$

if and only if $\dfrac{1}{\overline{\epsilon}(A) + \overline{\epsilon}(B)} < \eta(A \cup B) \leq \dfrac{1}{\overline{\epsilon}(A)} \bigwedge \dfrac{1}{\overline{\epsilon}(B)}$.

(2)

$$P\left(\bigcap_{\mathbf{i}\in\mathbf{A}\cup\mathbf{B}} \{U_{\mathbf{i}} > u\}\right) < P\left(\bigcap_{\mathbf{i}\in\mathbf{A}} \{U_{\mathbf{i}} > u\}\right) P\left(\bigcap_{\mathbf{i}\in\mathbf{B}} \{U_{\mathbf{i}} > u\}\right), \; u \geq \textit{some } u_0,$$

if and only if $0 < \eta(A \cup B) < \dfrac{1}{\overline{\epsilon}(A) + \overline{\epsilon}(B)}$.

Proof: To obtain (1) we remark that it follows for (2) that, as $u \uparrow 1$,

$$\eta(A \cup B) \sim \frac{\log P\left(U_{\mathbf{1}} > u\right)}{\log P\left(\bigcap_{\mathbf{i}\in\mathbf{A}\cup\mathbf{B}} \{U_{\mathbf{i}} > u\}\right)},$$

and we have, for sufficient large u,

$$\log P^{\overline{\epsilon}(A)}(U_{\mathbf{1}} > u) \bigwedge \log P^{\overline{\epsilon}(B)}(U_{\mathbf{1}} > u) \geq \log P\left(\bigcap_{\mathbf{i} \in \mathbf{A} \cup \mathbf{B}} \{U_{\mathbf{i}} > u\}\right) >$$

$$\log P^{\overline{\epsilon}(A)}(U_{\mathbf{1}} > u) + \log P^{\overline{\epsilon}(B)}(U_{\mathbf{1}} > u)$$

if and only if

$$\frac{1}{\overline{\epsilon}(A) + \overline{\epsilon}(B)} < \frac{\log P(U_{\mathbf{1}} > u)}{\log P\left(\bigcap_{\mathbf{i} \in \mathbf{A} \cup \mathbf{B}} \{U_{\mathbf{i}} > u\}\right)} \leq \frac{1}{\overline{\epsilon}(A)} \bigwedge \frac{1}{\overline{\epsilon}(B)}.$$

The statment in (2) follows analogously. □

We will now rescale $\eta(A \cup B)$ in order to obtain a coefficient $\overline{\chi}(A, B)$ with positive, negative or null values corresponding to "positive dependence", "negative dependence" and near independence of the above extremal events.

Definition 2.1 *For any set of locations A and B satisfying (2.2) let*

$$\overline{\chi}(A, B) = (\overline{\epsilon}(A) + \overline{\epsilon}(B))\eta(A \cup B) - 1.$$

The coefficient $\overline{\chi}(A, B)$ takes values in $\left(-1, \frac{\overline{\epsilon}(A) \bigwedge \overline{\epsilon}(B)}{\overline{\epsilon}(A) \bigvee \overline{\epsilon}(B)}\right]$.

In the case of $A = \{\mathbf{i}\}$ and $B = \{\mathbf{j}\}$ the above coefficient becomes $\overline{\chi}(\mathbf{i}, \mathbf{j}) = \mathbf{2}\eta(\mathbf{i}, \mathbf{j}) - \mathbf{1}$ considered in the references.

We now extend the interpretation of the bivariate diagnostic measures for pairwise dependence.

Let $\lambda(A, B)(t) = \dfrac{P\left(\bigcap_{\mathbf{i} \in \mathbf{A} \cup \mathbf{B}} \{F(X_{\mathbf{i}}) > 1 - t\}\right)}{P\left(\bigcap_{\mathbf{i} \in \mathbf{A}} \{F(X_{\mathbf{i}}) > 1 - t\}\right)}$ and $\lambda(A, B) = \lim_{t \downarrow 0} \lambda(A, B)(t)$.

Then

$$\lambda(A, B)(t) \sim \mathcal{L}_{A \cup B}(t^{-1}) t^{\frac{1 - \overline{\epsilon}(A)\eta(A \cup B)}{\eta(A \cup B)}}, \tag{2.3}$$

and

$$\lambda(B, A)(t) \sim \mathcal{L}_{A \cup B}(t^{-1}) t^{\frac{1 - \overline{\epsilon}(B)\eta(A \cup B)}{\eta(A \cup B)}},$$

with $0 \leq 1 - \overline{\epsilon}(A)\eta(A \cup B) < 1$ and $0 \leq 1 - \overline{\epsilon}(B)\eta(A \cup B) < 1$.

From the above asymptotic equivalences and the proposition 2.1 we can state the following conclusions about the events $\bigcap_{\mathbf{i}\in\mathbf{A}} \{X_{\mathbf{i}} > x\}$ and $\bigcap_{\mathbf{i}\in\mathbf{A}} \{X_{\mathbf{i}} > x\}$, with A and B satisfying (2.2):

a) If $\eta(A \cup B) = \frac{1}{\overline{\epsilon}(A)}$ or $\eta(A \cup B) = \frac{1}{\overline{\epsilon}(B)}$ (i.e. $\overline{\chi}(A,B) = \frac{\overline{\epsilon}(A) \bigwedge \overline{\epsilon}(B)}{\overline{\epsilon}(A) \bigvee \overline{\epsilon}(B)}$) and $\mathcal{L}_{A\cup B}(x) \to c > 0$, as $x \to \infty$, then $\lambda(A,B) = c$ or $\lambda(B,A) = c$ and we say that the events are asymptotically dependent of degree $c = \lambda(A,B) \bigvee \lambda(B,A)$.

b) If $\eta(A \cup B) = \frac{1}{\overline{\epsilon}(A)}$ or $\eta(A \cup B) = \frac{1}{\overline{\epsilon}(B)}$ (i.e. $\overline{\chi}(A,B) = \frac{\overline{\epsilon}(A) \bigwedge \overline{\epsilon}(B)}{\overline{\epsilon}(A) \bigvee \overline{\epsilon}(B)}$) and $\mathcal{L}_{A\cup B}(x) \to 0$, as $x \to \infty$, then $\lambda(A,B) = \lambda(B,A) = 0$ and we say that the events are asymptotically independent.

c) If $0 < \eta(A \cup B) < \frac{1}{\overline{\epsilon}(A)}$ and $0 < \eta(A \cup B) < \frac{1}{\overline{\epsilon}(B)}$ (i.e. is $0 < \eta(A \cup B) < \frac{1}{\overline{\epsilon}(A)} \bigwedge \frac{1}{\overline{\epsilon}(B)}$ and $\overline{\chi}(A,B) < \frac{\overline{\epsilon}(A) \bigwedge \overline{\epsilon}(B)}{\overline{\epsilon}(A) \bigvee \overline{\epsilon}(B)}$), then $\lambda(A,B) = \lambda(B,A) = 0$ and we find again the asymptotic independent.

In this case, we distinguish three situations:

c1) If $\frac{1}{\overline{\epsilon}(A)+\overline{\epsilon}(B)} < \eta(A \cup B)$ (i.e. $\overline{\chi}(A,B) > 0$) then the extremal events tend to occur more frequently than under the exact independence.

c2) If $\eta(A \cup B) < \frac{1}{\overline{\epsilon}(A)+\overline{\epsilon}(B)}$ (i.e. $\overline{\chi}(A,B) < 0$) then the extremal events tend to occur less frequently than under the exact independence.

c3) If $\eta(A \cup B) = \frac{1}{\overline{\epsilon}(A)+\overline{\epsilon}(B)}$ (i.e. $\overline{\chi}(A,B) = 0$) then

$$P\left(\bigcap_{\mathbf{i}\in\mathbf{A}\cup\mathbf{B}} \{F(X_{\mathbf{i}}) > 1 - t\}\right) \sim \mathcal{L}_{A\cup B}(t^{-1}) t^{\overline{\epsilon}(A)} t^{\overline{\epsilon}(B)},$$

and we say that the events are near independent being exactly independent when $\mathcal{L}_{A\cup B}(x) = 1$.

We can assume in (2.2) only the asymptotically equivalence $\sim$ instead the equalities or, more generally, work only with the initial assumption (2.1) on A and B. Even in this last case we will find analogous results to those in the Proposition 2.1.:

(1)

$$P\left(\bigcap_{\mathbf{i}\in\mathbf{A}\cup\mathbf{B}} \{U_{\mathbf{i}} > u\}\right) > P\left(\bigcap_{\mathbf{i}\in\mathbf{A}} \{U_{\mathbf{i}} > u\}\right) P\left(\bigcap_{\mathbf{i}\in\mathbf{B}} \{U_{\mathbf{i}} > u\}\right), \; u \geq \text{some } u_0,$$

if and only if $\dfrac{1}{\frac{1}{\eta(A)}+\frac{1}{\eta(B)}} < \eta(A\cup B) \leq \eta(A)\bigwedge\eta(B)$.

(2)

$$P\left(\bigcap_{\mathbf{i}\in\mathbf{A}\cup\mathbf{B}}\{U_{\mathbf{i}}>u\}\right) < P\left(\bigcap_{\mathbf{i}\in\mathbf{A}}\{U_{\mathbf{i}}>u\}\right)P\left(\bigcap_{\mathbf{i}\in\mathbf{B}}\{U_{\mathbf{i}}>u\}\right),\ u\geq \text{some } u_0,$$

if and only if $0<\eta(A\cup B)<\dfrac{1}{\frac{1}{\eta(A)}+\frac{1}{\eta(B)}}$.

(3) If $\frac{1}{\eta(A\cup B)}=\dfrac{1}{\eta(A)}+\dfrac{1}{\eta(B)}$ then

$$P\left(\bigcap_{\mathbf{i}\in\mathbf{A}\cup\mathbf{B}}\{F(X_{\mathbf{i}})>1-t\}\right)\sim$$

$$\frac{\mathcal{L}_{A\cup B}(t^{-1})}{\mathcal{L}_A(t^{-1})\mathcal{L}_B(t^{-1})}P\left(\bigcap_{\mathbf{i}\in\mathbf{A}}\{F(X_{\mathbf{i}})>1-t\}\right)P\left(\bigcap_{\mathbf{i}\in\mathbf{B}}\{F(X_{\mathbf{i}})>1-t\}\right).$$

However, in order to obtain a discussion for $\lambda(A,B)$ and $\lambda(B,A)$ as stated in a), b) and c) we need to make additional assumptions such as slow variation of the quotients $\frac{\mathcal{L}_{A\cup B}(x)}{\mathcal{L}_A(x)}$, $\frac{\mathcal{L}_{A\cup B}(x)}{\mathcal{L}_B(x)}$ and $\frac{\mathcal{L}_{A\cup B}(x)}{\mathcal{L}_A(x)\mathcal{L}_B(x)}$ and on the existence of their limits, as $x\to\infty$, since

$$\lambda(A,B)(t)\sim\frac{\mathcal{L}_{A\cup B}(t^{-1})}{\mathcal{L}_A(t^{-1})}t^{\frac{1}{\eta(A\cup B)}-\frac{1}{\eta(A)}}$$

and

$$\lambda(B,A)(t)\sim\frac{\mathcal{L}_{A\cup B}(t^{-1})}{\mathcal{L}_B(t^{-1})}t^{\frac{1}{\eta(A\cup B)}-\frac{1}{\eta(B)}}.$$

In our opinion, the local conditions (2.2) provide the natural way to extend the bivariate measures of pairwise dependence since they are trivially satisfied when $A=\{\mathbf{i}\}$ and $B=\{\mathbf{j}\}$ and the coefficients $\overline{\epsilon}(A)$ and $\overline{\epsilon}(B)$ are the counterpart of $\epsilon(A)$ and $\epsilon(B)$ in modelling joint survivor distributions. They also provide a good motivation for the general case since the used arguments are easily modified.

The results concerning the range of values of $\eta(A\cup B)$ can be extended for the union of several sets of locations.

We will now apply these diagnostic measures of asymptotic independence to a particular random field which is a generalization to space processes of the modelling of duration of extremes, via minima of consecutive i.i.d. random variables, considered in Draisma (2001).

3 Example

Let $\mathbf{Y} = \{Y_\mathbf{t}\}_{\mathbf{t}\in\mathbb{Z}^2}$ be an i.i.d. random field and define $\mathbf{X} = \{X_\mathbf{i} = \min\{Y_\mathbf{i},\ Y_\mathbf{j},\ \mathbf{j} \in V^{(1)}(\mathbf{i})\}\}_{\mathbf{i}\in\mathbb{Z}^2}$, where $V^{(n)}(\mathbf{i}) = \{\mathbf{j} \in \mathbb{Z}^2 :\ \max\{|j_s - i_s|,\ s = 1,2\} = n\}$, $n \geq 1$.

Let $A = \{\mathbf{i}\}$ and $B = V^{(n)}(\mathbf{i})$. We can compute the sequences

$$\eta(n) = \eta(\{\mathbf{i}\}, V^{(n)}(\mathbf{i})),\ n \geq 1,\ \text{and}\ \overline{\epsilon}(n) = \overline{\epsilon}(V^{(n)}(\mathbf{i})),\ n \geq 1,$$

of dependence coefficients which are independent of $\mathbf{i}$ by the stationarity of $\mathbf{X}$ and evaluate the dependence relation between $X_\mathbf{i}$ and its neighbors at the distance n.

We have $\mathcal{L}_{\{\mathbf{i}\}\cup\mathbf{V}^{(\mathbf{n})}(\mathbf{i})}(t) = 1$, $n \geq 1$;

$\eta(1) = \frac{9}{25}$ and $\overline{\epsilon}(V^{(1)}(\mathbf{i})) = \frac{25}{9}$;

$\eta(2) = \frac{9}{49}$ and $\overline{\epsilon}(V^{(2)}(\mathbf{i})) = \frac{48}{9}$;

$\eta(n) = \frac{1}{1+\overline{\epsilon}(V^{(n)}(\mathbf{i}))}$, for $n \geq 3$.

Therefore $\{X_\mathbf{i} > x\}$ and $\bigcap\limits_{\mathbf{i}\in\mathbf{V}^{(1)}(\mathbf{i})} \{X_\mathbf{i} > x\}$ are asymptotically dependent with degree 1, $\{X_\mathbf{i} > x\}$ and $\bigcap\limits_{\mathbf{i}\in\mathbf{V}^{(2)}(\mathbf{i})} \{X_\mathbf{i} > x\}$ are asymptotically independent and tend to occur more frequently than under the exact independence.

The last equalities ($n \geq 3$) correspond to exact independence, as expected from the definition of the 2-dependent random field $\mathbf{X}$.

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