Point processes of exceedances by random fields

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Abstract: Random fields on $\mathbb{Z}_+^d$, with long range weak dependence for each coordinate at a time, usually present clustering of high values based on Poisson distributions for positions of the clusters. The asymptotic theory for point processes of exceedances of high values is developed.

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1. Introduction

Let $\mathbf{X} = \{X_{\mathbf{n}}\}_{\mathbf{n}\geq\mathbf{1}}$ be a stationary random field on Z_+^d, where Z_+ is the set of all positive integers and $d \geq 2$. We shall consider the conditions and results for $d = 2$ since it is notationally simpler and the proofs for higher dimensions follow analogous arguments. The inequality $(i_1, i_2) \leq (n_1, n_2)$ means $i_k \leq n_k, k = 1, 2$, and $\frac{\mathbf{i}}{\mathbf{n}} = \left(\frac{i_1}{n_1}, \frac{i_2}{n_2}\right)$.
For a family of real levels $\{u_{\mathbf{n}}\}_{\mathbf{n}\geq\mathbf{1}}$ and a subset $\mathbf{I}$ of the rectangle of points $\mathbf{R_n} = \{1, \ldots, n_1\} \times \{1, \ldots, n_2\}$, we will denote the event $\{X_{\mathbf{i}} \leq u_{\mathbf{n}}, i \in \mathbf{I}\}$ by $\{M_{\mathbf{n}}(\mathbf{I}) \leq u_{\mathbf{n}}\}$ or simply by $\{M_{\mathbf{n}} \leq u_{\mathbf{n}}\}$ when $\mathbf{I} = \mathbf{R_n}$. For each $i = 1, 2$, we say the pair $\mathbf{I}_1 \subset Z_+^2$ and $\mathbf{I}_2 \subset Z_+^2$ is in $S_i(l)$ if the distance between $\Pi_i(\mathbf{I}_1)$ and $\Pi_i(\mathbf{I}_2)$ is greater or equal to l, where $\Pi_i, i = 1, 2$, denote the cartesian projections.

In this paper we study the convergence of the sequence of point processes of exceedances of $u_{\mathbf{n}}$ by a random field, when there is "high local dependence" in the random field so that one exceedance is likely to be followed by others. The result is a clustering of exceedances, leading to a compounding of events in the limiting point process.
To include cases where clustering occurs, we require a modest strengthening of the coordinatewise-mixing condition introduced in [6] and [2], which exploits the past and future separation one coordinate at a time.

Let $Y \in B_{\mathbf{I}}(u_{\mathbf{n}})$ denote that the random variable Y is measurable to the $\sigma-$field generated by the events $\{X_{\mathbf{i}} \leq u_{\mathbf{n}}\}$ with $i \in \mathbf{I}$. We shall assume that there are sequences of integer valued constants $\{k_{n_i}\}_{n_i \geq 1}$, $\{l_{n_i}\}_{n_i \geq 1}$, $i = 1, 2$, such that, as $\mathbf{n} = (n_1, n_2) \longrightarrow \infty$, we have

$$(k_{n_1}, k_{n_2}) \longrightarrow \infty, \ \left(\frac{k_{n_1} l_{n_1}}{n_1}, \frac{k_{n_2} l_{n_2}}{n_2}\right) \longrightarrow \mathbf{0}, (k_{n_1}\Delta_1, k_{n_1} k_{n_2} \Delta_2) \longrightarrow \mathbf{0}, \tag{1.1}$$

where $\Delta_i, i = 1, 2$, are the components of the mixing coefficient defined as follows:

$$\Delta_1 = sup \left\{ |E(YZ) - E(Y)E(Z)| : Y \in \mathcal{B}_{\mathbf{I}_1}(u_{\mathbf{n}}), Z \in \mathcal{B}_{\mathbf{I}_2}(u_{\mathbf{n}}), 0 \leq Y, Z \leq 1 \right\},$$

where the supremum is taken over pairs $\mathbf{I}_1$ and $\mathbf{I}_2$ in $S_1(l_{n_1})$,

$$\Delta_2 = sup \left\{ |E(YZ) - E(Y)E(Z)| : Y \in \mathcal{B}_{\mathbf{I}_1}(u_{\mathbf{n}}), Z \in \mathcal{B}_{\mathbf{I}_2}(u_{\mathbf{n}}), 0 \leq Y, Z \leq 1 \right\},$$

where the supremum is taken over pairs $\mathbf{I}_1$ and $\mathbf{I}_2$ in $S_2(l_{n_2})$.
Then we say that $\mathbf{X}$ satisfies the $\Delta(u_{\mathbf{n}})$ -condition.

In section 2 we derive some results on the asymptotic independence of clustered exceedances. As a corollary we find the maximum can be regarded as the maximum of a random field of approximately independent variables. Therefore the extremal types theorem holds for such stationary random fields.
Section 3 shows that any limiting point process for exceedances is necessarily a compound Poisson process.
In section 4 we show that the limiting distributions of the clusters size and the maximum characterize this compound Poisson process.
We end this section with an example which illustrate the theory.
In the proofs of our results we follow the main steps to obtain the limiting behavior of the exceedance point process for stationary sequences in [4], with a specific approach for the random fields.

2. Asymptotic independence of clustered exceedances

Let $\mathbf{r_n} = \left(\frac{n_1}{k_{n_1}}, \frac{n_2}{k_{n_2}}\right)$ for some $\mathbf{k_n} = (k_{n_1}, k_{n_2})$ satisfying (1.1). The exceedances of $u_\mathbf{n}$ by $X_\mathbf{i}$ with $\mathbf{i} \in J_{\mathbf{n},\mathbf{s}} = \{(s_1-1)\, r_{n_1}+1, \ldots, s_1 r_{n_1}\} \times \{(s_2-1)\, r_{n_2}+1, \ldots, s_2 r_{n_2}\}$, for some $\mathbf{s} \leq \mathbf{k_n}$, are regarded as forming a cluster.

The following lemma shows that exceedances over disjoint rectangles behave asymptotically as independent.

Lemma 2.1: *Suppose that the condition $\Delta(u_\mathbf{n})$ holds for $\mathbf{X}$ and $J_{\mathbf{n},\mathbf{i}} = J_{n_1,i_1} \times J_{n_2,i_2}$, $i_1 = 1, \ldots, k_{n_1}, i_2 = 1, \ldots, k_{n_2}$, are $k_{n_1}k_{n_2}$ disjoint rectangles such that $\sharp \bigcup_{\mathbf{i} \leq k_\mathbf{n}} J_{\mathbf{n},\mathbf{i}} \sim n_1 n_2$. Then, for any non-negative continuous or step function f on $[0,1]^2$,*

$$d_\mathbf{n} = E\left(exp\left(-\sum_{\mathbf{i}\leq\mathbf{n}} f\left(\frac{\mathbf{i}}{\mathbf{n}}\right) 1_{\{X_\mathbf{i}>u_\mathbf{n}\}}\right)\right) - \prod_{\mathbf{s}\leq\mathbf{k_n}} E\left(exp\left(-\sum_{\mathbf{i}\in\mathbf{J_{n,s}}} f\left(\frac{\mathbf{i}}{\mathbf{n}}\right) 1_{\{X_\mathbf{i}>u_\mathbf{n}\}}\right)\right) \underset{\mathbf{n}\to\infty}{\longrightarrow} 0, \tag{2.1}$$

where $1_\mathbf{A}$ denotes the indicator of the event $\mathbf{A}$.

Proof: Assume that f is not identically zero and that each $\Pi_s(J_{\mathbf{n},\mathbf{i}})$ consists of at least l_{n_s} integers, for each $s = 1, 2$. Let $\overline{I}_i^{(s)}$ be the interval of the largest elements in $\Pi_s(J_{\mathbf{n},\mathbf{i}})$, $J^*_{\mathbf{n},\mathbf{i}} = \left\{\mathbf{j} : j_1 \in \overline{I}_i^{(1)} \vee j_2 \in \overline{I}_i^{(2)}\right\}$ and $\mathbf{I}_{\mathbf{n},\mathbf{i}} = J_{\mathbf{n},\mathbf{i}} \backslash J^*_{\mathbf{n},\mathbf{i}}$.
It is sufficient to show that for any $\mathbf{S} \subset Z^2_+$, there exists a further path $\mathbf{S}' \subset \mathbf{S}$ through which $d_\mathbf{n} \underset{\mathbf{n}\to\infty}{\longrightarrow} 0$.
Let

$$\left\{c_\mathbf{n} = \left(inf_{\mathbf{I}\in\mathbf{M}(l_{n_1},l_{n_2})} E\left(exp\left(-\sum_{\mathbf{i}\in\mathbf{I}} 1_{\{X_\mathbf{i}>u_\mathbf{n}\}}\right)\right)\right)^{k_{n_1}k_{n_2}} : \mathbf{n} \in \mathbf{S}\right\}, \tag{2.2}$$

where $\mathbf{M}(l_{n_1}, l_{n_2}) = \left\{\mathbf{I} \subset Z^2_+ : |\Pi_1(\mathbf{I})| \leq l_{n_1} \vee |\Pi_2(\mathbf{I})| \leq l_{n_2}\right\}$. Since the set given in (2.2) contains infinitely many numbers in $[0,1]^2$, there exists $\mathbf{S}' \subset \mathbf{S}$ such that, for some c, $c_\mathbf{n} \to c$ through $\mathbf{S}'$.

To conclude (2.1) we should show that $d_{\mathbf{n}} \xrightarrow[\mathbf{n}\to\infty]{} 0$ through $\mathbf{S}'$.
Consider separately the following two cases:
(a) If $c=1$, then by the triangle inequality, $d_{\mathbf{n}}$ is bounded in absolute value by

$$\left|E\left(exp\left(-\sum_{\mathbf{i}\leq\mathbf{n}} f\left(\frac{\mathbf{i}}{\mathbf{n}}\right)1_{\{X_{\mathbf{i}}>u_{\mathbf{n}}\}}\right)\right) - E\left(exp\left(-\sum_{\mathbf{s}\leq\mathbf{k_n}}\sum_{\mathbf{i}\in\mathbf{I_{n,s}}} f\left(\frac{\mathbf{i}}{\mathbf{n}}\right)1_{\{X_{\mathbf{i}}>u_{\mathbf{n}}\}}\right)\right)\right|$$
$$+\left|E\left(exp\left(-\sum_{\mathbf{s}\leq\mathbf{k_n}}\sum_{\mathbf{i}\in\mathbf{I_{n,s}}} f\left(\frac{\mathbf{i}}{\mathbf{n}}\right)1_{\{X_{\mathbf{i}}>u_{\mathbf{n}}\}}\right)\right) - \prod_{\mathbf{s}\leq\mathbf{k_n}} E\left(exp\left(-\sum_{\mathbf{i}\in\mathbf{I_{n,s}}} f\left(\frac{\mathbf{i}}{\mathbf{n}}\right)1_{\{X_{\mathbf{i}}>u_{\mathbf{n}}\}}\right)\right)\right|$$
$$+\left|\prod_{\mathbf{s}\leq\mathbf{k_n}} E\left(exp\left(-\sum_{\mathbf{i}\in\mathbf{I_{n,s}}} f\left(\frac{\mathbf{i}}{\mathbf{n}}\right)1_{\{X_{\mathbf{i}}>u_{\mathbf{n}}\}}\right)\right) - \prod_{\mathbf{s}\leq\mathbf{k_n}} E\left(exp\left(-\sum_{\mathbf{i}\in\mathbf{J_{n,s}}} f\left(\frac{\mathbf{i}}{\mathbf{n}}\right)1_{\{X_{\mathbf{i}}>u_{\mathbf{n}}\}}\right)\right)\right| \tag{2.3}$$

Since f is bounded by some integer A, the first term in (2.3) is bounded by

$$\begin{aligned}
& E\left(\left|\prod_{\mathbf{s}\leq\mathbf{k}_n} exp\left(-\sum_{\mathbf{i}\in\mathbf{I_{n,s}}} f\left(\frac{\mathbf{i}}{\mathbf{n}}\right)1_{\{X_{\mathbf{i}}>u_{\mathbf{n}}\}}\right)\prod_{\mathbf{s}\leq\mathbf{k}_n} exp\left(-\sum_{\mathbf{i}\in\mathbf{J^*_{n,s}}} f\left(\frac{\mathbf{i}}{\mathbf{n}}\right)1_{\{X_{\mathbf{i}}>u_{\mathbf{n}}\}}\right)\right.\right. \\
& \qquad \left.\left. -\prod_{\mathbf{s}\leq\mathbf{k}_n} E\left(exp\left(-\sum_{\mathbf{i}\in\mathbf{I_{n,s}}} f\left(\frac{\mathbf{i}}{\mathbf{n}}\right)1_{\{X_{\mathbf{i}}>u_{\mathbf{n}}\}}\right)\right)\right|\right) \\
\leq\ & E\left(\sum_{\mathbf{s}\leq\mathbf{k}_n}\left(1-exp\left(-\sum_{\mathbf{i}\in\mathbf{J^*_{n,s}}} f\left(\frac{\mathbf{i}}{\mathbf{n}}\right)1_{\{X_{\mathbf{i}}>u_{\mathbf{n}}\}}\right)\right)\right) \\
\leq\ & A\sum_{\mathbf{s}\leq\mathbf{k}_n} 1-E\left(exp\left(-\sum_{\mathbf{i}\in\mathbf{J^*_{n,s}}} 1_{\{X_{\mathbf{i}}>u_{\mathbf{n}}\}}\right)\right) \\
\leq\ & Ak_{n_1}k_{n_2}\left(1-inf_{\mathbf{I}\in\mathbf{M}(l_{n_1},l_{n_2})} exp\left(-\sum_{\mathbf{i}\in\mathbf{I}} 1_{\{X_{\mathbf{i}}>u_{\mathbf{n}}\}}\right)\right)
\end{aligned}$$

which tends to zero when $\mathbf{n}\to\infty$ along $\mathbf{S}'$. In fact, $c=1$ is equivalent to

$$k_{n_1}k_{n_2}\left(1-inf_{\mathbf{I}\in\mathbf{M}(l_{n_1},l_{n_2})} exp\left(-\sum_{\mathbf{i}\in\mathbf{I}} 1_{\{X_{\mathbf{i}}>u_{\mathbf{n}}\}}\right)\right) \longrightarrow 0.$$

The third term in (2.3) is bounded by

$$E\left(\sum_{\mathbf{s}\leq\mathbf{k}_n}\left(1-exp\left(-\sum_{\mathbf{i}\in\mathbf{J^*_{n,s}}} f\left(\frac{\mathbf{i}}{\mathbf{n}}\right)1_{\{X_{\mathbf{i}}>u_{\mathbf{n}}\}}\right)\right)\right),$$

and the second term in (2.3) is bounded by $k_{n_1}\Delta_1 + k_{n_1}k_{n_2}\Delta_2$. Both majorants tend to zero and we conclude (2.1) in this first case.

(b) $c < 1$

Let $\mathbf{I} \subset [0,1]^2$ such that $inf_{\mathbf{x}\in\mathbf{I}} f(\mathbf{x}) \geq \alpha > 0$. Write $\mathbf{nI} = \{(n_1x_1, n_2x_2) : \mathbf{x} \in \mathbf{I}\}$. For each $\mathbf{n}$, in each $\mathbf{J}_{\mathbf{n},\mathbf{i}} \cap \mathbf{nI}$ which contains more than $4l_{n_1}l_{n_2}$ integers place $\theta_{\mathbf{n}}^{(\mathbf{i})} = \theta_{n_1}^{(i_1)}\theta_{n_2}^{(i_2)}$ rectangles $\mathbf{R}_{\mathbf{n},\mathbf{j}}^{(i)} = R_{\mathbf{n},j_1}^{(i)} \times R_{\mathbf{n},j_2}^{(i)}$, $j_1 = 1,\ldots,\theta_{n_1}^{(i)}$, $j_2 = 1,\ldots,\theta_{n_2}^{(i)}$ of $l_{n_1}l_{n_2}$ integers, where the rectangles belong to $S_1(l_{n_1}) \cap S_2(l_{n_2})$ and the $\theta_{n_1}^{(i_1)}$ and $\theta_{n_2}^{(i_2)}$ are chosen so that $\theta_{\mathbf{n}} \stackrel{def}{=} \sum_{\mathbf{i}\leq\mathbf{k_n}} \theta_{\mathbf{n}}^{(\mathbf{i})}$ satisfies

$$\frac{\theta_{\mathbf{n}}}{k_{n_1}k_{n_2}} \longrightarrow \infty, \qquad \sum_{i_1=1}^{k_{n_1}} \theta_{n_1}^{(i_1)}\Delta_1 \longrightarrow 0 \;\text{ and }\; \theta_{\mathbf{n}}\Delta_1\Delta_2 \longrightarrow 0,$$

$\mathbf{n} \to \infty$ through $\mathbf{S}'$.

Then

$$\begin{aligned}
& E\left(exp\left(-\sum_{\mathbf{i}\leq\mathbf{n}} f\left(\frac{\mathbf{i}}{\mathbf{n}}\right) 1_{\{X_{\mathbf{i}}>u_{\mathbf{n}}\}}\right)\right) \\
\leq\ & E\left(exp\left(-\sum_{\mathbf{s}\leq\mathbf{k_n}}\sum_{\mathbf{j}\leq\theta_{\mathbf{n}}^{(\mathbf{s})}}\sum_{\mathbf{i}\in\mathbf{R}_{\mathbf{n},\mathbf{j}}^{(\mathbf{s})}} f\left(\frac{\mathbf{i}}{\mathbf{n}}\right) 1_{\{X_{\mathbf{i}}>u_{\mathbf{n}}\}}\right)\right) \\
\leq\ & E^{\theta_{\mathbf{n}}}\left(exp\left(-\alpha \sum_{\mathbf{i}\in\{1,\ldots,l_{n_1}\}\times\{1,\ldots,l_{n_2}\}} f\left(\frac{\mathbf{i}}{\mathbf{n}}\right) 1_{\{X_{\mathbf{i}}>u_{\mathbf{n}}\}}\right)\right) + \sum_{i_1=1}^{k_{n_1}} \theta_{n_1}^{(i_1)}\Delta_1 + \theta_{\mathbf{n}}\Delta_1\Delta_2 \\
\leq\ & \left(E\left(exp\left(-\sum_{\mathbf{i}\in\{1,\ldots,l_{n_1}\}\times\{1,\ldots,l_{n_2}\}} 1_{\{X_{\mathbf{i}}>u_{\mathbf{n}}\}}\right)\right)\right)^{\alpha\theta_{\mathbf{n}}} + o(1) \\
\leq\ & c_n^{\alpha\frac{\theta_{\mathbf{n}}}{k_{n_1}k_{n_2}}} + o(1) = o(1).
\end{aligned}$$

The other term of $d_{\mathbf{n}}$ is bounded by

$$\begin{aligned}
& \prod_{\mathbf{s}\leq\mathbf{k_n}}\left(\prod_{\mathbf{j}\leq\theta_{\mathbf{n}}^{(\mathbf{s})}} E\left(exp\left(-\sum_{\mathbf{i}\in\mathbf{R}_{\mathbf{n},\mathbf{j}}^{(\mathbf{s})}} f\left(\frac{\mathbf{i}}{\mathbf{n}}\right) 1_{\{X_{\mathbf{i}}>u_{\mathbf{n}}\}}\right)\right)\right) + o(1) \\
\leq\ & c_n^{\alpha\frac{\theta_{\mathbf{n}}}{k_{n_1}k_{n_2}}} + o(1)
\end{aligned}$$

which tends to zero as previously. Thus both terms in $d_{\mathbf{n}}$ tend to zero, concluding the proof. $\square$

As hold for point processes of exceedances over the real line, the Lemma 2.1 still holds with $a_{\mathbf{n}}f$ instead f, where $a_{\mathbf{n}}$ are non-negative constants. Then, by taking $f \equiv 1$ and $a_{\mathbf{n}} \to \infty$ in such way that $k_{n_1}k_{n_2}exp(-a_{\mathbf{n}}) \to 0$ we get the lemma of asymptotic independence of maxima over disjoint rectangles.

Lemma 2.2: *Suppose that the random field* $\mathbf{X}$ *verifies the coordinatewise-mixing condition* $\Delta(u_{\mathbf{n}})$ *and* $J_{\mathbf{n},\mathbf{i}} = J_{n_1,i_1} \times J_{n_2,i_2}$, $i_1 = 1,\ldots,k_{n_1}$, $i_2 = 1,\ldots,k_{n_2}$, *are* $k_{n_1}k_{n_2}$ *disjoint rectangles such that* $\sharp \bigcup_{\mathbf{i}\le k_{\mathbf{n}}} J_{\mathbf{n},\mathbf{i}} \sim n_1 n_2$. *Then*

$$P\left(M_{\mathbf{n}} \le u_{\mathbf{n}}\right) - \prod_{\mathbf{s}\le\mathbf{k_n}} P\left(M_{\mathbf{n}}\left(\mathbf{J}_{\mathbf{n},\mathbf{s}}\right) \le u_{\mathbf{n}}\right) \xrightarrow[\mathbf{n}\to\infty]{} 0.$$

□

Lemma 2.2 was proved in [6] (see also [7] and [8]) with a weaker mixing condition than $\Delta(u_{\mathbf{n}})$. However, in this context it can be obtained as a corollary of Lemma 2.1.

3. Convergence of the sequence of point processes of exceedances

For applications in the extreme value theory, the main result of [4] on the sequence of point processes of exceedances can be extended for

$$S_{\mathbf{n}}(\mathbf{B}) = \sum_{\mathbf{i}\le\mathbf{n}} 1_{\{X_{\mathbf{i}}>u_{\mathbf{n}}\}}\delta_{\frac{\mathbf{i}}{\mathbf{n}}}(\mathbf{B}),\ \mathbf{B} \subset [0,1]^2.$$

Proposition 3.1: *Suppose that* $\Delta(u_{\mathbf{n}})$ *holds for* $\mathbf{X}$ *and* $\{S_{\mathbf{n}}\}_{\mathbf{n}\ge\mathbf{1}}$ *converges in distribution to some point process* S. *Then* S *is stationary, has independent increments and therefore has Laplace transform given by*

$$L_S(f) = exp\left(-\alpha\int_{[0,1]^2} f - \int_{[0,1]^2}\int_0^{+\infty}\left(1 - e^{-yf(x,y)}\right)d\nu(y)d(x,y)\right),$$

with $\alpha = 0$ and $\nu(.)$ is a finite measure concentrated on the positive integers Z_+.

Proof: The stationarity of S follows from the stationarity of $S_{\mathbf{n}}$ and its convergence to S. For each $k \in Z_+$, let $\mathbf{B}_1, \mathbf{B}_2, \ldots, \mathbf{B}_k$ be disjoint rectangles in $[0,1]^2$. Since, by Lemma 2.1,

$$\begin{aligned} L_{(S(\mathbf{B}_1),\ldots,S(\mathbf{B}_k))}(t_1,\ldots,t_k) &= \lim_{\mathbf{n}} L_{S_{\mathbf{n}}}\left(\sum_{j=1}^{k} t_j 1_{\mathbf{B}_j}\right) \\ &= \lim_{\mathbf{n}} E\left(\prod_{j=1}^{k} exp\left(-t_j \sum_{\mathbf{i}\in\mathbf{n}\mathbf{B}_j} 1_{\{X_{\mathbf{i}}>u_{\mathbf{n}}\}}\right)\right) \\ &= \lim_{\mathbf{n}} \prod_{j=1}^{k} L_{S_{\mathbf{n}}(\mathbf{B}_j)}(t_j) \\ &= \prod_{j=1}^{k} L_{S(\mathbf{B}_j)}(t_j), \end{aligned}$$

we conclude that S has independent increments.
The result follows, since each stationary process with independent increments has the following representation

$$L_S(f) = exp\left(-\alpha \int_{[0,1]^2} f d(x,z) - \int_{[0,1]^2} \int_0^{+\infty} \left(1 - e^{-yf(x,z)}\right) d\nu(y) d(x,z)\right)$$

and in the case that S is a point process we have $\alpha = 0$ and $\nu(\cdot)$ is a finite measure concentrated on the positive integers Z_+. $\square$

By writting $\nu(0,\infty) = \nu$ and π for the probability distribution $\frac{\nu(.)}{\nu(0,\infty)}$, we get as a corollary that the limiting point process is a compound Poisson point process with Poisson rate ν and distribution of multiplicities π, being its Laplace transform

$$L_S(f) = exp\left(-\nu \int_{[0,1]^2} \left(1 - \sum_{j=1}^{\infty} \pi(j) e^{-jf(x,y)}\right) d(x,y)\right).$$

Thus S is a point process with masses $\mathbf{j}$ at points (x,y) where $((x,y),\mathbf{j})$ are points of a Poisson process in $[0,1]^2 \times (0,\infty)$ with intensity measure $\nu m \times \pi, m$ denoting the Lebesgue measure.

4. A sufficient condition for convergence of the sequence of point processes of exceedances

The number of exceedances in the cluster $J_{\mathbf{n},\mathbf{s}} = \{(s_1-1)\,r_{n_1}+1,\ldots,s_1 r_{n_1}\}\times\{(s_2-1)\,r_{n_2}+1,\ldots,s_2 r_{n_2}\}$ is $S_{\mathbf{n}}\left(\frac{J_{\mathbf{n},\mathbf{s}}}{\mathbf{n}}\right) = \sum_{\mathbf{i}\in J_{\mathbf{n},\mathbf{s}}} 1_{\{X_{\mathbf{i}}>u_{\mathbf{n}}\}}$ and the distribution $\pi_{\mathbf{n}}$ of cluster sizes is defined by

$$\pi_{\mathbf{n}}(j) = P\left(\sum_{\mathbf{i}\leq\mathbf{r_n}} 1_{\{X_{\mathbf{i}}>u_{\mathbf{n}}\}} = j \mid \sum_{\mathbf{i}\leq\mathbf{r_n}}\sum 1_{\{X_{\mathbf{i}}>u_{\mathbf{n}}\}} > 0\right), \quad j=1,2,\ldots.$$

Now we show that, under under the condition $\Delta(u_{\mathbf{n}})$, in the limit S of $S_{\mathbf{n}}$ the multiplicity distribution π is the limit of the cluster size distribution $\pi_{\mathbf{n}}$ and the rate ν can be obtained from the limit of zero exceedances.

Proposition 4.1: *Suppose that $\Delta(u_{\mathbf{n}})$ holds for $\mathbf{X}$ and $\{S_{\mathbf{n}}\}_{\mathbf{n}\geq\mathbf{1}}$ converges in distribution to a compound Poisson point process S with Poisson rate ν and distribution of multiplicities π. Then*

$$\nu = -log \lim_{\mathbf{n}\to\infty} P\left(M_{\mathbf{n}} \leq u_{\mathbf{n}}\right) \tag{4.1}$$

and, if $\nu \neq 0$,

$$\pi(j) = \lim_{\mathbf{n}\to\infty} \pi_{\mathbf{n}}(j), \;\; j=1,2,\ldots, \tag{4.2}$$

for some $\mathbf{r_n} = \left(\frac{n_1}{k_{n_1}}, \frac{n_2}{k_{n_2}}\right)$ and $\mathbf{k_n} = (k_{n_1}, k_{n_2})$ satisfying (1.1).

Proof: Since the Laplace Transform of $S\left([0,1]^2\right)$ is given by

$$E\left(exp\left(-tS\left([0,1]^2\right)\right)\right) = exp\left(-\nu\left(1-\sum_{k=1}^{\infty}\pi(k)e^{-kt}\right)\right)$$

then

$$\begin{aligned}
\lim_{\mathbf{n}\to\infty} P\left(M_{\mathbf{n}} \leq u_{\mathbf{n}}\right) &= \lim_{\mathbf{n}\to\infty} P\left(S_{\mathbf{n}}\left([0,1]^2\right) = 0\right) \\
&= P\left(S\left([0,1]^2\right) = 0\right) \\
&= \lim_{t\to\infty} L_{S\left([0,1]^2\right)}(t) \\
&= \lim_{t\to\infty} exp\left(-\nu\left(1-\sum_{k=1}^{\infty}\pi(k)e^{-kt}\right)\right) \\
&= exp(-\nu).
\end{aligned}$$

To show the other convergence, i.e., $\pi(j) = \lim_{\mathbf{n}\to\infty} \pi_{\mathbf{n}}(j)$, note that, by Lemma 2.1,

$$\lim_{\mathbf{n}\to\infty} E\left(exp\left(-tS_{\mathbf{n}}\left([0,1]^2\right)\right)\right) = \lim_{\mathbf{n}\to\infty} E^{k_{n_1}k_{n_2}}\left(exp\left(-t\sum_{\mathbf{i}\leq\mathbf{r_n}} 1_{\{X_{\mathbf{i}}>u_{\mathbf{n}}\}}\right)\right)$$

and, by the lemma of asymptotic independence of maxima over disjoint rectangles, we have

$$\begin{aligned}
& E\left(exp\left(-t\sum_{\mathbf{i}\leq\mathbf{r_n}} 1_{\{X_{\mathbf{i}}>u_{\mathbf{n}}\}}\right)\right) \\
= \; & 1 - P\left(M_{\mathbf{r_n}} > u_{\mathbf{n}}\right)\left(1 - \sum_{j=1}^{\infty} exp\left(-tj\right)\pi_{\mathbf{n}}(j)\right) \\
= \; & 1 - \frac{\nu}{k_{n_1}k_{n_2}}\left(1 - \sum_{j=1}^{\infty} exp\left(-tj\right)\pi_{\mathbf{n}}(j)\right)(1+o(1)).
\end{aligned}$$

Since

$$E^{k_{n_1}k_{n_2}}\left(exp\left(-t\sum_{\mathbf{i}\leq\mathbf{r_n}} 1_{\{X_{\mathbf{i}}>u_{\mathbf{n}}\}}\right)\right) \approx E\left(exp\left(-tS_{\mathbf{n}}\left([0,1]^2\right)\right)\right)$$

converges then by considering $\alpha_{\mathbf{n}} = \nu\left(1 - \sum_{j=1}^{\infty} exp\left(-tj\right)\pi_{\mathbf{n}}(j)\right)$,

$$\left(1 - \frac{\alpha_{\mathbf{n}}}{k_{n_1}k_{n_2}}\right)^{k_{n_1}k_{n_2}} \approx E^{k_{n_1}k_{n_2}}\left(exp\left(-t\sum_{\mathbf{i}\leq\mathbf{r_n}} 1_{\{X_{\mathbf{i}}>u_{\mathbf{n}}\}}\right)\right)$$

converges, so $\alpha_{\mathbf{n}}$ converges and consequently, for each t, $\sum_{j=1}^{\infty} exp\left(-tj\right)\pi_{\mathbf{n}}(j)$ converges, i.e., there exists π' such that $\pi'(j) = \lim_{\mathbf{n}\to\infty} \pi_{\mathbf{n}}(j)$, $j = 1, 2, ...$
Hence we have

$$\begin{aligned}
& \lim_{\mathbf{n}\to\infty} E\left(exp\left(-tS_{\mathbf{n}}\left([0,1]^2\right)\right)\right) \\
= \; & \lim_{\mathbf{n}\to\infty} E^{k_{n_1}k_{n_2}}\left(exp\left(-t\sum_{\mathbf{i}\leq\mathbf{r_n}} 1_{\{X_{\mathbf{i}}>u_{\mathbf{n}}\}}\right)\right) \\
= \; & \lim_{\mathbf{n}\to\infty}\left(1 - \frac{\alpha_{\mathbf{n}}}{k_{n_1}k_{n_2}}\right)^{k_{n_1}k_{n_2}} \\
= \; & exp\left(-\nu\left(1 - \sum_{j=1}^{\infty} exp\left(-tj\right)\pi'(j)\right)\right).
\end{aligned}$$

Since we showed that

$$\lim_{\mathbf{n}\to\infty} E\left(exp\left(-tS_{\mathbf{n}}\left([0,1]^2\right)\right)\right) = exp\left(-\nu\left(1-\sum_{j=1}^{\infty} exp\left(-tj\right)\pi(j)\right)\right)$$

it follows that $\pi' = \pi$. $\square$

The following technical result is necessary to obtain Proposition 4.2 and corresponds to Lemma 4. in [4] for stationary sequences. Both proofs follow the same arguments with obvious modifications.

Lemma 4.1: *Suppose that* $\lim_{\mathbf{n}\to\infty} P\left(M_{\mathbf{n}} \leq u_{\mathbf{n}}\right) = exp(-\nu), \nu \neq 0,$ *and* $\pi_{\mathbf{n}} \xrightarrow[\mathbf{n}\to\infty]{} \pi$. *For a fixed step function* f *on* $[0,1]^2$, *define a function* $T_{\mathbf{n}}$ *on* $[0,1]^2$ *by*

$$T_{\mathbf{n}}(\mathbf{t}) = \begin{cases} 1 - E\left(exp\left(-\sum_{\mathbf{j}\in \mathbf{J}_{\mathbf{n},\mathbf{s}}} f\left(\frac{\mathbf{j}}{\mathbf{n}}\right) 1_{\left\{X_{\mathbf{j}}>u_{\mathbf{n}}\right\}}\right)\right) & \text{if } \mathbf{nt} \in \mathbf{J}_{\mathbf{n},\mathbf{s}}, \mathbf{s} \leq \mathbf{k}_{\mathbf{n}} \\ 0 & \text{if } \mathbf{t} = \mathbf{0} \ \vee \mathbf{nt} \in \mathbf{R}_{\mathbf{n}} - \cup_{\mathbf{s}\leq \mathbf{k}_{\mathbf{n}}} \mathbf{J}_{\mathbf{n},\mathbf{s}} \end{cases}$$

where $\mathbf{J}_{\mathbf{n},\mathbf{s}} = \left(\left(s_1-1\right)r_{n_1}, s_1 r_{n_1}\right] \times \left(\left(s_2-1\right)r_{n_2}, s_2 r_{n_2}\right].$
Then, as $\mathbf{n} \to \infty$,

(i) $\frac{n_1 n_2}{r_{n_1} r_{n_2}} T_{\mathbf{n}}(\mathbf{t})$ *is uniformly bounded;*

(ii) $\frac{n_1 n_2}{r_{n_1} r_{n_2}} \int_{[0,1]^2} T_{\mathbf{n}}(\mathbf{t}) d\mathbf{t} \longrightarrow \nu \int_{[0,1]^2} \left(1 - \sum exp(-jf(\mathbf{t}))\, \pi(j)\right) d\mathbf{t}.$

$\square$

In the following proposition we present a sufficient condition for the convergence of the point process of exceedances based on the convergence of the probability of non-occurrence of exceedances of the level $u_{\mathbf{n}}$ by the variables $X_{\mathbf{i}}$ of the random field $\mathbf{X}$ and the cluster size distribution $\pi_{\mathbf{n}}$.

Proposition 4.2: *Suppose that the random field* $\mathbf{X}$ *satisfies the condition* $\Delta(u_{\mathbf{n}})$, *and (4.1) and (4.2) hold for some* $\mathbf{r}_{\mathbf{n}} = \left(\frac{n_1}{k_{n_1}}, \frac{n_2}{k_{n_2}}\right)$ *and* $\mathbf{k}_{\mathbf{n}} = \left(k_{n_1}, k_{n_2}\right)$ *satisfying (1.1).*
Then $\left\{S_{\mathbf{n}}\right\}_{\mathbf{n}\geq \mathbf{1}}$ *converges in distribution to a compound Poisson point process* S *with Poisson rate* ν *and distribution of multiplicities* π.

Proof: It suffices to show that $L_{S_{\mathbf{n}}}(f) \xrightarrow[\mathbf{n}\to\infty]{} L_S(f)$ for each non-negative step function f on $[0,1]^2$. With the notation of the previous lemma and by using Lemma 2.1, it follows

$$\begin{aligned}
\log L_{S_{\mathbf{n}}}(f) &= \log E\left(exp\left(-\sum_{\mathbf{j}\leq\mathbf{n}} f\left(\frac{\mathbf{j}}{\mathbf{n}}\right)1_{\{X_{\mathbf{j}}>u_{\mathbf{n}}\}}\right)\right)\\
&\approx \log\prod_{\mathbf{s}\leq\mathbf{k_n}} E\left(exp\left(-\sum_{\mathbf{j}\in\mathbf{J_{n,s}}} f\left(\frac{\mathbf{j}}{\mathbf{n}}\right)1_{\{X_{\mathbf{j}}>u_{\mathbf{n}}\}}\right)\right)\\
&= \frac{n_1n_2}{r_{n_1}r_{n_2}}\sum_{\mathbf{s}\leq\mathbf{k_n}}\frac{r_{n_1}r_{n_2}}{n_1n_2}log(1-T_{\mathbf{n}}(\mathbf{t}))\\
&= -\frac{n_1n_2}{r_{n_1}r_{n_2}}\int_{[0,1]^2} -log(1-T_{\mathbf{n}}(\mathbf{t}))d\mathbf{t}.
\end{aligned}$$

Since for large $\mathbf{n}$, $T_{\mathbf{n}}(\mathbf{t}) \longrightarrow 0$ and

$$|-\log(1-T_{\mathbf{n}}(\mathbf{t}))-T_{\mathbf{n}}(\mathbf{t})| \leq T_{\mathbf{n}}^2(\mathbf{t}) \longrightarrow 0$$

uniformly in $\mathbf{t}$, it follows from Lemma 4.1 that

$$\begin{aligned}
\lim_{\mathbf{n}\to\infty}\log L_{S_{\mathbf{n}}}(f) &= \lim_{\mathbf{n}\to\infty} -\frac{n_1n_2}{r_{n_1}r_{n_2}}\int_{[0,1]^2} T_{\mathbf{n}}(\mathbf{t})d\mathbf{t}\\
&= -\nu\int_{[0,1]^2}\left(1-\sum exp(-jf(\mathbf{t})\right)\pi(j)d\mathbf{t},\\
&= \log L_S(f).
\end{aligned}$$

□

The Poisson rate ν and the limiting multiplicity π present additional interesting properties for levels $\mathbf{u} \equiv \mathbf{u}^{(\tau)} = \left\{u_{\mathbf{n}}^{(\tau)}\right\}_{\mathbf{n}\geq\mathbf{1}}$ satisfying $n_1n_2P(X_{\mathbf{1}} > u_{\mathbf{n}}^{(\tau)}) \xrightarrow[\mathbf{n}\to\infty]{} \tau > 0$. By writting $S_{\mathbf{n}}^{(\tau)}$ for the point process of exceedances of $u_{\mathbf{n}}^{(\tau)}$, if $u_{\mathbf{n},1} \equiv u_{\mathbf{n},1}^{(\tau)}$ and $u_{\mathbf{n},2} \equiv u_{\mathbf{n},2}^{(\tau)}$, that is the two levels are normalized for the same τ, then for the corresponding point processes of exceedances $S_{\mathbf{n},1}^{(\tau)}$ and $S_{\mathbf{n},2}^{(\tau)}$ it holds

$$P\left(S_{\mathbf{n},1}^{(\tau)} \neq S_{\mathbf{n},2}^{(\tau)}\right) \leq n_1n_2\left|P(X_{\mathbf{1}} \leq u_{\mathbf{n},1}^{(\tau)}) - P(X_{\mathbf{1}} \leq u_{\mathbf{n},2}^{(\tau)})\right| \xrightarrow[\mathbf{n}\to\infty]{} 0.$$

Therefore we can use any convenient normalized levels.

The following result states that if the point process of exceedances of the normalized levels $\left\{u_{\mathbf{n}}^{(\tau)}\right\}$ has a limit for one τ it has a limit for all τ and the limit point process is a compound Poisson process with rate $\nu = \theta\tau$ with θ independent of τ. This parameter θ is called the extremal index of the random field $\mathbf{X}$, and was introduced in [2]. Specifically, $\mathbf{X}$ has extremal index θ if, for each $\tau > 0$, there exists $\left\{u_{\mathbf{n}}^{(\tau)}\right\}_{\mathbf{n}\geq\mathbf{1}}$ and

$$\lim_{\mathbf{n}\to\infty} P\left(M_n \leq u_{\mathbf{n}}^{(\tau)}\right) = \lim_{\mathbf{n}\to\infty} P\left(S_{\mathbf{n}}^{(\tau)}\left([0,1]^2\right) = 0\right) = e^{-\theta\tau}.$$

Proposition 4.3: *Suppose that, for each $\tau > 0$, $\Delta(\mathbf{u}^{(\tau)})$ holds for $\mathbf{X}$.*
If, for some $\tau_0 > 0$, $\left\{S_{\mathbf{n}}^{(\tau_0)}\right\}_{\mathbf{n}\geq\mathbf{1}}$ converges in distribution to some point process $S^{(\tau_0)}$, then for all $\tau > 0$, $\left\{S_{\mathbf{n}}^{(\tau)}\right\}_{\mathbf{n}\geq\mathbf{1}}$ converges in distribution to a compound Poisson process with Poisson rate $\nu = \theta\tau$, $\nu = -log \lim_{\mathbf{n}\to\infty} P\left(M_{\mathbf{n}} \leq u_{\mathbf{n}}^{(\tau)}\right)$, $0 \leq \theta \leq \left(\sum_{j\geq 1} j\pi(j)\right)^{-1} \leq 1$, θ and π being independent of τ.

Proof: Assume without loss of generality that $\tau_1 = 1$.Then, by Proposition 4.1.,

$$L_{S^{(1)}}(f) = exp\left(-\theta \int_{[0,1]^2} \left(1 - \sum_{k=1}^{\infty} \pi(k) e^{-kf(\mathbf{t})} d\mathbf{t}\right)\right)$$

with $\theta = -log \lim_{\mathbf{n}\to\infty} P\left(M_{\mathbf{n}} \leq u_{\mathbf{n}}^{(1)}\right)$.
To show that the result holds for each $\tau > 0$ it suffices to prove that, for each $\tau > 0$, there exists a $\delta > 0$ such that for each rectangle $\mathbf{I} \subset [0,1]^2$ with $m(\mathbf{I}) > \delta$, $S_{\mathbf{n}}^{(\tau)}(\mathbf{I})$ converges in distribution to a compound Poisson random variable with Laplace transform $exp\left\{-\theta\tau m(\mathbf{I})(1 - \sum_{k=1}^{\infty} \pi(k) e^{-k\mathbf{t}})\right\}$.
This is sufficient, since any finite number of disjoint rectangles $\mathbf{I}_i$, $1 \leq i \leq k$, in $[0,1]^2$, can be decomposed into disjoint rectangles $\mathbf{I}_{ij}$, $1 \leq j \leq n_i$, $1 \leq i \leq k$, each of which has measure lesser than δ, and thus by Lemma 2.1

$$\begin{aligned}
& \lim_{\mathbf{n}\to\infty} E\left(exp\left(-\sum_{i=1}^{k} s_i S_{\mathbf{n}}^{(\tau)}(\mathbf{I}_i)\right)\right) \\
= & \lim_{\mathbf{n}\to\infty} E\left(exp\left(\sum_{i=1}^{k}\sum_{j=1}^{n_i} s_i S_{\mathbf{n}}^{(\tau)}(\mathbf{I}_{ij})\right)\right) \\
= & \lim_{\mathbf{n}\to\infty} \prod_{i=1}^{k}\prod_{j=1}^{n_i} E\left(exp\left(-s_i S_{\mathbf{n}}^{(\tau)}(\mathbf{I}_{ij})\right)\right).
\end{aligned}$$

To show the existence of such δ, first let $\tau < 1$ and assume for convenience that $u_{\mathbf{n}'}^{(1)} \equiv u_{\mathbf{n}}^{(\tau)}$, where $\mathbf{n}' = \left(\frac{n_1}{\tau_1}, \frac{n_2}{\tau_2}\right)$, $\tau_1\tau_2 = \tau$.

Hence

$$
\begin{aligned}
& \left| E\left(exp(-sS_{\mathbf{n}}^{(\tau)}(\mathbf{I})\right) - E\left(exp\left(-sS_{\mathbf{n}'}^{(1)}\left([0, m(I_1)\tau_1] \times [0, m(I_2)\tau_2]\right)\right)\right)\right| \\
\leq \; & (4m(I_1) + 4m(I_2))\, max \left\{ 1 - F\left(u_{\mathbf{n}}^{(\tau)}\right), 1 - F\left(u_{\mathbf{n}'}^{(1)}\right) \right\} \longrightarrow 0.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
& \lim_{\mathbf{n}\to\infty} E\left(exp\left(-sS_{\mathbf{n}}^{(\tau)}(\mathbf{I})\right)\right) \\
= \; & \lim_{\mathbf{n}\to\infty} E\left(exp\left(-sS_{\mathbf{n}'}^{(1)}([0, m(I_1)\tau_1] \times [0, m(I_2)\tau_2])\right)\right) \\
= \; & exp(-\theta\tau m(\mathbf{I}) \left(1 - \sum_j exp(js)\pi(j) \right).
\end{aligned}
$$

This proves the required with $\delta = 1$. For $\tau > 1$, the proof is identical except that δ must be $\frac{1}{\tau}$.
By Fatou's Lemma it follows that

$$
\tau = \lim_{\mathbf{n}} n_1 n_2 P(X_1 \leq u_{\mathbf{n}}) = \lim_{\mathbf{n}} E\left(S_n^{(\tau)}\left([0,1]^2\right)\right) \geq E\left(S^{(\tau)}\left([0,1]^2\right)\right) = \theta\tau \sum_{j=1}^{\infty} j\pi(j),
$$

concluding the remaining stated inequality. □

If $\theta = 1$ the compound Poisson process reduces to a simple Poisson process. For example, i.i.d. random fields have extremal index $\theta = 1$. Much research has been made, by specific approach, on the asymptotic behavior of the maximum of a stationary Normal field under a variety of conditions ([10]). The classical limit still holds and the extremal index is equal to 1. A value $\theta < 1$ indicates clustering of exceedances of $u_{\mathbf{n}}$, as we illustrate in the next example.

Example: For each $\mathbf{i} = (i_1, i_2) \in Z_+^2$, let $b_s(\mathbf{i}), s = 1, 2, \ldots, 8$, be the neighbors of $\mathbf{i}$ defined as $b_1(\mathbf{i}) = (i_1 + 1, i_2)$, $b_2(\mathbf{i}) = \mathbf{i} + \mathbf{1}$, $b_3(\mathbf{i}) = (i_1, i_2 + 1)$, $b_4(\mathbf{i}) = (i_1 - 1, i_2 + 1)$, $b_5(\mathbf{i}) = (i_1 - 1, i_2)$, $b_6(\mathbf{i}) = \mathbf{i} - \mathbf{1}$, $b_7(\mathbf{i}) = (i_1, i_2 - 1)$ and $b_8(\mathbf{i}) = (i_1 + 1, i_2 - 1)$. For each $s = 1, 2, \ldots, 8$, we shall denote the s-crossing event $\left\{X_{\mathbf{i}} \leq u_{\mathbf{n}}, X_{b_s(\mathbf{i})} > u_{\mathbf{n}}\right\}$ by $B_{\mathbf{i}, b_s(\mathbf{i}), \mathbf{n}}$ where $X_{b_s(\mathbf{i})} = -\infty$ if $b_s(\mathbf{i}) \notin Z_+^2$.
Let $\mathbf{Y} = \{Y_{\mathbf{n}}\}_{\mathbf{n} \geq \mathbf{0}}$ be an i.i.d. random field with common distribution function $F_{\mathbf{Y}}$ and define

$$
X_{\mathbf{n}} = max\left\{Y_{\mathbf{n}}, Y_{b_5(\mathbf{n})}, Y_{b_6(\mathbf{n})}, Y_{b_7(\mathbf{n})}\right\}, \ \mathbf{n} \geq \mathbf{1}.
$$

Let $\{u_{\mathbf{n}}\}_{\mathbf{n} \geq \mathbf{1}}$ be such that $n_1 n_2 \left(1 - F_{\mathbf{Y}}\left(u_{\mathbf{n}}\right)\right) \xrightarrow[\mathbf{n}\to\infty]{} \tau > 0$. Then, we have $u_{\mathbf{n}} \equiv u_{\mathbf{n}}^{(\tau_1)}$ for $\mathbf{X}$ with $\tau_1 = 4\tau$ and $\mathbf{X}$ has extremal index $\theta = \frac{1}{4}$. Moreover $\pi(4) = \lim_{\mathbf{n}\to\infty} \pi_{\mathbf{n}}(4) = 1$ and clusters are asymptotically squares of four exceedances. For each $s = 1, 2, \ldots, 8$, in each cluster we can´t expect more than an s-crossing event. □

In [3] (see also [9]) we present local dependence conditions under which the extremal index can be calculated from the joint distribution of a finite number of variables.

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