Adaptive Probability Weighted Moments Estimation*

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Abstract

In this paper, we make use of *probability weighted moments* of largest observations, in order to build classes of estimators of the *extreme value index*, the primary parameter in *statistics of extremes*. Due to the specificity of these estimators, and contrarily to what happens with the most common estimator of a positive extreme value index, the Hill estimator, a direct estimation of the optimal sample fraction, done on the basis of estimates of scale and shape second-order parameters, is problematic. Again, the use of bootstrap computer intensive methods helps us to provide an adaptive choice of the optimal number of order statistics to be used in the estimation. We also apply the developed methodology to a data set in the field of insurance.

Keywords. Heavy tails; statistics of extremes; extreme value index; adaptive semi-parametric estimation; bias reduction; location/scale invariant estimation.

1 Introduction and preliminaries

The *extreme value index* (EVI) is the parameter $\gamma \in \mathbb{R}$ in the general *extreme value* (EV) distribution function (d.f.)

$$EV_\gamma(x) := \begin{cases} \exp(-(1+\gamma x)^{-1/\gamma}), & 1+\gamma x > 0 \quad \text{if} \quad \gamma \neq 0 \\ \exp(-\exp(-x)), & x \in \mathbb{R} \quad \text{if} \quad \gamma = 0. \end{cases} \tag{1}$$

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Let $(X_1, \ldots, X_n)$ denote a sample of size n from either independent, identically distributed (i.i.d.) or even weakly dependent random variables (r.v.'s) and consider the associated sample of ascending order statistics (o.s.'s) $(X_{1:n} \le \cdots \le X_{n:n})$. The EV d.f., in (1), appears as the limiting d.f., whenever such a non-degenerate limit exists, of the maximum $X_{n:n}$, suitably linearly normalized. We then say that F is in the *domain of attraction* for maximum values of the general EV d.f., in (1), and use the notation $F \in \mathcal{D}_{\mathcal{M}}(EV_\gamma)$.

We shall deal with heavy-tails, i.e. a positive EVI. Then the right-tail function is of regular variation with an index of regular variation equal to $-1/\gamma$, i.e.

$$F \in \mathcal{D}_{\mathcal{M}}(EV_\gamma)_{\gamma>0} \quad \Longleftrightarrow \quad \overline{F} := 1 - F \in RV_{-1/\gamma}, \tag{2}$$

where the notation RV_β stands for the class of *regularly varying* functions at infinity with an *index of regular variation* equal to β, i.e., positive measurable functions g such that $\lim\limits_{t\to\infty} g(tx)/g(t) = x^\beta$, for all $x > 0$.

With the notation

$$U(t) := F^{\leftarrow}(1 - 1/t), \ t \ge 1, \qquad F^{\leftarrow}(y) := \inf\{x : F(x) \ge y\}, \tag{3}$$

condition (2) is equivalent to saying that $U \in RV_\gamma$.

1.1 The estimators under study

One of the first classes of semi-parametric estimators of a positive EVI was considered in Hill (1975). Hill's estimators are based on the log-excesses over an intermediate o.s., $X_{n-k:n}$, with the functional form

$$\hat\gamma_{k,n}^{H} := \frac{1}{k} \sum_{i=1}^{k} \{\ln X_{n-i+1:n} - \ln X_{n-k:n}\}, \quad k = 1, 2, \ldots, n-1. \tag{4}$$

Consistency is achieved in the whole $\mathcal{D}_{\mathcal{M}}(EV_\gamma)_{\gamma\ge 0}$ provided that $X_{n-k:n}$ is an intermediate o.s., i.e., we need to have

$$k = k_n \to \infty \quad \text{and} \quad k/n \to 0, \ \text{as } n \to \infty. \tag{5}$$

We shall also deal with *Pareto probability weighted moments* (PPWM) EVI-estimators, recently introduced in Caeiro and Gomes (2009). They are valid for heavy right-tails, compare favourably with the Hill estimator, in (4), and are given by

$$\widehat{\gamma}_{k,n}^{PPWM} := 1 - \frac{\hat a_1(k)}{\hat a_0(k) - \hat a_1(k)}, \tag{6}$$

with

$$\hat{a}_0(k) := \frac{1}{k}\sum_{i=1}^{k} X_{n-i+1:n} \quad \text{and} \quad \hat{a}_1(k) := \frac{1}{k}\sum_{i=1}^{k} \frac{i}{k}\, X_{n-i+1:n}.$$

Again, consistency is achieved under the first-order framework in (2) and intermediate k-values, i.e., whenever (5) holds.

In order to derive the asymptotic normality of the estimators either in (4) or in (6), it is often assumed the validity of a second-order condition either on $\overline{F}$, in (2), or on U, in (3), like

$$\lim_{t\to\infty} \frac{\ln U(tx) - \ln U(t) - \gamma \ln x}{A(t)} = \frac{x^\rho - 1}{\rho}, \tag{7}$$

where $\rho \leq 0$ is a second-order parameter, which measures the rate of convergence in the first-order condition, (2). If the limit in (7) exists, it is necessarily of the above mentioned type and $|A| \in RV_\rho$ (Geluk and de Haan, 1987). If we assume the validity of the second-order framework in (7), these EVI-estimators are asymptotically normal, provided that $\sqrt{k}A(n/k) \to \lambda_{_A}$, finite, as $n \to \infty$, with A given in (7). Indeed, if we denote $\hat{\gamma}^\bullet_{k,n}$, either the Hill estimator in (4) or the PPWM estimator in (6), we have, with $Z^\bullet_k$ asymptotically standard normal and for adequate $(b_\bullet,\ \sigma_\bullet) \in (\mathbb{R},\ \mathbb{R}^+)$, the validity of the asymptotic distributional representation

$$\hat{\gamma}^\bullet_{k,n} \overset{d}{=} \gamma + \sigma_\bullet Z^\bullet_k/\sqrt{k} + b_\bullet\ A(n/k)(1 + o_p(1)), \quad \text{as} \quad n \to \infty. \tag{8}$$

1.2 Scope of the article

In this article, after a brief review, in Section 2, of the role of the bootstrap methodology in the estimation of optimal sample fractions, we provide an algorithm for the adaptive estimation of the EVI through the PPWM EVI-estimators, also valid for the Hill estimators. In Section 3, we apply such a data-driven estimation to a data set in the field of insurance.

2 The bootstrap methodology and optimal levels

Under the second-order framework, in (7), but with $\rho < 0$, let us parameterize the function A as $A(t) = \gamma\beta t^\rho$, where β and ρ are generalized scale and shape second-order parameters.

Given any semi-parametric EVI-estimator, $\hat{\gamma}^{\bullet}_{k,n}$, let us denote

$$k_0^{\bullet} \equiv k_0^{\hat{\gamma}^{\bullet}}(n) := \arg\min_k MSE(\hat{\gamma}^{\bullet}_{k,n}),$$

with MSE standing for *mean squared error.*

2.1 Adaptive estimation of the EVI

With $\mathbb{E}$ denoting the mean value operator, a possible substitute for the MSE of any classical EVI-estimator $\hat{\gamma}^{\bullet}_{k,n}$ is, cf. equation (8),

$$AMSE(\hat{\gamma}^{\bullet}_{k,n}) := \mathbb{E}\big(\sigma_{\bullet}\ \overline{Z}_k/\sqrt{k} + b_{\bullet}\ A(n/k)\big)^2 = \sigma_{\bullet}^2/k + b_{\bullet}^2\ \gamma^2\ \beta^2\ (n/k)^{2\rho},$$

depending on n and k, and with $AMSE$ standing for *asymptotic mean squared error.* We get (Dekkers and de Haan, 1993)

$$\begin{aligned} k_{0|\hat{\gamma}^{\bullet}}(n) &:= \arg\min_k AMSE\big(\hat{\gamma}^{\bullet}_{k,n}\big) \\ &= \big((-2\rho)\ b_{\bullet}^2\ \gamma^2\beta^2\ n^{2\rho}/\sigma_{\bullet}^2\big)^{-1/(1-2\rho)} = k_0^{\hat{\gamma}^{\bullet}}(n)(1+o(1)), \end{aligned} \tag{9}$$

For the Hill estimator, we have, in (8), $\sigma_{H} = \gamma$ and $b_{H} = 1/(1-\rho)$. Consequently, with $(\hat{\beta}, \hat{\rho})$ any consistent estimator of the vector (β, ρ) of second-order parameters, (9) justifies asymptotically the estimator

$$\hat{k}_0^H := \big[\big((1-\hat{\rho})^2 n^{-2\hat{\rho}}/(-2\hat{\rho}\hat{\beta}^2)\big)^{1/(1-2\hat{\rho})}\big], \tag{10}$$

where, as usual, $[x]$ denotes the integer part of x. Moreover, provided that $\sqrt{k}\ (n/k)^{\rho} \to \lambda$, finite, and with $b_{k,n,\rho} = 1 + \beta(n/k)^{\rho}/(1-\rho)$, $\sqrt{k}\big\{\hat{\gamma}^H_{k,n}/\gamma - b_{k,n,\rho}\big\}$ is approximately $\mathcal{N}(0,1)$. We may then get approximate $100(1-\alpha)\%$ confidence intervals (CI's) for γ,

$$\left(\frac{\hat{\gamma}^H_{k,n}}{b_{k,n,\rho} + \frac{\xi_{1-\alpha/2}}{\sqrt{k}}},\ \frac{\hat{\gamma}^H_{k,n}}{b_{k,n,\rho} - \frac{\xi_{1-\alpha/2}}{\sqrt{k}}}\right), \tag{11}$$

where ξ_p is the p-quantile of a $\mathcal{N}(0,1)$ d.f. If $\lambda = 0$, we may replace in (11) the bias summand $\beta(n/k)^{\rho}/(1-\rho)$ by 0, i.e., we should consider $b_{k,n,\rho} = 1$, in (11).

The same does not happen with the PPWM EVI-estimators, with an asymptotic variance (σ_{PPWM}) and a dominant component of bias (b_{PPWM}) dependent on γ. In this situation, it is sensible to use the bootstrap methodology for the adaptive PPWM EVI-estimation. Just as in Gomes and Oliveira (2001), for the estimation of γ through the Hill estimator, and in Gomes *et al.* (2009), for adaptive reduced-bias estimation, let us consider the auxiliary statistic,

$$T^\bullet_{k,n} := \hat\gamma^\bullet_{[k/2],n} - \hat\gamma^\bullet_{k,n}, \quad k = 2, \ldots, n-1. \tag{12}$$

On the basis of results similar to the ones in Gomes *et al.* (2000) and Gomes and Oliveira (2001), we can get, for the auxiliary statistic $T^\bullet_{k,n}$, in (12), the asymptotic distributional representation,

$$T^\bullet_{k,n} \overset{d}{=} \sigma_\bullet\ Q^\bullet_k/\sqrt{k} + b_\bullet\ (2^\rho - 1)\ A(n/k) + o_p(A(n/k)),$$

with $Q^\bullet_k$ asymptotically standard normal, and $(b_\bullet, \sigma_\bullet)$ given in (8). The $AMSE$ of $T^\bullet_{k,n}$ is thus minimal at a level $k_{0|T^\bullet}(n)$ such that $\sqrt{k}\ A(n/k) \to \lambda'_{_A} \neq 0$, i.e. a level of the type of the one in (9), with $b_\bullet$ replaced by $b_\bullet(2^\rho - 1)$, and we consequently have

$$k_{0|\hat\gamma^\bullet}(n) = k_{0|T^\bullet}(n)\,(1-2^\rho)^{\frac{1}{1-2\rho}}\,(1+o(1)).$$

Then, given the sample $\underline{X}_n = (X_1, \ldots, X_n)$ from an unknown model F, consider for any $n_1 = O(n^{1-\epsilon})$, with $0 < \epsilon < 1$, the bootstrap sample $\underline{X}^*_{n_1} = (X^*_1, \ldots, X^*_{n_1})$, from the model $F^*_n(x) = \frac{1}{n}\sum_{i=1}^n I_{\{X_i \le x\}}$, the empirical d.f. associated with the original sample $\underline{X}_n$. Next, associate to that bootstrap sample the corresponding bootstrap auxiliary statistic, denoted $T^*_{k_1,n_1}$, $1 < k_1 < n_1$. Then, with the obvious notation $k^*_{0|T}(n_1) = \arg\min_{k_1} AMSE\big(T^*_{k_1,n_1}\big)$, $k^*_{0|T}(n_1)/k_{0|T}(n) = (n_1/n)^{-\frac{2\ \rho}{1-2\ \rho}}\,(1+o(1))$, Consequently, for another sample size n_2, and for every $\alpha > 1$,

$$\frac{\big(k^*_{0|T}(n_1)\big)^\alpha}{k^*_{0|T}(n_2)}\ \left(\frac{n_1^\alpha}{n^\alpha}\frac{n}{n_2}\right)^{-\frac{c\ \rho}{1-c\ \rho}} = \big\{k_{0|T}(n)\big\}^{\alpha-1}\,(1+o(1)).$$

It is then enough to choose $n_2 = n\,(n_1/n)^\alpha$, to have independence of ρ. If we put $n_2 = n_1^2/n$, i.e., $\alpha = 2$, we have

$$\frac{\big(k^*_{0|T}(n_1)\big)^2}{k^*_{0|T}(n_2)} = k_{0|T}(n)(1+o(1)),\ \text{ as } n \to \infty.$$

We are now able to estimate $k_0^{\widehat{\gamma}}(n)$, on the basis of any estimate $\hat\rho$ of ρ. With $\hat k^*_{0T}$ denoting the sample counterpart of k^*_{0T}, $\hat\rho$ the ρ-estimate in Step **3.** of the algorithm, and taking into account (9), we can build the k_0-estimate,

$$\hat k^\bullet_{0*} \equiv \hat k^\bullet_{0*}(n; n_1) := \min\left(n-1,\ \left[\frac{\big(1-2^{\hat\rho}\big)^{\frac{1}{1-2\hat\rho}}\big(\hat k^*_{0|T}(n_1)\big)^2}{\hat k^*_{0|T}([n_1^2/n]+1)}\right]+1\right), \tag{13}$$

and the γ-estimate

$$\hat{\gamma}_*^\bullet \equiv \hat{\gamma}_*^\bullet(n;n_1) := \hat{\gamma}_{\hat{k}_{0*}^\bullet(n;n_1),n}. \tag{14}$$

A few practical questions, some of them with answers out of the scope of this paper, may be raised under the set-up developed: How does the asymptotic method work for moderate sample sizes? What is the type of the sample path of the new estimator for different values of the tuning parameter n_1? Is the method strongly dependent on the choice of n_1? Although aware of the theoretical need to have $n_1 = o(n)$, what happens if we choose $n_1 = n$?

2.2 An algorithm for the adaptive EVI-estimation

The estimates $(\hat{\beta}, \hat{\rho})$, of the vector (β, ρ) of second-order parameters, are the ones already used in previous papers, like Gomes *et al.* (2009). Now, and with $\hat{\gamma}_{k,n}^{PPWM}$ defined in (6), the algorithm is the following:

1. Given a sample $(x_1, x_2, \ldots, x_n)$, compute, for tuning parameters $\tau = 0$ and $\tau = 1$, the observed values of $\hat{\rho}_\tau(k)$ introduced and studied in Fraga Alves *et al.* (2003).

2. Consider $\{\hat{\rho}_\tau(k)\}_{k\in\mathcal{K}}$, with $\mathcal{K} = ([n^{0.995}],\ [n^{0.999}])$, compute their median, denoted η_τ, and compute $I_\tau := \sum_{k\in\mathcal{K}} \left(\hat{\rho}_\tau(k) - \eta_\tau\right)^2$, $\tau = 0, 1$. Next choose the *tuning parameter* $\tau^* = 0$ if $I_0 \le I_1$; otherwise, choose $\tau^* = 1$.

3. Work with $\hat{\rho} \equiv \hat{\rho}_{\tau^*} = \hat{\rho}_{\tau^*}(k_1)$ and $\hat{\beta} \equiv \hat{\beta}_{\tau^*} := \hat{\beta}_{\hat{\rho}_{\tau^*}}(k_1)$, $k_1 = [n^{0.999}]$ and $\hat{\beta}_{\hat{\rho}}(k)$ given in Gomes and Martins (2002).

4. Compute $\hat{\gamma}_{k,n}^{PPWM}$, $k = 1, 2, \cdots, n-1$.

5. Next, consider a sub-sample size $n_1 = o(n)$, and $n_2 = [n_1^2/n] + 1$.

6. For l from 1 until B, generate independently B bootstrap samples $(x_1^*, \ldots, x_{n_2}^*)$ and $(x_1^*, \ldots, x_{n_2}^*, x_{n_2+1}^*, \ldots, x_{n_1}^*)$, of sizes n_2 and n_1, respectively, from the empirical d.f. $F_n^*(x) = \frac{1}{n}\sum_{i=1}^n I_{\{X_i \le x\}}$ associated with the observed sample $(x_1, \ldots, x_n)$.

7. Denoting $T_{k,n}^*$ the bootstrap counterpart of $T_{k,n}^{PPWM}$, defined in (12), obtain, $1 \le l \le B$, $t_{k,n_1,l}^*$, $1 < k < n_1$, and $t_{k,n_2,l}^*$, $1 < k < n_2$, the observed values of the statistics T_{k,n_i}^*, $i = 1, 2$. For $k = 2, \ldots, n_i - 1$, compute

$$MSE^*(n_i, k) = \frac{1}{B}\sum_{l=1}^{B} \left(t_{k,n_i,l}^*\right)^2,$$

and obtain $\hat{k}^*_{0|T}(n_i) := \arg\min_{1<k<n_i} MSE^*(n_i, k),, \ i = 1, 2.$

8. Compute the threshold estimate $\hat{k}_{0*} \equiv \hat{k}^{PPWM}_{0*}$, in (13).
9. Obtain $PPWM^* \equiv \hat{\gamma}^{PPWM}_* \equiv \hat{\gamma}^{PPWM}_*(n; n_1) := \hat{\gamma}_{\hat{k}_{0*},n}$, already provided in (14).

A similar procedure can be used for the bootstrap data-driven estimation through the Hill estimator, in (4). Note also that bootstrap confidence intervals are easily associated with the estimates presented, through the replication of steps from 6. up to 9. of this algorithm r times.

3 A case study in the field of insurance

We shall next consider an illustration of the performance of the adaptive PPWM EVI-estimates under study, comparatively with the same methodology applied to the Hill EVI-estimates, again through the analysis of automobile claim amounts exceeding 1,200,000 Euro over the period 1988-2001, gathered from several European insurance companies co-operating with the same re-insurer (Secura Belgian Re). This data set was already studied in Beirlant *et al.* (2004), Vandewalle and Beirlant (2006), Beirlant *et al.* (2008) and Gomes *et al.* (2009), as an example to excess-of-loss reinsurance rating and heavy-tailed distributions in car insurance. A preliminary graphical analysis of the data, x_i, $1 \le i \le n$, $n = 371$, leads us to an immediate conclusion that data have been censored to the left and that the right-tail of the underlying model is quite heavy. The sample paths of the ρ-estimates associated with $\tau = 0$ and $\tau = 1$ lead us to choose, on the basis of any stability criterion for large k, the estimate associated with $\tau = 0$. The algorithm here presented led us to the ρ-estimate $\hat{\rho}_0 = -0.74$, obtained at the level $k_1 = [n^{0.999}] = 368$. The associated β-estimate was $\hat{\beta}_0 = 0.80$. For the Hill estimator, we got the estimate $\hat{k}^H_0 = 55$, with $\hat{k}^H_0$ provided in (10), and an associated γ-estimate equal to 0.291. The associated approximate 95% confidence interval, in (11), is $(0.2115, 0.3432)$, with a size 0.1317.

The application of the algorithm presented in Section 2.2 of this paper, with a sub-sample size $n_1 = [n^{0.955}] = 284$, and $B = 250$ bootstrap generations, led us to $\hat{k}^{PPWM}_{0*} = 58$ and to the adaptive PPWM EVI-estimate $PPWM^* \equiv \hat{\gamma}^{PPWM}_* = 0.272$. This same algorithm applied to the Hill estimates leads us to $\hat{k}^H_{0*} = 52$ and to the adaptive Hill EVI-estimate

$H^* \equiv \hat{\gamma}_*^H = 0.299$. These values are pictured in Figure 1, where we also present the estimates under study as a function of k. The most adequate estimate seems neatly to be the one associated with the PPWM methodology, as detected in a comparative study of the bootstrap Hill and PPWM provided by this algorithm, a topic out of the scope of this paper.

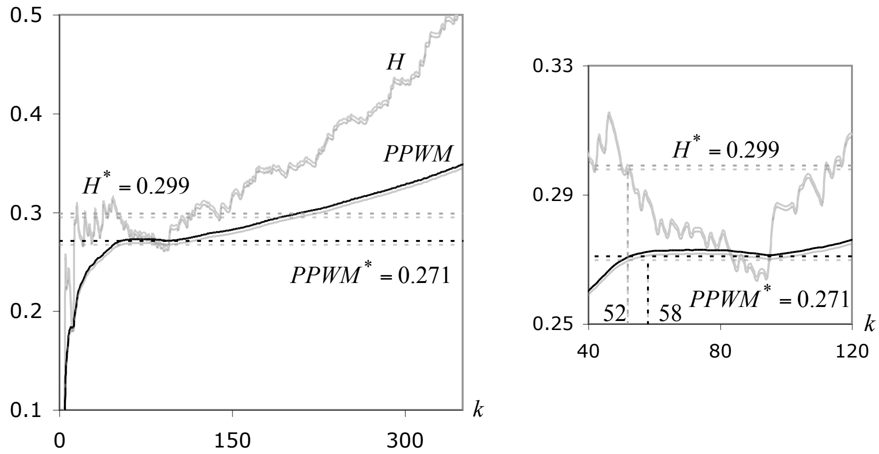

Figure 1: H and PPWM EVI-estimates for the SECURA data, as a function of k (*left*), and a zoomed figure fo $40 \leq k \leq 120$ (*right*).

3.1 Resistance of the methodology to changes in the sub-sample size n_1

In Figure 2, we picture at the left, as a function of the sub-sample size n_1, ranging from $n_1 = n^{0.95} = 275$ until $n_1 = n^{0.9999} = 370$, the estimates of the OSF for the adaptive bootstrap estimation of γ through the Hill and the PPWM estimators, in (4) and (6), respectively. The associated bootstrap EVI-estimates are pictured at the right.

The bootstrap PPWM EVI-estimates are indeed quite stable as a function of the sub-sample size n_1 (see Figure 2, at the right), varying from a minimum value equal to 0.272 until 0.273, with a median equal to 0.273, not a long way from the value we have obtained for the bootstrap γ-estimate associated to the arbitrarily chosen sub-sample size $n_1 = n^{0.955} = 284$, equal to 0.272. We can indeed guarantee the two decimal figures, i.e. the estimate 0.27. The bootstrap estimates of the OSF for the estimation of γ through the PPWM estimator, in (6), vary from 15.4% until 17%, with a median equal to 15.9%. For the bootstrap Hill EVI-estimates the volatility is higher. We get estimates from a minimum value equal to

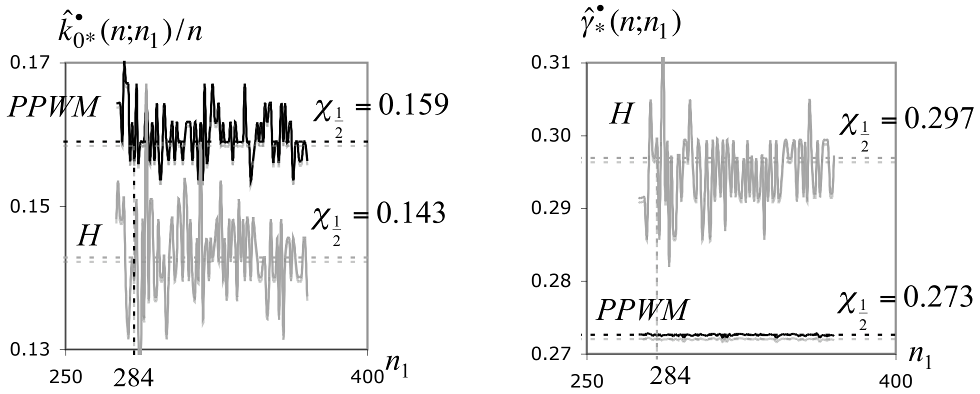

Figure 2: Estimates of the OSF's $\hat{k}_0^{PPWM}/n$ (*left*) and the bootstrap adaptive extreme value index estimates $\hat{\gamma}_*^{PPWM}$ (*right*), as functions of the sub-sample size n_1, for the SECURA data.

0.283 until a maximum value equal to 0.315. The median of these values is 0.297, also not a long way from the value we obtain for the bootstrap γ-estimate associated with the same arbitrarily chosen sub-sample size $n_1 = 284$, equal to 0.299. The volatility of the OSF's for the Hill estimation is similar to the one we get for the PPWM EVI estimation, ranging from 12.4% until 16.7%, with a median equal to 14.3%.

The running of the above mentioned algorithm $r = 100$ times, for $n_1 = n^{0.955}$, provided the median and average estimates, as well as the 95% bootstrap confidence intervals for γ presented in the first row of Table 1. The 95% confidence intervals are based on the quantiles with probability 0.025 and 0.975 of the 100 replicates. These values are not a long way from the ones presented in the second row of Table 1, related with the equivalent quantities associated with the bootstrap estimates obtained in this Section for the 96 values of n_1, $275 \le n_1 \le 370$.

$\hat{\gamma}^{PPWM}$ Median/Average	95% PPWM-CI's for γ	$\hat{\gamma}^{H}$ Median/Average	95% H-CI's for γ
0.2726 / 0.2725	(0.2715, 0.2728)	0.2969 / 0.2949	(0.2826, 0.3133)
0.2726 / 0.2726	(0.2722, 0.2728)	0.2969 / 0.2952	(0.2863, 0.3050)

Table 1: Adaptive bootstrap estimates and 95% confidence intervals for γ.

The size of the confidence intervals as well as the above mentioned simulation study are in favour of the PPWM estimation, as expected. As already detected in previous papers, and in the most diversified comparisons, the Hill estimates are clearly over-estimating the true value of the *extreme value index*.

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