A note on the excesses over a high threshold*

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Abstract. In this note we are interested in the derivation of the asymptotic distributional properties of the maximum likelihood estimator of a positive tail index γ, on the basis of the excesses over a high random threshold.

AMS 2000 subject classification. Primary 62G32; Secondary 65C05.
Keywords and phrases. *Statistics of Extremes*, *Semi-parametric estimation*, *Generalized Jackknife statistics.*

1 Some comments on the excesses and the Generalized Pareto model

Heavy tail models appear often in practice. A model F is a heavy tail model whenever the *tail function*, $1-F$, is a regularly varying function with a negative index of regular variation $\alpha = -1/\gamma$, i.e., for every $x > 0$,

$$\lim_{t\to\infty} \frac{1-F(tx)}{1-F(t)} = x^{-1/\gamma}.$$

The parameter $\gamma(>0)$ is the *tail index*, one of the most relevant parameters of rare events.

In a context of heavy tails, and with the notation $U(t) = F^{\leftarrow}(1-1/t),\ t \geq 1$, $F^{\leftarrow}(y) = \inf\{x : F(x) \geq y\}$ the generalized inverse function of the underlying model F, the first order parameter (or tail index) γ (>0) appears, for every $x > 0$, as the limiting value, as $t \to \infty$, of

$$\frac{\ln U(tx) - \ln U(t)}{\ln x},$$

i.e., with the usual notation RV_α for the class of regularly varying functions with index of regularly variation α,

$$1 - F \in RV_{-1/\gamma} \quad \text{if and only if} \quad U \in RV_\gamma. \tag{1.1}$$

*Research partially supported by FCT / POCTI / FEDER.

The second order parameter ρ (≤ 0) is the non-positive value which appears in the limiting relation

$$\lim_{t\to\infty} \frac{\frac{U(tx)}{U(t)} - x^\gamma}{A(t)} = x^\gamma \ \frac{x^\rho - 1}{\rho}, \tag{1.2}$$

which we assume to hold for every $x > 0$, and where $|A(t)|$ is of regular variation with index ρ (Geluk and de Haan, 1987).

For intermediate k, i.e., a seguence of integers $k = k_n$ such that

$$k = k_n \to \infty, \quad k_n = o(n), \text{ as } n \to \infty, \tag{1.3}$$

let us think on the excesses over the random high level $X_{n-k:n}$,

$$V_i = X_{n-i+1:n} - X_{n-k:n}, \quad 1 \leq i \leq k. \tag{1.4}$$

From the definition of the function U and from the fact the $F(X)$ is a uniform random variable (r.v.), we get the representation $X_{i:n} = U(Y_{i:n})$ where Y is a unit Pareto r.v., i.e., $F_Y(y) = 1 - y^{-1}$, $y \geq 1$, and since for $j > i$, $\frac{Y_{j:n}}{Y_{i:n}} \stackrel{d}{=} Y_{j-i:n-i}$, $\ln Y_{i:n} \stackrel{d}{=} E_{i:n}$ where E denotes a standard exponential r.v. and $Y_{n-k:n} \sim n/k$, we may indeed write, whenever we are under the second order framework in (1.2),

$$\begin{aligned} V_i &= U(Y_{n-k:n}) \left\{ \frac{U(Y_{n-i+1:n})}{U(Y_{n-k:n})} - 1 \right\} \\ &= U(n/k) \left\{ Y^\gamma_{k-i+1:k} - 1 + A(n/k) Y^\gamma_{k-i+1:k} \frac{Y^\rho_{k-i+1:k} - 1}{\rho} (1 + o_p(1)) \right\} (1+o_p(1)) \end{aligned}$$

Then, with $\delta = \gamma U(n/k)$, we have

$$V_i = X_{n-i+1:n} - X_{n-k:n} \sim \delta \ \frac{Y^\gamma_{k-i+1:k} - 1}{\gamma}, \quad 1 \leq i \leq k, \tag{1.5}$$

i.e., the excesses V_i may be assumed to be the k order statistics in a sample of size k from a Generalized Pareto model, with distribtion function

$$GP(x; \gamma, \delta) = 1 - \left(1 + \gamma \ \frac{x}{\delta}\right)^{-1/\gamma}, \quad x > 0 \quad (\gamma > 0), \tag{1.6}$$

and γ and δ may be estimated through *Maximum Likelihood* (ML). Since for ML-estimation in a Generalized Pareto model, it is easier the re-parametrization in $(\gamma, \alpha) = (\gamma, \gamma/\delta)$ (Davison, 1984) we shall assume the excesses V_i to come from the model

$$GP(x; \gamma, \alpha) = 1 - (1 + \alpha \ x)^{-1/\gamma}, \quad x > 0 \quad (\gamma > 0), \tag{1.7}$$

i.e., there exists γ and α such that, for $1 \leq i \leq k$,

$$\alpha \ V_i = \left\{ Y^\gamma_{k-i+1:k} - 1 + A(n/k) Y^\gamma_{k-i+1:k} \frac{Y^\rho_{k-i+1:k} - 1}{\rho} (1 + o_p(1)) \right\} (1 + o_p(1)) \tag{1.8}$$

2 The ML estimation in a Generalized Pareto model

Let us assume we have access to a sample $\underline{V} = (V_i, 1 \le i \le k)$ from the Generalized Pareto model in (5.2).

The joint maximization, in γ and α, of the log-likelihood of the excesses,

$$\ln L(\gamma, \alpha; V_i, 1 \le i \le k) = k \ln \alpha - k \ln \gamma - \left(\frac{1}{\gamma} + 1\right) \sum_{i=1}^{k} \ln(1 + \alpha\ V_i),$$

leads us to an explicit expression for $\widehat{\gamma}$ as a function of $\widehat{\alpha}$ given by

$$\widehat{\gamma}(k) := \frac{1}{k} \sum_{i=1}^{k}\ \ln(1 + \widehat{\alpha}\ V_i) =: \widehat{A}. \tag{2.1}$$

Then, introducing the notation:

$$A = \tfrac{1}{k} \textstyle\sum_{i=1}^{k}\ \ln(1 + \alpha\ V_i), \quad \widehat{A} = \tfrac{1}{k} \textstyle\sum_{i=1}^{k}\ \ln(1 + \widehat{\alpha}\ V_i),$$

$$B = \tfrac{1}{k} \textstyle\sum_{i=1}^{k}\ \tfrac{\alpha\ V_i}{1+\alpha V_i}, \qquad \widehat{B} = \tfrac{1}{k} \textstyle\sum_{i=1}^{k}\ \tfrac{\widehat{\alpha}\ V_i}{1+\widehat{\alpha} V_i},$$

$$C = \tfrac{1}{k} \textstyle\sum_{i=1}^{k}\ \tfrac{\alpha\ V_i}{(1+\alpha\ V_i)^2}, \qquad \widehat{C} = \tfrac{1}{k} \textstyle\sum_{i=1}^{k}\ \tfrac{\widehat{\alpha}\ V_i}{(1+\widehat{\alpha}\ V_i)^2},$$

and noticing that

$$\frac{\partial A}{\partial \alpha} = \frac{B}{\alpha}, \qquad \frac{\partial B}{\partial \alpha} = \frac{C}{\alpha}$$

the ML estimator of α is solution of the equation

$$\widehat{B} + \frac{\widehat{B}}{\widehat{A}} - 1 \equiv 0 \tag{2.2}$$

We may thus write for the ML equations:

$$\widehat{\gamma}^{GP}(k) = \widehat{A} = A + \frac{\partial A}{\partial \alpha}\, (\widehat{\alpha} - \alpha)\, (1 + o_p(1)) = A + B\ \frac{\widehat{\alpha} - \alpha}{\alpha} (1 + o_p(1)), \tag{2.3}$$

and, since $\partial\, (B + B/A - 1)\, / \partial \alpha = (C + C/A - (B/A)^2)/\alpha$, $\widehat{\alpha} \equiv \widehat{\alpha}^{GP}(k)$ is such that

$$\widehat{B} + \frac{\widehat{B}}{\widehat{A}} - 1 \equiv 0 = B + \frac{B}{A} - 1 + \frac{\widehat{\alpha} - \alpha}{\alpha} \left(C + \frac{C}{A} - \left(\frac{B}{A}\right)^2 \right) (1 + o_p(1)), \tag{2.4}$$

i.e.,

$$\frac{\widehat{\alpha} - \alpha}{\alpha} = \frac{1 - B - B/A}{C + C/A - (B/A)^2}\ (1 + o_p(1)). \tag{2.5}$$

3 The asymptotic behaviour of the random variables under play

With $\{E_i\}_{i\geq 1}$ denoting a sequence of independent, identically distributed (i.i.d.), standard exponential random variables, let us use the notation

$$P_k := \sqrt{k}\left(\frac{1}{k}\sum_{i=1}^{k} E_i - 1\right), \tag{3.1}$$

$$Q_k := \frac{(1+\gamma)\sqrt{(1+2\gamma)k}}{\gamma}\left(\frac{1}{k}\sum_{i=1}^{k} e^{-\gamma E_i} - \frac{1}{1+\gamma}\right), \tag{3.2}$$

which are asymptotically standard normal r.v.'s.

Let us also introduce the notation,

$$c_j = \frac{1}{(1+j\gamma)(1-\rho+j\gamma)}, \quad j = 0, 1, 2, \tag{3.3}$$

We first state the following straighforward result:

Proposition 3.1. *The second order structure between the r.v.'s P_k and Q_k is given by*

$$Cov(P_k, Q_k) = -\frac{\sqrt{1+2\gamma}}{1+\gamma}. \tag{3.4}$$

Proof. The result comes straightforwardly from the fact that $E(e^{-\gamma E}) = \frac{1}{1+\gamma}$ and $E(E\ e^{-\gamma E}) = \frac{1}{(1+\gamma)^2}$. Consequently $Cov(E, e^{-\gamma E}) = \frac{\gamma}{(1+\gamma)^2}$ □

For the r.v.'s under play we have the validity of the following distributional representations,

Theorem 3.1. *If the second order condition (1.2) holds, if $k = k_n$ is a sequence of intermediate positive integers, i.e., (1.3) holds, we have the validity of the following distributional representations:*

$$A \stackrel{d}{=} \gamma + \frac{\gamma}{\sqrt{k}}\ P_k + c_0\ A(n/k)(1 + o_p(1)), \tag{3.5}$$

$$B \stackrel{d}{=} \frac{\gamma}{1+\gamma} - \frac{\gamma}{(1+\gamma)\sqrt{(1+2\gamma)\ k}}\ Q_k + c_1\ A(n/k)(1 + o_p(1)), \tag{3.6}$$

and

$$C \stackrel{d}{=} \frac{\gamma}{(1+\gamma)(1+2\gamma)} + o_p(1). \tag{3.7}$$

Consequently, we have

$$1-B-\frac{B}{A} \stackrel{d}{=} \frac{1}{\sqrt{k}}\left(\frac{Q_k}{\sqrt{1+2\gamma}}+\frac{P_k}{1+\gamma}\right)+\frac{c_0-c_1(1+\gamma)^2}{\gamma(1+\gamma)}\, A(n/k)(1+o_p(1)) \quad (3.8)$$

and

$$C+\frac{C}{A}-\left(\frac{B}{A}\right)^2 \stackrel{d}{=} \frac{\gamma^2}{(1+\gamma)^2(1+2\gamma)}+o_p(1). \quad (3.9)$$

Proof. Let us think first on A. Since

$$\begin{aligned}
\alpha\ V_i &= Y^{\gamma}_{k-i+1:k}-1+A(n/k)\ Y^{\gamma}_{k-i+1:k}\ \frac{Y^{\rho}_{k-i+1:k}-1}{\rho}(1+o_p(1)) \\
1+\alpha\ V_i &= Y^{\gamma}_{k-i+1:k}\left(1+A(n/k)\ \frac{Y^{\rho}_{k-i+1:k}-1}{\rho}(1+o_p(1))\right),
\end{aligned}$$

and we may write

$$\begin{aligned}
A &= \frac{1}{k}\sum_{i=1}^{k}\ln(1+\alpha\ V_i) \\
&= \frac{\gamma}{k}\sum_{i=1}^{k}E_i+\frac{1}{k}\sum_{i=1}^{k}\frac{Y_i^{\rho}-1}{\rho}\ A(n/k)\ (1+o_p(1)).
\end{aligned}$$

Since $E\left[\frac{Y^{\rho}-1}{\rho}\right]=\frac{1}{1-\rho}$, (3.5) follows. Analogously, we have

$$\begin{aligned}
\frac{\alpha\ V_i}{1+\alpha\ V_i} &= \frac{Y^{\gamma}_{k-i+1:k}-1+A(n/k)\ Y^{\gamma}_{k-i+1:k}\frac{Y^{\rho}_{k-i+1:k}-1}{\rho}(1+o_p(1))}{Y^{\gamma}_{k-i+1:k}\left(1+A(n/k)\ \frac{Y^{\rho}_{k-i+1:k}-1}{\rho}(1+o_p(1))\right)} \\
&= \left(1+Y^{-\gamma}_{k-i+1:k}+A(n/k)\frac{Y^{\rho}_{k-i+1:k}-1}{\rho}(1+o_p(1))\right) \\
&\qquad \times\left(1-A(n/k)\ \frac{Y^{\rho}_{k-i+1:k}-1}{\rho}(1+o_p(1))\right) \\
&= 1-Y^{-\gamma}_{k-i+1:k}+A(n/k)\ Y^{-\gamma}_{k-i+1:k}\frac{Y^{\rho}_{k-i+1:k}-1}{\rho}(1+o_p(1)).
\end{aligned}$$

Then

$$\begin{aligned}
B &= \frac{1}{k}\sum_{i=1}^{k}\frac{\alpha\ V_i}{1+\alpha\ V_i} \\
&= 1-\frac{1}{k}\sum_{i=1}^{k}e^{-\gamma\ E_i}+\frac{1}{k}\sum_{i=1}^{k}\ Y_i^{-\gamma}\ \frac{Y_i^{\rho}-1}{\rho}\ A(n/k)(1+o_p(1)).
\end{aligned}$$

Since $E\left[Y^{-\gamma}\frac{Y^{\rho}-1}{\rho}\right]=\frac{1}{(1+\gamma)(1-\rho+\gamma)}$, (3.6) follows as well. Finally, $\frac{\alpha\ V_i}{(1+\alpha\ V_i)^2}$ may be written as

$$\begin{aligned}
&\frac{1-Y_{k-i+1:k}^{-\gamma}+A(n/k)\ Y_{k-i+1:k}^{-\gamma}\frac{Y_{k-i+1:k}^{\rho}-1}{\rho}(1+o_p(1))}{Y_{k-i+1:k}^{\gamma}\left(1+A(n/k)\ \frac{Y_{k-i+1:k}^{\rho}-1}{\rho}(1+o_p(1))\right)}\\
=&\left(Y_{k-i+1:k}^{-\gamma}(1-Y_{k-i+1:k}^{-\gamma})+A(n/k)Y_{k-i+1:k}^{-2\gamma}\frac{Y_{k-i+1:k}^{\rho}-1}{\rho}(1+o_p(1))\right)\\
&\times\left(1-A(n/k)\ \frac{Y_{k-i+1:k}^{\rho}-1}{\rho}(1+o_p(1))\right)\\
=&\ Y_{k-i+1:k}^{-\gamma}(1-Y_{k-i+1:k}^{-\gamma})\\
&+A(n/k)\left(2Y_{k-i+1:k}^{-2\gamma}-Y_{k-i+1:k}^{-\gamma}\right)\frac{Y_{k-i+1:k}^{\rho}-1}{\rho}(1+o_p(1)).
\end{aligned}$$

Consequently

$$\begin{aligned}
C &= \frac{1}{k}\sum_{i=1}^{k}\frac{\alpha\ V_i}{(1+\alpha\ V_i)^2}\\
&= \frac{1}{k}\sum_{i=1}^{k}e^{-\gamma\ E_i}\left(1-e^{-\gamma\ E_i}\right)+\frac{1}{k}\sum_{i=1}^{k}\left(Y_i^{-2\gamma}-Y_i^{-\gamma}\right)\frac{Y_i^{\rho}-1}{\rho}(1+o_p(1))
\end{aligned}$$

and (3.7) follows, as well as the remaining of the theorem. □

4 Back to the ML estimators

We may state the following:

Theorem 4.1. *If the second order condition (1.2) and if (1.3) hold, we have the following asymptotic distributional representations:*

$$\frac{\widehat{\alpha}^{GP}(k)-\alpha}{\alpha}\stackrel{d}{=}\frac{(1+\gamma)\sqrt{1+2\gamma}}{\gamma\sqrt{k}}R_k+\frac{\rho(1+\gamma)(1+2\gamma)}{\gamma^2(1-\rho)(1-\rho+\gamma)}\ A(n/k)(1+o_p(1)),\quad(4.1)$$

and

$$\widehat{\gamma}^{GP}(k)\stackrel{d}{=}\gamma+\frac{(1+\gamma)}{\sqrt{k}}S_k+\frac{(1+\gamma)(\gamma+\rho)}{\gamma(1-\rho)(1-\rho+\gamma)}\ A(n/k)(1+o_p(1)),\quad(4.2)$$

with R_k and S_k asymptotically standard normal random variables.

Proof. In Theorem 3.1 we have got

$$1 - B - \frac{B}{A} \stackrel{d}{=} \frac{1}{\sqrt{k}} \left(\frac{Q_k}{\sqrt{1+2\gamma}} + \frac{P_k}{1+\gamma} \right) + \frac{c_0 - c_1(1+\gamma)^2}{\gamma(1+\gamma)} \; A(n/k)(1+o_p(1))$$

Since $C = \frac{\gamma^2}{(1+\gamma)^2(1+2\gamma)} + o_p(1)$ and $\frac{\widehat{\alpha}^{GP}(k)-\alpha}{\alpha} = \frac{1-B-B/A}{C+C/A-(B/A)^2}$, we get

$$\begin{aligned} \frac{\widehat{\alpha}^{GP}(k) - \alpha}{\alpha} \stackrel{d}{=} & \frac{(1+\gamma)^2(1+2\gamma)}{\gamma^2\sqrt{k}} \left(\frac{Q_k}{\sqrt{1+2\gamma}} + \frac{P_k}{1+\gamma} \right) \\ & + \frac{(1+\gamma)(1+2\gamma)(c_0 - c_1(1+\gamma)^2}{\gamma^3} \; A(n/k)(1+o_p(1)), \end{aligned}$$

and, from the covariance structure between P_k and Q_k, (4.1) follows.

Finally, since we have $\widehat{\gamma}^{GP}(k) = A + B\ \frac{\widehat{\alpha}^{GP}(k)-\alpha}{\alpha}$, we get

$$\begin{aligned} \widehat{\gamma}^{GP}(k) \stackrel{d}{=} & \gamma + \frac{\gamma}{\sqrt{k}}\ P_k + c_0\ A(n/k) + \frac{(1+\gamma)(1+2\gamma)}{\gamma\sqrt{k}} \left(\frac{Q_k}{\sqrt{1+2\gamma}} + \frac{P_k}{1+\gamma} \right) \\ & + \frac{(1+2\gamma)(c_0 - c_1(1+\gamma)^2}{\gamma^2} \; A(n/k)(1+o_p(1)) \end{aligned}$$

$$\begin{aligned} \stackrel{d}{=} & \gamma + \frac{1+\gamma}{\gamma\sqrt{k}} \left((1+\gamma)P_k + \sqrt{1+2\gamma} Q_k \right) \\ & + \frac{(1+\gamma)^2(c_0 - c_1(1+2\gamma)}{\gamma^2} \; A(n/k)(1+o_p(1)), \end{aligned}$$

from which (4.2) follows. □

Remark 4.1. *Smith (1987) has already got these same results. The conclusion of his Theorem 3.2 gives in our notation, and whenever* $\sqrt{k}\ A(n/k) \underset{n\to\infty}{\longrightarrow} \lambda$*, finite*

$$\sqrt{k}\left(\widehat{\gamma}^{GP}(k) - \gamma\right) \stackrel{d}{\longrightarrow} Normal\left[\frac{\lambda(1+\gamma)(\gamma+\rho)}{\gamma(1-\rho)(1+\gamma-\rho)}, (1+\gamma)^2 \right]$$

5 An alternative approach to reduce bias in the PML estimation

We have written before

$$V_i = X_{n-k:n} \left\{ Y^{\gamma}_{k-i+1:k} - 1 + A(n/k) Y^{\gamma}_{k-i+1:k} \frac{Y^{\rho}_{k-i+1:k} - 1}{\rho} (1 + o_p(1)) \right\}.$$

Let us write now

$$\begin{aligned}
V_i &= X_{n-k:n}\left\{Y_{k-i+1:k}^{\gamma}\left(1+A(n/k)\frac{Y_{k-i+1:k}^{\rho}-1}{\rho}(1+o_p(1))\right)-1\right\} \\
&= X_{n-k:n}\left\{e^{\gamma\ln Y_{k-i+1:k}+A(n/k)\frac{Y_{k-i+1:k}^{\rho}-1}{\rho}}-1+o_p(A(n/k))\right\} \\
&= X_{n-k:n}\left\{e^{\gamma\ln Y_{k-i+1:k}\left(1+\frac{A(n/k)}{\gamma}\frac{Y_{k-i+1:k}^{\rho}-1}{\rho\ln Y_{k-i+1:k}}\right)}-1\right\} \\
&= X_{n-k:n}\left\{Y_{k-i+1:k}^{\gamma\left(1+\frac{A(n/k)}{\gamma}\frac{Y_{k-i+1:k}^{\rho}-1}{\rho\ln Y_{k-i+1:k}}\right)}-1+o_p(A(n/k))\right\} \\
&= X_{n-k:n}\left\{Y_{k-i+1:k}^{\gamma e^{\frac{A(n/k)}{\gamma}\frac{Y_{k-i+1:k}^{\rho}-1}{\rho\ln Y_{k-i+1:k}}}}-1+o_p(A(n/k))\right\}.
\end{aligned}$$

Since

$$\frac{Y_{k-i+1:k}^{\rho}-1}{\rho\ln Y_{k-i+1:k}}\sim-\frac{(i/k)^{-\rho}-1}{\rho\ln(i/k)}=:\psi_{ik}$$

we expect to get a less biased estimator if we assume that the random excess V_i comes from a GP model with a shape parameter not equal to γ, as it is usually done, but dependent on i and k, and more specifically given by

$$\gamma_{ik}=\gamma\ e^{D\ \left(\frac{n}{k}\right)^{\rho}\psi(i,k)}=\gamma\ e^{A^*(n/k)\psi(i,k)},\quad A^*(n/k)=\frac{A(n/k)}{\gamma} \tag{5.1}$$

assuming thus that we are in Hall's class of models, with $A(t)=C\ t^{\rho}$, $D=C/\gamma$.

We are thus going to assume that there exists a parameter α such that $V_i = X_{n-i+1:n}-X_{n-k:n}$ comes from the Generalized Pareto model, with distribtion function

$$GP(x;\gamma,\alpha)=1-(1+\alpha\ x)^{-1/\gamma_{ik}},\quad x>0\quad(\gamma_{ik}>0), \tag{5.2}$$

for every $1\le i\le k$.

The likelihood function of the V_i's is then

$$L(\gamma,A^*;\underline{v})=\frac{\alpha^k}{\gamma^k}\prod_{i=1}^{k}e^{-A^*(n/k)\psi_{ik}}\,(1+\alpha\ v_i)^{-\frac{1}{\gamma}A^*(n/k)\psi_{ik}-1},$$

and consequently we have a log-likelihood given by

$$\ln L(\gamma, A^*; \underline{v}) = k \ln \alpha - k \ln \gamma - A^*(n/k) \sum_{i=1}^{k} \psi_{ik} - \sum_{i=1}^{k} \ln(1 + \alpha\ v_i) \\ - \frac{1}{\gamma} \sum_{i=1}^{k} e^{-A^*(n/k)\psi_{ik}} \ln(1 + \alpha\ v_i).$$

We then get an explicit expression for $\widehat{\gamma}^{PML}(k)$, given by

$$\widehat{\gamma}^{PML}(k) := \frac{1}{k} \sum_{i=1}^{k} e^{-A^*(n/k)\psi_{ik}} \ln(1 + \alpha V_i). \tag{5.3}$$

Remark 5.1. *As noticed by Laurens de Haan, since α may be chosen equal to $1/X_{n-k:n}$ we may think also in a weighted combination of the log-excesses, i.e. on*

$$\widehat{\gamma}^{WH}(k) := \frac{1}{k} \sum_{i=1}^{k} e^{\widehat{D}(n/k)^{\widehat{\rho}}\ \frac{(i/k)^{-\widehat{\rho}}-1}{\widehat{\rho}\ln(i/k)}} \ln\left(\frac{X_{n-i+1:n}}{X_{n-k:n}}\right), \tag{5.4}$$

where with WH we denote Weighted Hill.

Let us assume everything is known, apart from γ. We may write

$$\ln(1 + \alpha V_i) = \gamma \ln Y_{k-i+1:k} \left(1 + \frac{A(n/k)}{\gamma}\ \frac{Y_{k-i+1:k}^{\rho} - 1}{\rho \ln Y_{k-i+1:k}} (1 + o_p(1))\right),$$

and consequently $\widehat{\gamma}^{PML}(k)$ may be written as

$$\frac{\gamma}{k} \sum_{i=1}^{k} \ln Y_{k-i+1:k} \left(1 + \frac{A(n/k)}{\gamma}\ \frac{Y_{k-i+1:k}^{\rho} - 1}{\rho \ln Y_{k-i+1:k}} (1 + o_p(1))\right) \left(1 - \frac{A(n/k)}{\gamma} \psi_{ik}\right),$$

and consequently, with

$$B_k := \frac{1}{k} \sum_{i=1}^{k} \frac{Y_{k-i+1:k}^{\rho} - 1}{\rho} - \frac{1}{k} \sum_{i=1}^{k} \psi_{ik}\ E_{k-i+1:k} =: B_{k1} - B_{k2} \tag{5.5}$$

we have

$$\widehat{\gamma}^{PML}(k) = \frac{\gamma}{k} \sum_{i=1}^{k} E_i + A(n/k)\ B_k\ (1 + o_p(1))$$

Let us srudy the asymptotic behaviour of the random variable B_k in (5.5): the weak law of large numbers enables to say that both B_{k1} and B_{k2} converge in probability towards their mean values. We have

$$E[B_{k1}] = E[(Y^{\rho} - 1)/\rho] = 1/(1 - \rho).$$

On the other hand

$$
\begin{aligned}
E[B_{k2}] &= \frac{1}{k}\sum_{i=1}^{k}\psi_{ik}\ E[E_{k-i+1:k}] = -\frac{1}{\rho}\left(\frac{1}{k}\sum_{i=1}^{k}\frac{(i/k)^{-\rho}-1}{\ln(i/k)}\sum_{j=i}^{k}\frac{1}{j}\right) \\
&= -\frac{1}{\rho}\left(\frac{1}{k}\sum_{j=1}^{k}\frac{1}{j}\sum_{i=1}^{j}\frac{\left(\frac{i}{j}\right)^{-\rho}\left(\frac{j}{k}\right)^{-\rho}-1}{\ln(i/j)+ln(j/k)}\right) \\
&\underset{k\to\infty}{\longrightarrow} -\frac{1}{\rho}\iint\limits_{[0,1]\times[0,1]}\frac{(xy)^{-\rho}-1}{\ln(xy)}\,dxdy = -\int_0^1\frac{v^{-\rho}-1}{\rho\ln v}\,dv\int_v^1\frac{1}{u}\,du \\
&= \int_0^1\frac{v^{-\rho}-1}{\rho}\,dv = \frac{1}{1-\rho}.
\end{aligned}
$$

Consequently B_k converges in probability towards 0 as $k\to\infty$ and

$$
\widehat{\gamma}^{PML}(k) \stackrel{d}{=} \gamma + \frac{\gamma}{\sqrt{k}}P_k + o_p(A(n/k)), \tag{5.6}
$$

i.e., the usual dominant component of bias, which is for the classical estimators of the tail index of the order of $A(n/k)$ is null.

The main problems to be dealt with are now related to the maximum likelihood estimation of (α, D, ρ).

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