The Generalized Pareto process; with application

Ana Ferreira*
ISA, Universidade Técnica de Lisboa and CEAUL
Tapada da Ajuda 1349-017 Lisboa, Portugal

Laurens de Haan
Erasmus University Rotterdam and CEAUL
P.O. Box 1738, 3000 DR Rotterdam, The Netherlands

February 9, 2012

Abstract

In extreme value statistics the peaks-over-threshold method is widely used. The method is based on the Generalized Pareto distribution ([1], [8] in univariate theory and e.g. [3], [14] in multivariate theory) characterizing probabilities of exceedances over high thresholds. We present a generalization of this concept in the space of continuous functions. We call this the Generalized Pareto process. Different from earlier papers our definition is not based on a distribution function but on functional properties.

As an application we use the theory to produce wind fields connected to disastrous storms on the basis of observed extreme but not disastrous storms.

Keywords: domain of attraction, extreme value theory, generalized Pareto process, max-stable processes, regular variation

1 Introduction

We say that a stochastic process X in $C(S)$ (the space of continuous functions whith S a compact subset of $\mathbb{R}^d$) is in the domain of attraction of a max-stable process if there are continuous functions $a_s(n)$ positive and $b_s(n)$ on S such that the processes

$$\left\{ \max_{i \le n} \frac{X_i(s) - b_s(n)}{a_s(n)} \right\}_{s \in S},$$

with $X, X_1, \ldots, X_n$ independent and identically distributed, converge in distribution to a max-stable process Y in $C(S)$. Necessary and sufficient conditions for this to happen are: uniform convergence of the marginal distributions and a convergence of measures (in fact a form of regular variation):

$$\lim_{t\to\infty} tP\left(\eta_t \in A\right) = \nu(A) \tag{1.1}$$

where $\eta_t(s) := \left(1+\gamma(s)\frac{X(s)-b_s(t)}{a_s(t)}\right)^{1/\gamma(s)}$ for all $s \in S$, ν is a homogeneous (of order -1) measure on $C^+(S) := \{f \in C(S) : f \geq 0\}$ and A any Borel set of $C^+(S)$ with the properties: $\nu\left(\partial A\right) = 0$ and $\inf\{\sup_{s\in S} f(s) : f \in A\} > 0$ (de Haan and Lin (2001), cf. de Haan and Ferreira Section 9.5). The functions $a_s(n)$ and $b_s(n)$ are chosen in such a way that the marginal distributions are in standard form: $\exp -\left(1+\gamma(s)x\right)^{-1/\gamma(s)}$. Here γ is a continuous function. In particular one may take $b_s(t) := \inf\{x : P(X(s) \leq x) \geq 1 - 1/t\}$. This is how we choose $b_s(t)$ from now on.

From (1.1) it follows that

$$\frac{P\left(\left(1+\gamma(\cdot)\frac{X(\cdot)-b_\cdot(t)}{a_\cdot(t)}\right)^{1/\gamma(\cdot)} \in A\right)}{P\left(\sup_{s\in S}\frac{X(s)-b_s(t)}{a_s(t)} > 0\right)}$$

converges as $t \to \infty$ and so does

$$P\left(\left(1+\gamma(\cdot)\frac{X(\cdot)-b_\cdot(t)}{a_\cdot(t)}\right)^{1/\gamma(\cdot)} \in A \Big| \sup_{s\in S}\frac{X(s)-b_s(t)}{a_s(t)} > 0\right).$$

The limit constitutes a probability distribution on $C^+(S)$.

This reasoning is quite similar to how one gets the generalized Pareto distributions in $\mathbb{R}$ (Balkema and de Haan, 1974) and in $\mathbb{R}^d$ (Rootzén and Tajvidi, 2006; Falk, Hüsler and Reiss, 2010). It leads to what we call generalized Pareto processes.

As in the finite dimensional context it is convenient to study first generalized Pareto processes in a standardized form. This is done in Section 2. The general process is discussed in Section 3 and the domains of attraction in Section 4.

In finite dimensional space the peaks-over-threshold method for estimating distribution tails is well known (see e.g. Coles' (2001) book, Chapters 4 and 8). In the same vein, in Section 5, we show that by using the stability property of generalized Pareto processes one can create extreme storm fields starting form independent and identically observations of storm fields.

A note on notation. Operations like $w_1 + w_2$ or $w_1 \wedge w_2$ with $w_1, w_2 \in C(S)$ mean respectively $\{w_1(s) + w_2(s)\}_{s\in S}$ and $\{w_1(s) \wedge w_2(s)\}_{s\in S}$. Then with abuse of notation, operations like $w + x$ or $w \wedge x$ with $w \in C(S)$ and $x \in \mathbb{R}$ mean respectively $\{w(s) + x\}_{s\in S}$ and $\{w(s) \wedge x\}_{s\in S}$. Similarly for products and powers. Then e.g. we shall simply write $\left(1+\gamma\frac{X-b(t)}{a(t)}\right)^{1/\gamma}$ for $\left\{\left(1+\gamma(s)\frac{X(s)-b_s(t)}{a_s(t)}\right)^{1/\gamma(s)}\right\}_{s\in S}$, when $X = \{X(s)\}_{s\in S}$, $a(t) = \{a_s(t)\}_{s\in S}$, $b(t) = \{b_s(t)\}_{s\in S}$ and $\gamma = \{\gamma(s)\}_{s\in S}$.

2 The simple Pareto process

Let $C^+(S)$ be the space of non-negative real continuous functions in S, with S some compact subset of $\mathbb{R}^d$. We denote the Borel subsets of a metric space by $\mathcal{B}(\cdot)$.

Theorem 2.1. *Let W be a stochastic process in $C^+(S)$ and ω_0 a positive constant. The following three statements are equivalent:*

1. *(POT - peaks-over-threshold - stability)*
   - *(a)* $E\left(W(s)/\sup_{s\in S} W(s)\right) > 0$ *for all* $s \in S$,
   - *(b)* $P\left(\sup_{s\in S} W(s)/\omega_0 > x\right) = x^{-1}$, *for* $x > 1$ *(standard Pareto distribution),*
   - *(c)*
   $$P\left(\frac{\omega_0\, W}{\sup_{s\in S} W(s)} \in B \Big| \sup_{s\in S} W(s) > r\right) = P\left(\frac{\omega_0\, W}{\sup_{s\in S} W(s)} \in B\right), \tag{2.1}$$
   for all $r > \omega_0$ *and* $B \in \mathcal{B}\left(\bar{C}^+_{\omega_0}(S)\right)$ *with*
   $$\bar{C}^+_{\omega_0}(S) := \{f \in C^+(S) : \sup_{s\in S} f(s) = \omega_0\}. \tag{2.2}$$

2. *(Random functions)*
   - *(a)* $P\left(\sup_{s\in S} W(s) \geq \omega_0\right) = 1$,
   - *(b)* $E\left(W(s)/\sup_{s\in S} W(s)\right) > 0$ *for all* $s \in S$,
   - *(c)*
   $$P(W \in rA) = r^{-1} P(W \in A), \tag{2.3}$$
   for all $r > 1$ *and* $A \in \mathcal{B}\left(C^+_{\omega_0}(S)\right)$, *where* rA *means the set* $\{rf, f \in A\}$, *and*
   $$C^+_{\omega_0}(S) := \{f \in C^+(S) : \sup_{s\in S} f(s) \geq \omega_0\}. \tag{2.4}$$

3. *(Constructive approach)* $W(s) = YV(s)$, *for all* $s \in S$, *for some* Y *and* $V = \{V(s)\}_{s\in S}$ *verifying:*
   - *(a)* $V \in C^+(S)$ *is a stochastic process verifying* $\sup_{s\in S} V(s) = \omega_0$ *a.s., and* $EV(s) > 0$ *for all* $s \in S$,
   - *(b)* Y *is a standard Pareto random variable,* $F_Y(y) = 1 - 1/y$, $y > 1$,
   - *(c)* Y *and* V *are independent.*

Definition 2.1. *The process W characterized in Theorem 2.1, with threshold parameter ω_0, is called simple Pareto process. The probability measure in* (2.1) *i.e.,*

$$Q(B) = P\left(\frac{\omega_0\, W}{\sup_{s\in S} W(s)} \in B\right), \quad \text{for } B \in \mathcal{B}\left(\bar{C}^+_{\omega_0}(S)\right), \tag{2.5}$$

is called the spectral measure.

Proof of Theorem 2.1. We start by proving that *1.* implies *3.* By compactness and continuity, $\sup_{s\in S} W(s) < \infty$ a.s. Take:

$$Y = \frac{\sup_{s\in S} W(s)}{\omega_0} \quad \text{and} \quad V = \frac{\omega_0 W}{\sup_{s\in S} W(s)}.$$

Then (a), (b) and (c) are straightforward.

Next we prove that *3.* implies *2.* Let

$$A_{r,B} = \left\{ f \in C^+(S) : \sup_{s\in S} f(s)/\omega_0 > r,\ \frac{\omega_0 f}{\sup_{s\in S} f(s)} \in B \right\} = r \times A_{1,B},$$

for all $r > 1$ and $B \in \mathcal{B}\left(\bar{C}^+_{\omega_0}(S)\right)$. Then,

$$\begin{aligned} P\left(W \in A_{r,B}\right) &= P\left(\sup_{s\in S} W(s)/\omega_0 > r,\ \frac{\omega_0 W}{\sup_{s\in S} W(s)} \in B\right) \\ &= P(Y > r,\, V \in B) = P(Y > r)\, P(V \in B) \\ &= \frac{1}{r} P\left(\sup_{s\in S} W(s)/\omega_0 > 1,\ \frac{\omega_0 W}{\sup_{s\in S} W(s)} \in B\right) = \frac{1}{r} P\left(W \in A_{1,B}\right) \end{aligned}$$

using in particular the independence of Y and V and $P\left(\sup_{s\in S} W(s)/\omega_0 > 1\right) = 1$. Since $P(tA) = t^{-1}P(A)$ holds for any of the above sets, it holds for all Borel sets in the statement.

Finally, check that *2.* implies *1.* For any $r \geq 1$,

$$\begin{aligned} &P\left(\frac{\sup_{s\in S} W(s)}{\omega_0} > r\right) \\ &\quad = P\left(\frac{\sup_{s\in S} W(s)}{\omega_0} > r,\ \frac{\omega_0 W}{\sup_{s\in S} W(s)} \in \bar{C}^+_{\omega_0}(S)\right) = \frac{1}{r} P\left(\bar{C}^+_{\omega_0}(S)\right) = \frac{1}{r} \end{aligned}$$

and also for any $B \in \mathcal{B}\left(\bar{C}^+_{\omega_0}(S)\right)$,

$$\begin{aligned} &P\left(\sup_{s\in S} W(s)/\omega_0 > r,\ \frac{\omega_0 W}{\sup_{s\in S} W(s)} \in B\right) \\ &= \frac{1}{r} P\left(\sup_{s\in S} W(s)/\omega_0 > 1,\ \frac{\omega_0 W}{\sup_{s\in S} W(s)} \in B\right) = \frac{1}{r} P\left(\frac{\omega_0 W}{\sup_{s\in S} W(s)} \in B\right) \end{aligned}$$

since $\sup_{s\in S} W(s) > \omega_0$ holds a.s. That is, it follows that $\sup_{s\in S} W(s)/\omega_0$ is univariate Pareto distributed and, $\sup_{s\in S} W(s)$ and $W/\sup_{s\in S} W(s)$ are independent. □

The following properties are direct consequences:

Corollary 2.1. *For any simple Pareto process W, the random variable $\omega_0^{-1} \sup_{s\in S} W(s)$ has standard Pareto distribution.*

Corollary 2.2. *$W \in C^+(S)$ is a simple Pareto process if and only if any of the two equivalent statements hold:*

1. (a) $E\left(W(s)/\sup_{s\in S} W(s)\right) > 0$ *for all* $s \in S$,

   (b) $P\left(\sup_{s\in S} W(s)/\omega_0 > x\right) = x^{-1}$, *for* $x > 1$,

   (c)
$$P\left(W \in rA \big| \sup_{s\in S} W(s) > r\omega_0\right) = P\left(W \in A\right) \tag{2.6}$$
   for all $r > 1$ *and* $A \in \mathcal{B}\left(C^+_{\omega_0}(S)\right)$.

2. (a) $E\left(W(s)/\sup_{s\in S} W(s)\right) > 0$ *for all* $s \in S$,

   (b)
$$P\left(\sup_{s\in S} \frac{W(s)}{\omega_0} > r, \frac{\omega_0\, W}{\sup_{s\in S} W(s)} \in B\right) = \frac{Q(B)}{r}, \tag{2.7}$$
   for all $r > 1$ *and* $B \in \mathcal{B}\left(\bar{C}^+_{\omega_0}(S)\right)$.

From (2.6) we see that the probability distribution of W serves in fact as the exponent measure in max-stable processes (cf. de Haan and Ferreira (2006), Section 9.3). Characterization 2. suggests ways for testing and modelling Pareto processes.

Let $w, W \in C^+(S)$. Conditions $W \leq (>) w$ define the sets $\{f \in C^+(S) : f(s) \leq (>) w(s)$ for all $s \in S\}$ and $(W \not\leq w)$ defines the set $\{f \in C^+(S) : f(s) > w(s)$ for at least one $s \in S\}$; in the later if additionally $\inf_{s\in S} w(s) > \omega_0$ then $\{\sup_{s\in S} f \in C^+(S) : f(s) > w(s)$ for at least one $s \in S\} > \omega_0$. Note also that $W > w$ is not the complement of $W \leq w$.

Proposition 2.1. *(Distribution functions) Let $w, W \in C^+(S)$, with W simple Pareto process. Then,*

$$P(W \leq w) = E\left(\sup_{s\in S} \frac{V(s)}{w(s) \wedge \omega_0}\right) - E\left(\sup_{s\in S} \frac{V(s)}{w(s)}\right) \tag{2.8}$$

with the difference interpreted as zero if $\inf_{s\in S} w(s) = 0$. *In particular,* $P(W \not\leq \omega_0) = 1$.

Corollary 2.3. *For all $w \in C^+(S)$ such that* $\sup_{s\in\mathbb{R}} w(s) > \omega_0$,

$$P(W > w) = E\left(\inf_{s\in S} \frac{V(s)}{w(s)}\right) = \omega_0 E\left(\frac{\inf_{s\in S} W(s)/w(s)}{\sup_{s\in S} W(s)}\right). \tag{2.9}$$

Corollary 2.4. *For $x \in \mathbb{R}$,*

$$P\left(W(s) > x \text{ for all } s \in S | W > \omega_0\right) = \frac{\omega_0}{x}, \quad x > \omega_0, \tag{2.10}$$

$$P\left(W(s) > x | W(s) > \omega_0\right) = \frac{\omega_0}{x}, \quad x > \omega_0, \quad \text{for all } s \in S. \tag{2.11}$$

Relation (2.8) is an analog of Definition 2.1 in Rootzén and Tajvidi (2006), and also (2.11) is in agreement with their results on lower-dimensional distributions.

Proof of Proposition 2.1. $P\left(\sup_{s\in S} W(s) \geq \omega_0\right) = 1$ implies that, for all $0 \leq \varepsilon < \omega_0$ there exists $s \in S$ such that $W(s) > \varepsilon$ with probability one, which implies, for $w \in C^+(S)$ such that $\inf_{s\in S} w(s) = 0$,

$$P(W \not\leq w) = P\left(\{f \in C^+(S) : f(s) > w(s) \text{ for at least one } s\}\right) = 1.$$

Hence from now on take w such that $\inf_{s\in S} w(s) > 0$. If even $\inf_{s\in S} w(s) \geq \omega_0$,

$$P(W \leq w) = P(YV \leq w) = P\left(Y \leq \inf_{s\in S} \frac{w(s)}{V(s)}\right) = 1 - E\left(\sup_{s\in S} \frac{V(s)}{w(s)}\right) \tag{2.12}$$

hence,

$$P(W \not\leq w) = E\left(\sup_{s\in S} \frac{V(s)}{w(s)}\right) = \omega_0 E\left(\frac{\sup_{s\in S} W(s)/w(s)}{\sup_{s\in S} W(s)}\right). \tag{2.13}$$

On the other hand, the homogeneity property (2.3) allows us to extend the probability measure of W that lives on the space $C^+_{\omega_0}(S)$ to all of $C^+(S)$. We call the resulting measure ν. It plays the same role as the exponent measure ν in the theory of max-stable processes (cf. de Haan and Ferreira (2006), Section 9.3). Then, for $B \subset C^+(S)$,

$$P(W \in B) = \nu\{f \in C^+(S) : f \in B, \sup_{s\in S} f(s) > \omega_0\}. \tag{2.14}$$

In order to determine $P(W \leq w)$ for any $w \in C^+(S)$ we use (2.13), (2.14) and Theorem 2.1:

$$\begin{aligned}\nu\left\{f \in C^+(S) : f \not\leq w\right\} &= \frac{\omega_0}{\inf_{s\in S} w(s)} \nu\left\{f \in C^+(S) : f \not\leq \frac{w\,\omega_0}{\inf_{s\in S} w(s)}\right\} \\ &= \frac{\omega_0}{\inf_{s\in S} w(s)} P\left(W \not\leq \frac{w\,\omega_0}{\inf_{s\in S} w(s)}\right) = E\left(\sup_{s\in S} \frac{V(s)}{w(s)}\right),\end{aligned}$$

hence,

$$\begin{aligned}&P(W \not\leq w) = \nu\left\{f \in C^+(S) : f \not\leq w, f \not\leq \omega_0\right\} \\ &= \nu\left\{f \in C^+(S) : f \not\leq w\right\} + \nu\left\{f \in C^+(S) : f \not\leq \omega_0\right\} - \nu\left\{f \in C^+(S) : f \not\leq w \wedge \omega_0\right\} \\ &= E\left(\sup_{s\in S} \frac{V(s)}{w(s)}\right) + 1 - E\left(\sup_{s\in S} \frac{V(s)}{w(s) \wedge \omega_0}\right).\end{aligned}$$

□

Proof of Corollary 2.3. Similar to (2.12). □

Proof of Corollary 2.4. Relation (2.10) is direct from Corollary 2.3.

For (2.11): for $x \in \mathbb{R}$,

$$P\left(W(s) > x\right) = P\left(YV(s) > x\right) = P\left(Y > \frac{x}{V(s)}\right)$$
$$= E\left(\frac{V(s)}{x} \wedge 1\right) = \int_{\{v(s)/x \leq 1\}} \frac{v(s)}{x} dF_{V(s)} + P\left(V(s) > x\right)$$

which simplifies to $x^{-1}EV(s)$ for $x > \omega_0$. The rest follows. □

Finally, we give two simple examples.

Example 2.1. 1. *One of the simplest process is when $W \equiv Y$, that is a Pareto random variable governing through the whole space. Then Q concentrates on the constant function $w \equiv 1$. Some examples of probabilities in space are then, for all w with $\inf_{s\in\mathbb{R}} w(s) > 1$,*

$$P(W \leq w) = P\left(Y \leq \inf_{s\in\mathbb{R}} w(s)\right) = 1 - \frac{1}{\inf_{s\in\mathbb{R}} w(s)}$$

and, for w with $\sup_{s\in\mathbb{R}} w(s) > 1$,

$$P(W > w) = P\left(Y > \sup_{s\in\mathbb{R}} w(s)\right) = \frac{1}{\sup_{s\in\mathbb{R}} w(s)}.$$

Of course all the univariate marginals are standard Pareto.

2. *Another simple situation is with $\{W(s)\}_{s\in\mathbb{R}} = \{Y\, v(s)\}_{s\in\mathbb{R}}$ with $\{v(s)\}_{s\in\mathbb{R}}$ some positive deterministic continuous function with finite supremum (for instance suppose some behaviour described by the curve v subjected to some independent random impact Y). Then we have, for w such that $\inf_{s\in\mathbb{R}} w(s)/v(s) > 1$,*

$$P(W \leq w) = P\left(Y \leq \inf_{s\in\mathbb{R}} \frac{w(s)}{v(s)}\right) = 1 - \sup_{s\in\mathbb{R}} \frac{w(s)}{x(s)}$$

and, for w such that $\sup_{s\in\mathbb{R}} w(s)/v(s) > 1$,

$$P(W > w) = P\left(Y > \sup_{s\in\mathbb{R}} \frac{w(s)}{v(s)}\right) = \inf_{s\in\mathbb{R}} \frac{v(s)}{w(s)}.$$

The univariate marginal distributions are Pareto ($v(s)$), i.e. $F_{W(s)}(y) = 1 - v(s)/y$, $y > v(s)$, $s \in \mathbb{R}$. For example, contour levels $w(s) = c\,v(s)$ for some constant $c > \omega_0$, have exceedance probabilities $P(W > w)$ also governed by a univariate Pareto distribution.

To end this section, we link the finite dimensional distributions of the simple Pareto process with the finite dimensional distributions of simple max-stable processes. For completeness we define max-stable processes and review a representation for these processes.

Definition 2.2. *A process* $\eta = \{\eta(s)\}_{s\in\mathbb{R}} \in C(\mathbb{R})$ *with non-degenerate marginals is called max-stable if, for* $\eta_1, \eta_2, \dots,$ *i.i.d. copies of* η*, there are real continuous functions* $c_n = \{c_s(n)\}_{s\in\mathbb{R}} > 0$ *and* $d_n = \{d_s(n)\}_{s\in\mathbb{R}}$ *such that,*

$$\max_{1\le i\le n} \frac{\eta_i - d_n}{c_n} \stackrel{d}{=} \eta \qquad \text{for all } n = 1, 2, \dots .$$

It is called simple if its marginal distributions are standard Fréchet, and then it will be denoted by $\bar\eta$*.*

Proposition 2.2 (Penrose(1992))**.** *All simple max-stable processes in* $C^+(\bar{\mathbb{R}})$ *can be generated in the following way. Consider a Poisson point process on* $(0,\infty]$ *with mean measure* $r^{-2}\,dr$*. Let* $\{Z_i\}_{i=1}^{\infty}$ *be a realization of this point process. Further consider i.i.d. stochastic processes* $V_1, V_2, \dots$ *in* $C^+(\bar{\mathbb{R}})$ *with* $EV_1(s) = 1$ *for all* $s \in \bar{\mathbb{R}}$ *and* $E\sup_{s\in\bar{\mathbb{R}}} V(s) < \infty$*. Then*

$$\bar\eta =^d \max_{i=1,2,\dots} Z_i\, V_i.$$

Conversely each process with this representation is simple max-stable (and one can take V *such that* $\sup_{s\in\bar{\mathbb{R}}} V(s) = c$ *a.s. with* $c > 0$*).*

The spectral measure of the max-stable process is defined as the probability measure associated to V. The process V for max-stable processes plays the same role as the process V for Pareto processes in Theorem 2.1. The connection between finite dimensional distribution functions is discussed next.

The finite dimensional distributions of η where computed in de Haan (1984). From the given representation we have, for $s_1, \dots, s_n \in \mathbb{R}$, $x_1, \dots, x_n \in \mathbb{R}$, for all $n \in \mathbb{N}$,

$$G(x_1,\dots,x_n) = P(\bar\eta(s_1) \le x_1, \dots, \bar\eta(s_n) \le x_n) = \exp\left(-E \max_{1\le i\le n} \frac{V(s_i)}{x_i}\right). \tag{2.15}$$

Now note that (2.12) particularized to finite-dimensions give, for $s_1, \dots, s_n \in \mathbb{R}$, $x_1, \dots, x_n \in \mathbb{R}$, with $x_i > \omega_0$, $i = 1, \dots, n$,

$$\begin{aligned} P(W(s_1) \le x_1, \dots, W(s_n) \le x_n) &= P(YV(s_1) \le x_1, \dots, YV(s_n) \le x_n) \\ &= P\left(Y \le \min_{1\le i\le n} \frac{x_i}{V(s_i)}\right) = 1 - E \max_{1\le i\le n} \frac{V(s_i)}{x_i}. \end{aligned} \tag{2.16}$$

More generally for $K_1, K_2, \dots, K_d$ compact subsets of $\mathbb{R}$ and $x_1, x_2, \dots, x_d > \omega_0$,

$$P\left(W(s) \le x_i, \text{ for all } s \in K_i, i = 1, 2, \dots, d\right) = 1 - E \max_{1\le i\le d}\left(x_i^{-1} \sup_{s\in K_i} V(s)\right).$$

Comparing simple Pareto and simple max-stable processes, if the two processes have the same spectral measure, (2.16) equals '$1 + \log G$', i.e. the typical distribution-relation between extreme value and Pareto distributions holds in the given region. The definition of a multivariate Pareto distribution in Michel

(2008) (cf. also Falk et al. (2010)) is: any multivariate distribution function that can be represented by $1+\log G$, with G any multivariate extreme value distribution, in a neighborhood of the right endpoint of G. Hence we cover this situation.

From the finite dimensional distributions one may check that independence in the Pareto process among any points is impossible. For any two points $s_1, s_2 \in \mathbb{R}$, $P(W(s_1) > c, W(s_2) > c) = c^{-1}E\left(V(s_1) \wedge V(s_2)\right)$, $P(W(s_i) > c) = EV(s_i)/c > 0$, $i = 1, 2$, and $E\left(V(s_1) \wedge V(s_2)\right) = EV(s_1)EV(s_2)/c > 0$, for all $c > \omega_0$ is impossible.

All the above results extend in a natural way to $\bar{\mathbb{R}} := [-\infty, +\infty]$.

3 The generalized Pareto process

Let $C(S)$ be the space of real continuous functions on S with $S \subset \mathbb{R}^d$ compact. The more general processes with continuous extreme value index function $\gamma = \{\gamma(s)\}_{s\in S}$, location and scale functions $\mu = \{\mu(s)\}_{s\in S}$ and $\sigma = \{\sigma(s)\}_{s\in S}$ is defined as:

Definition 3.1. *Let W be a simple Pareto process, $\mu, \sigma, \gamma \in C(S)$ with $\sigma > 0$. The generalized Pareto process $W_{\mu,\sigma,\gamma} \in C(S)$ is given by,*

$$W_{\mu,\sigma,\gamma} = \mu + \sigma\,\frac{W^\gamma - 1}{\gamma} \tag{3.1}$$

with all operations taken componentwise (recall the convention explained in the end of Section 1).

The correspondent to Corollary 2.1 is,

Corollary 3.1. *The random variable* $\sup_{s\in S}\left\{\left(1+\gamma(s)\frac{W_{\mu,\sigma,\gamma}(s)-\mu(s)}{\sigma(s)}\right)^{1/\gamma(s)}\right\}\omega_0^{-1}$ *has standard Pareto distribution.*

Related with stability or homogeneity properties we have:

Proposition 3.1. *For any Pareto process $W_{\mu,\sigma,\gamma}$,*

$$P\left(\left(1+\gamma\frac{W_{\mu,\sigma,\gamma}-\mu}{\sigma}\right)^{1/\gamma} \in rA\right) = r^{-1}P\left(\left(1+\gamma\frac{W_{\mu,\sigma,\gamma}-\mu}{\sigma}\right)^{1/\gamma} \in A\right), \tag{3.2}$$

for all $r > 1$ and $A \in \mathcal{B}\left(C^+_{\omega_0}(S)\right)$. Moreover, there exist normalizing functions $u(r)$ and $s(r)$ such that

$$P\left(\left(1+\gamma\frac{W_{\mu,\sigma,\gamma}-u(r)}{s(r)}\right)^{1/\gamma} \in A \Big| \sup_{s\in S}\left(1+\gamma\frac{W_{\mu,\sigma,\gamma}-u(r)}{s(r)}\right)^{1/\gamma} > \omega_0\right)$$
$$= P\left(\left(1+\gamma\frac{W_{\mu,\sigma,\gamma}-\mu}{\sigma}\right)^{1/\gamma} \in A\right), \tag{3.3}$$

for all $r>1$ *and* $A\in\mathcal{B}\left(C^{+}_{\omega_0}(S)\right)$.

Conversely, (3.3) *and* $\sup_{s\in S}\left\{\left(1+\gamma(s)\frac{W_{\mu,\sigma,\gamma}(s)-\mu(s)}{\sigma(s)}\right)^{1/\gamma(s)}\right\}\omega_0^{-1}$ *being standard Pareto distributed, imply* (3.2).

Proof. Relation (3.2) is direct from Definition 3.1 and (2.3). Then, with $u(r)=\mu+\sigma(r^{\gamma}-1)/\gamma$ and $s(r)=\sigma r^{\gamma}$ and, for all $r>1$ and $A\in\mathcal{B}\left(C^{+}_{\omega_0}(S)\right)$,

$$\begin{aligned}
&P\left(\left(1+\gamma\frac{W_{\mu,\sigma,\gamma}-u(r)}{s(r)}\right)^{1/\gamma}\in A\Big|\left(1+\gamma\frac{W_{\mu,\sigma,\gamma}-u(r)}{s(r)}\right)^{1/\gamma}\not\leq\omega_0\right)\\
&\quad= P\left(\left(1+\gamma\frac{W_{\mu,\sigma,\gamma}-\mu}{\sigma}\right)^{1/\gamma}\in rA\Big|\left(1+\gamma\frac{W_{\mu,\sigma,\gamma}-\mu}{\sigma}\right)^{1/\gamma}\not\leq r\omega_0\right)\\
&\quad= \frac{P\left(\left(1+\gamma\frac{W_{\mu,\sigma,\gamma}-\mu}{\sigma}\right)^{1/\gamma}\in rA\right)}{P\left(\left(1+\gamma\frac{W_{\mu,\sigma,\gamma}-\mu}{\sigma}\right)^{1/\gamma}\not\leq r\omega_0\right)}=P\left(\left(1+\gamma\frac{W_{\mu,\sigma,\gamma}-\mu}{\sigma}\right)^{1/\gamma}\in A\right)
\end{aligned}$$

by (3.2) and Corollary 3.1.

Conversely, for all $r>1$ and $A\in\mathcal{B}\left(C^{+}_{\omega_0}(S)\right)$,

$$\begin{aligned}
&\frac{P\left(\left(1+\gamma\frac{W_{\mu,\sigma,\gamma}-u(r)}{s(r)}\right)^{1/\gamma}\in A\right)}{P\left(\sup_{s\in\mathbb{R}}\left(1+\gamma\frac{W_{\mu,\sigma,\gamma}-u(r)}{s(r)}\right)^{1/\gamma}\omega_0^{-1}>1\right)}\\
&\quad=\frac{P\left(\left(1+\gamma\frac{W_{\mu,\sigma,\gamma}-\mu}{\sigma}\right)^{1/\gamma}\in rA\right)}{r^{-1}}=P\left(\left(1+\gamma\frac{W_{\mu,\sigma,\gamma}-\mu}{\sigma}\right)^{1/\gamma}\in A\right)
\end{aligned}$$

by (3.3) and $\sup_{s\in S}\left\{\left(1+\gamma(s)\frac{W_{\mu,\sigma,\gamma}(s)-\mu(s)}{\sigma(s)}\right)^{1/\gamma(s)}\right\}\omega_0^{-1}$ being standard Pareto distributed. □

Example 3.1. *Relation* (3.3) *may be viewed as the correspondent to the stability property in Rootzén and Tajvidi (2006), cf. their relation (6). E.g. in the 'simple' case one checks that*

$$\begin{aligned}
&P\left(\frac{\left(\frac{W(s_i)}{\omega_0}-1\right)-(t-1)}{t}\leq\frac{x_i}{\omega_0},i=1,\dots n\Big|\frac{\left(\frac{W(s_i)}{\omega_0}-1\right)-(t-1)}{t}\not\leq 0,i=1,\dots n\right)\\
&\qquad\qquad=P\left(\frac{W(s_i)}{\omega_0}-1\leq\frac{x_i}{\omega_0},i=1,\dots n\right),
\end{aligned}$$

for all x_i, $i=1,\dots,n$.

The corresponding to Proposition 2.1 on distribution functions is now:

$$P(W_{\mu,\sigma,\gamma} \le w) = E\left\{\sup_{s\in\mathbb{R}} V(s)\left(\left(1+\gamma(s)\frac{w(s)-\mu(s)}{\sigma(s)}\right)^{1/\gamma(s)} \wedge \omega_0\right)^{-1}\right\}$$
$$- E\left\{\sup_{s\in\mathbb{R}} V(s)\left(1+\gamma(s)\frac{w(s)-\mu(s)}{\sigma(s)}\right)^{-1/\gamma(s)}\right\},$$

for $1+\gamma(w-\mu)/\sigma \in C^+(S)$ and the difference taken as zero if $\inf_{s\in S}\{1+\gamma(s)(w(s)-\mu(s))/\sigma(s)\}=0$.

4 Domain of attraction

The maximum domain of attraction of extreme value distributions in infinite-dimensional space has been characterized in de Haan and Lin (2001). This result leads directly to a characterization of the domain of attraction of a generalized Pareto process.

Let $C(S)$ be the space of real continuous functions in S, with $S \subset \mathbb{R}^d$ some compact subset, equipped with the supremum norm. The convergences below $\to^d$ denote weak or convergence in distribution. Denote by $\bar\eta = \{\bar\eta(s)\}_{s\in S}$ any simple max-stable process in $C^+(S)$ (cf. Definition 2.2). Any max-stable process $\eta = \{\eta(s)\}_{s\in S}$ in $C(S)$ can be represented by $\eta = (\bar\eta^\gamma - 1)/\gamma$, for some $\bar\eta$ and continuous function $\gamma = \{\gamma(s)\}_{s\in S}$. For simplicity we always take here

$$C_1^+(S) = \{f \in C^+(S) : \sup_{s\in S} f(s) \ge 1\},$$

i.e. consider the constant ω_0 introduced in Section 2 equal to 1.

The maximum domain of attraction condition in $C(S)$ can be stated as:

Condition 4.1. *For $X, X_1, X_2, \ldots$ i.i.d. random elements of $C(S)$, there exists a max-stable stochastic process $\eta \in C(S)$ with continuous index function γ, and $a_s(n) > 0$ and $b_s(n)$ in $C(S)$ such that*

$$\left\{\max_{1\le i\le n}\frac{X_i(s)-b_s(n)}{a_s(n)}\right\}_{s\in S} \to^d \{\eta(s)\}_{s\in S} \tag{4.1}$$

on $C(S)$. The normalizing functions are w.l.g. chosen in such a way that $-\log P(\eta(s)\le x) = (1+\gamma(s)x)^{-1/\gamma(s)}$ for all x with $1+\gamma(s)x > 0$, $s \in S$.

Next is an equivalent characterization of the domain of attraction condition, in terms of 'exceedances'. Suppose the marginal distribution functions $F_s(x) = P(X(s) \le x)$ are continuous in x, for all S. To simplify notation write the normalized process,

$$\eta_t = \left(1+\gamma\frac{X-b(t)}{a(t)}\right)_+^{1/\gamma},$$

for some normalizing functions in $C(S)$, $a(t) = \{a_s(t)\}_{s\in S} > 0$ and $b(t) = \{b_s(t)\}_{s\in S}$.

Condition 4.2. *For $X \in C(S)$ suppose, for some $a(t) > 0$ and $b(t)$ in $C(S)$,*

$$\lim_{t\to\infty} tP\left(\frac{X(s) - b_s(t)}{a_s(t)} > x\right) = (1 + \gamma(s)x)^{-1/\gamma(s)}, \quad 1 + \gamma(s)x > 0, \tag{4.2}$$

uniformly in s,

$$\lim_{t\to\infty} \frac{P\left(\sup_{s\in S} \eta_t(s) > x\right)}{P\left(\sup_{s\in S} \eta_t(s) > 1\right)} = \frac{1}{x}, \quad \text{for all } x > 1, \tag{4.3}$$

and

$$\lim_{t\to\infty} P\left(\frac{\eta_t}{\sup_{s\in S} \eta_t(s)} \in B \Big| \sup_{s\in S} \eta_t(s) > 1\right) = Q(B), \tag{4.4}$$

for each $B \in \mathcal{B}\left(\bar{C}_1^+(S)\right)$ with $Q(\partial B) = 0$, with Q some probability measure on $\bar{C}_1^+(S)$.

Note that this is the same as for max-stable processes; cf. Theorem 9.5.1 in de Haan and Ferreira (2006). Note also that (4.3)–(4.4) specify a simple Pareto process in the limit.

Theorem 4.1. *Conditions 4.1 and 4.2 are equivalent and the corresponding limiting processes share the same spectral measure.*

Proof. Cf. Theorem 9.5.1 in de Haan and Ferreira (2006). The normalization there is

$$\{\tilde{\eta}_t(s)\}_{s\in S} = \left\{\frac{1}{t(1 - F_s(X(s)))}\right\}_{s\in S},$$

but the results are the same. □

The following are direct consequences:

Corollary 4.1. *Any max-stable process verifies Condition 4.2.*

Corollary 4.2. *Relations* (4.3)–(4.4) *imply*

$$\lim_{t\to\infty} P\left(\eta_t \in A \Big| \sup_{s\in S} \eta_t(s) > 1\right) = P(W \in A),$$

with $A \in \mathcal{B}\left(C_1^+(S)\right)$, $P(\partial A) = 0$ and W some simple Pareto process.

The converse statement of Corollary 4.2 is as follows:

Theorem 4.2. *Suppose that there exists a continuous function $\tilde{b}(u) = \{\tilde{b}_s(u)\}_{s\in S}$ that is increasing and with the property that $P(X(s) > \tilde{b}_s(u)$ for some $s \in S) \to 0$ as $u \to \infty$, and a positive continuous function $\tilde{a}(u) = \{\tilde{a}_s(u)\}_{s\in S} > 0$ such that*

$$\lim_{u\to\infty} P\left(\frac{X - \tilde{b}(u)}{\tilde{a}(u)} \in A \Big| X(s) - \tilde{b}_s(u) > 0 \text{ for some } s \in S\right) = \tilde{P}(A),$$

for all $A \in \mathcal{B}(C(S))$ and $\tilde{P}(\partial A) = 0$. Then Conditions 4.1 and 4.2 are fulfilled.

Proof. By the conditions on $\tilde{b}(u)$, we can determine $q = q(t)$ such that $P(X(s) > \tilde{b}_s(q(t))$ for some $s \in S) = 1/t$. Then with $b_s(t) = \tilde{b}_s(q(t))$ and $a_s(t) = \tilde{a}_s(q(t))$,

$$\lim_{t\to\infty} tP\left(\frac{X-b(t)}{a(t)} \in C \text{ and } X(s) > b_s(t) \text{ for some } s \in S\right) = \tilde{P}(C),$$

for all $C \in \mathcal{B}(C(S))$ and $\tilde{P}(\partial C) = 0$. In particular, if $\inf\{\sup_{s\in S} f(s) : f \in C\} > 0$ we have

$$\lim_{t\to\infty} tP\left(\frac{X-b(t)}{a(t)} \in C\right) = \tilde{P}(C). \tag{4.5}$$

We proceed as usual in extreme value theory. Fix for the moment $s \in S$. It follows that for $x > 0$

$$\lim_{t\to\infty} tP\left(X_s(t) > b_s(t) + xa_s(t)\right) = \tilde{P}\{f : f(s) > x\}.$$

Let U_s be the inverse function of $1/P(X(s) > x)$ and V_s be the inverse function of $1/\tilde{P}\{f : f(s) > x\}$. Then

$$\lim_{t\to\infty} \frac{U_s(tx) - b_s(t)}{a_s(t)} = V_s(x), \qquad \text{for } x > 0.$$

It follows that for some real $\gamma(s)$ and all $x > 0$

$$\lim_{t\to\infty} \frac{b_s(tx) - b_s(t)}{a_s(t)} = \frac{x^{\gamma(s)} - 1}{\gamma(s)} \quad \text{and} \quad \lim_{t\to\infty} \frac{a_s(tx)}{a_s(t)} = x^{\gamma(s)}. \tag{4.6}$$

Since the limit process has continuous paths, the function γ must be continuous on S.

Now replace t in (4.5) by ct where $c > 0$. Then

$$\lim_{t\to\infty} tP\left(\frac{b_s(t) - b_s(tc)}{a_s(tc)} + \frac{a_s(t)}{a_s(tc)}\,\frac{X-b(t)}{a(t)} \in C\right) = \frac{1}{c}\tilde{P}(C)$$

hence, by (4.6)

$$\lim_{t\to\infty} tP\left(\left(1+\gamma\frac{X-b(t)}{a(t)}\right)^{1/\gamma} \in c\,(1+\gamma C)^{1/\gamma}\right) = \frac{1}{c}\tilde{P}(C)$$

and by (4.5)

$$\lim_{t\to\infty} tP\left(\left(1+\gamma\frac{X-b(t)}{a(t)}\right)^{1/\gamma} \in (1+\gamma C)^{1/\gamma}\right) = \tilde{P}(C).$$

Write $P(A) = \tilde{P}\left(\left(A^{\gamma} - 1\right)/\gamma\right)$. Then

$$\lim_{t\to\infty} tP\left(\eta_t \in A\right) = P(A),$$

with $P(cA) = c^{-1}P(A)$, for all $c > 0$ and $A \in \mathcal{B}(C(S))$ such that $\inf\{\sup_{s\in S} f(s) : f \in A\} > 1$ and $P(\partial A) = 0$. The rest is exactly like the proof of the equivalence between (2b) and (2c) of Theorem 9.5.1 in de Haan and Ferreira (2006). $\square$

By Corollary 3.1 and Proposition 3.1 it follows:

Corollary 4.3. *Any Pareto process is in the domain of attraction of a max-stable process with the same spectral measure.*

Example 4.1. *The finite dimensional distributions of the moving maximum processes obtained in de Haan and Pereira (2006) can be applied to obtain the finite dimensional distributions of the correspondent Pareto process, in the appropriate region.*

Example 4.2 (Regular variation (de Haan and Lin 2001, Hult and Lindskog 2005))**.** *A stochastic process X in $C(S)$ is regularly varying if and only if there exists an $\alpha > 0$ and a probability measure Q such that,*

$$\frac{P\left(\sup_{s\in S} X(s) > tx,\ X/\sup_{s\in S} X(s) \in \cdot\right)}{P\left(\sup_{s\in S} X(s) > t\right)} \to^d x^{-\alpha}\, Q(\cdot), \quad x > 0,\ t \to \infty, \tag{4.7}$$

on $\{f \in C(S) : \sup_{s\in S} f(s) = 1\}$. Hence, a regularly varying process such that (4.2) holds for the marginals, verifies Condition 4.2 with $\gamma = 1/\alpha$, $b(t) = t$ and $a(t) = t/\alpha$; note that the index function is constant in this case.

On the other hand, the normalized process $t\eta_t$ (or $t\tilde{\eta}_t$) with η_t verifying (4.3)–(4.4), verifies regular variation with $\alpha = 1$ and spectral measure Q on $C_1^+(S)$.

Remark 4.1. *Our analysis is also valid - mutatis mutandis - in the finite-dimensional set-up. The definition in that case is the same as in Rootzén and Tajvidi (2006). The difference is that their analysis is entirely based on distribution functions whereas ours is more structural. Here are some remarks.*

Let $\bar{F} = 1-F$ with F some $d-$variate distribution function, $\mathbf{x} = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d$, and $\mathbf{u}(\cdot) = (u_1(\cdot), u_2(\cdot), \ldots, u_d(\cdot))$ and σ the normalizing functions considered in Rootzén and Tajvidi (2006) (see e.g. their definition of $\mathbf{X_u}$). By using $\sigma(xt)/\sigma(t) \to (x^{\gamma_1}, x^{\gamma_2}, \ldots, x^{\gamma_d})$ and $(\mathbf{u}(xt) - \mathbf{u}(t))/\sigma(t) \to \left(\frac{x^{\gamma_1}-1}{\gamma}_1, \frac{x^{\gamma_2}-1}{\gamma}_2, \ldots, \frac{x^{\gamma_d}-1}{\gamma}_d\right)$, $t \to \infty$, for some reals $\gamma_1, \gamma_2, \ldots, \gamma_d$ (cf. proof of Theorem 2.1(ii)) and by

$$\bar{F}^*(\mathbf{x}) := \bar{F}\left(u_1(x_1), u_2(x_2), \ldots, u_d(x_d)\right),$$

one simplifies their relation (19) to

$$t\bar{F}^*(t\mathbf{x}) \to -\log G\left(\frac{x_1^{\gamma_1}-1}{\gamma_1}, \frac{x_2^{\gamma_2}-1}{\gamma_2}, \ldots, \frac{x_d^{\gamma_d}-1}{\gamma_d}\right),$$

and one simplifies their relation (6) to

$$P\left(\mathbf{X}^* \leq t\mathbf{x} | \mathbf{X}^* \not\leq t\mathbf{1}\right) = P\left(\mathbf{X}^* \leq \mathbf{x}\right)$$

for $t \geq 1$. Hence one can take $\mathbf{u}(t) := \left(\frac{t^{\gamma_1}-1}{\gamma_1}, \frac{t^{\gamma_2}-1}{\gamma_2}, \ldots, \frac{t^{\gamma_d}-1}{\gamma_d}\right)$ in Theorem 2.2 of that paper.

In Theorem 2.2 (ii) it is not sufficient to require (6) for $\mathbf{x} > \mathbf{0}$. *For example the probability distribution in* $\mathbb{R}^2$ *given by*

$$P(X > x \text{ or } Y > y) = \left(\frac{1}{2}e^{-2(x\vee 0)} + \frac{1}{2}e^{-2(y\vee 0)}\right)^{1/2}, \qquad x \vee y \geq 0,$$

satisfies (6) for $\mathbf{x} > \mathbf{0}$ *but not for all* $\mathbf{x}$ *and it is not a generalized Pareto distribution.*

5 Application

'Deltares' is an advisory organization of the Dutch government concerning (among others) the safety of the coastal defenses against severe wind storms. One studies the impact of severe storms on the coast, storms that are so severe that they have never been observed. In order to see how these storms look like it is planned to produce wind fields on and around the North Sea using certain climate models. These climate models produce independent and identically distributed (i.i.d.) wind fields similar to the ones that could be observed (but that are only partially observed). Since the model runs during a limited time, some of the wind fields will be connected with storms of a certain severity but we do not expect to see really disastrous storms that could endanger the coastal defenses. The question put forward by Deltares is: can we get an idea how the really disastrous wind fields look like on the basis of the 'observed' wind fields? We want to show that this can be done using the generalized Pareto process.

Consider i.i.d. continuous stochastic processes $\{X_i(s)\}_{s\in S}$ where S is a compact subset of $\mathbb{R}^d$. Suppose that the probability distribution of the process is in the domain of attraction of some max-stable process i.e., there exist functions $a_s(n) > 0$ and $b_s(n)$ $(s \in S)$ such that the sequence of processes

$$\left\{\max_{1\leq i\leq n} \frac{X_i(s) - b_s(n)}{a_s(n)}\right\}_{s\in S}$$

converges to a continuous process, say Y, in distribution in $C(S)$. Then Y is a max-stable process.

According to Theorem 4.1 the process X_1 is then in the domain of attraction of the corresponding generalized Pareto process i.e. from relations (4.3)–(4.4) with $R_X(t) := \sup_{s\in S}\left(1 + \gamma(s)\frac{X(s)-b_s(t)}{a_s(t)}\right)^{1/\gamma(s)}$,

$$\lim_{t\to\infty} P\left(\left(1 + \gamma\frac{X - b(t)}{a(t)}\right)^{1/\gamma} \in R_X(t)B \text{ and } R_X(t) > x \middle| R_X(t) > 1\right) = x^{-1}Q(B) \quad (5.1)$$

for $x > 1$ and $B \in \mathcal{B}\left(\bar{C}_1^+(S)\right)$ with $Q(\partial B) = 0$.

This statement leads to a peaks-over-threshold method in this framework:

Let $X_1, X_2, \ldots, X_n$ be i.i.d. and let the underlying distribution satisfy the conditions above. Select from the normalized processes

$$\left\{ \left(1 + \gamma \frac{X - b(t)}{a(t)} \right)^{1/\gamma} \right\}_{i=1}^{n}$$

those that satisfy $X_i(s) > b_s(t)$ for some $s \in S$ i.e. for which $R_{X_i}(t) > 1$. Here t is a large value and of course $b_s(t)$ can be taken as $\inf\{y : P(X_1(s) > y) \leq 1/t\}$. Let us denote the selected (normalized) processes as $\{X_j^{(1)}\}_{j=1}^{r}$ They are still i.i.d. and follow approximately the distribution given by (5.1):

$$P\left(X_j^{(1)} \in R_{X_j^{(1)}}(t)B \text{ and } R_{X_j^{(1)}}(t) > x \right) = x^{-1}Q(B).$$

This means that the processes $\{X_j^{(1)}\}_{j=1}^{r}$ are approximately generalized Pareto processes.

Next, in order to use Proposition 3.1, we multiply these processes by a (large) factor t_0. Define for $j = 1, 2, \ldots, r$,

$$X_j^{(2)}(s) := t_0 X_j^{(1)}(s) \qquad \text{for } s \in S.$$

This brings the processes to a higher level without changing the distribution essentially (by the homogeneity property). Finally we undo the normalization and define for $j = 1, 2, \ldots, r$

$$X_j^{(3)}(s) := a_s(t) \frac{\left(X_j^{(2)}(s) \right)^{\gamma(s)} - 1}{\gamma(s)} + b_s(t)$$

for $s \in S$.

We claim that the processes $\{X_j^{(3)}\}_{j=1}^{r}$ are peaks-over-threshold processes with respect to a much higher threshold (namely $b_s(tt_0)$) than the processes $\{X_j^{(1)}\}_{j=1}^{r}$ (with threshold $b_s(t)$). In order to prove this we need the following properties (cf. relation (9.5.2) page 312 of the Haan and Ferreira (2006)),

$$\lim_{t \to \infty} \frac{b_s(tx) - b_s(t)}{a_s(t)} = \frac{x^{\gamma(s)} - 1}{\gamma(s)} \tag{5.2}$$

and

$$\lim_{t \to \infty} \frac{a_s(tx)}{a_s(t)} = x^{\gamma(s)} \tag{5.3}$$

uniformly for $s \in S$ and locally uniformly in x. The derivation is straightforward:

$$
\begin{aligned}
&P\left(\frac{X^{(3)}-b(tt_0)}{a(tt_0)}\in A\right)\\
&= P\left\{\frac{a(t)}{a(tt_0)}\frac{\left[t_0\left(1+\gamma\frac{X^{(1)}-b(t)}{a(t)}\right)^{1/\gamma}\right]^{\gamma}-1}{\gamma}+\frac{b(t)-b(tt_0)}{a(tt_0)}\in A\right\}\\
&\approx P\left\{t_0^{-\gamma}\frac{\left[t_0^{\gamma}\left(1+\gamma\frac{X^{(1)}-b(t)}{a(t)}\right)\right]-1}{\gamma}+\frac{t_0^{-\gamma}-1}{\gamma}\in A\right\}\\
&= P\left\{\frac{\left(1+\gamma\frac{X^{(1)}-b(t)}{a(t)}\right)-t_0^{-\gamma}}{\gamma}+\frac{t_0^{-\gamma}-1}{\gamma}\in A\right\}\\
&= P\left(\frac{X^{(1)}-b(t)}{a(t)}\in A\right).
\end{aligned}
$$

Finally we comment that this is the third application of the peaks-over-threshold method. The other two are: estimation of the exceedance probability of a high level and high quantile estimation (Coles, 2001).

5.1 Simulations

We exemplify the lifting procedure with the process $X(s) = Z(s)^{\gamma(s)}$, with $\gamma(s) = 1 - s(1-s)^2$, $s \in [0,1]$, and Z is the moving maximum process with standard gaussian density. The Z process can be easily simulated in R-package due to Ribatet (2012). In Figures 1–2 are represented the 11 out of 20 of these processes for which $R_{X_i}(t) > 1$ i.e. $\{X_j^{(1)}\}_{j=1}^{11}$ and the corresponding lifted processes $\{X_j^{(3)}\}_{j=1}^{11}$ with $t_0 = 10$.

Acknowledgements. Research partially supported by FCT Project PTDC /MAT /112770 /2009.

References

[1] Balkema, A.A. and de Haan, L. (1974) Residual life time at great age. *Ann. Probab.* **2**, 792–804

[2] Coles, S. (2001) An Introduction to Statistical Modeling of Extreme Values, Springer.

[3] Falk, M., Hüsler, J. and Reiss, R.-D. (2010) Laws of Small Numbers: Extremes and Rare Events, Birkhäuser, Springer.

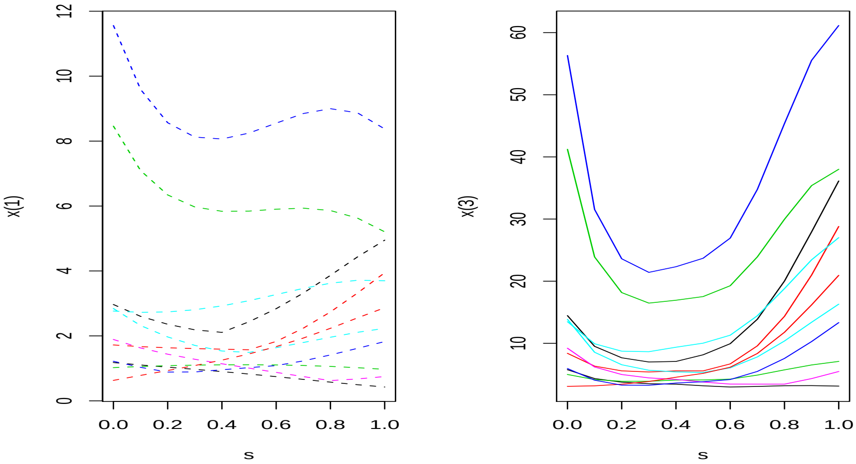

Figure 1: (a) $\{X_j^{(1)}\}_{j=1}^{11}$ obtained from moving maximum process with standard gaussian density; (b) the corresponding lifted processes $\{X_j^{(3)}\}_{j=1}^{11}$ with $t_0 = 10$

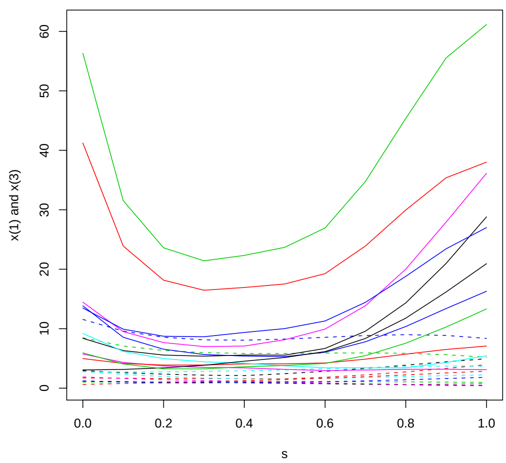

Figure 2: All processes: $\{X_j^{(1)}\}_{j=1}^{11}$ - slashed lines - and $\{X_j^{(3)}\}_{j=1}^{11}$ - continuous lines

[4] de Haan, L. (1984) A spectral representation for max-stable processes. *Ann. Prob.* **12**, 1194–1204

[5] de Haan, L. e Pickands, J. (1986) Stationary min-stable stochastic processes. *P*robab. Th. Rel. Fields **72**, 477–492

[6] de Haan, L. and Ferreira, A. (2006) Extreme Value Theory: An Introduction. Springer, Boston.

[7] Hult, H. and Lindskog, F. (2005) Extremal behaviour of regularly varying stochastic processes. *Stoch. Proc. Appl.* **1**, 249–274

[8] Pickands, J. III (1975) Statistical inference using extreme order statistics. *Ann. Statist.* **3**, 119-131.

[9] de Haan, L. e Pereira, T.T. (2006) Spatial Extremes: the stationary case. *A*nn. Statist. 34, 146–168

[10] de Haan, L. and Lin, T. (2001) On convergence toward an extreme value distribution in $C[0,1]$. *Ann. Prob.* **29**, 467–483

[11] Michel, R. (2008) Some notes on multivariate generalized Pareto distributions. *J. Multiv. Analysis* **99**, 1288–1301

[12] Penrose, M.D. (1992) Semi-min-stable processes. *Ann. Probab.* **20**, 1450–1463

[13] Ribatet, M. (2012) Package 'SpatialExtremes' (http://spatialextremes.r-forge.r-project.org/)

[14] H. Rootzén and N. Tajvidi (2006) Multivariate generalized Pareto distributions. *Bernoulli* **12**, 917–930