The total median in Statistical Quality Control*

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Abstract. In industry, most of the process observations are assumed to come from a normal population, but usually we merely want to control the process mean value. It is thus sensible to find control statistics, which are "robust" to monitor the process mean, giving the expected rate of false alarms whenever that mean is close to the target value, although not under a normal regime. Simulation studies for a few symmetric and asymmetric distributions allow us to suggest the total median as a robust median estimator. We shall here analyse such a robustness as well as the robustness of the total median chart comparatively to the sample mean chart to control the mean value of a symmetric underlying parent. Some indication is also provided on the comparative out-of-control behaviour of the two charts.

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1 Introduction

Control charts are tools widely used in industry to detect abnormal behaviour in manufacturing processes, as may be seen in Montgomery (1996). In general we assume that the process observations come, whenever the process is in-control, from a normal population with targets at a mean value μ_0 and a standard deviation σ_0. However, even if it is sensible to assume, on the basis of both theoretical and practical reasons, that the process is normal, there is often a possibility of having disturbances in the data. We then need to find efficient and robust estimators to monitor the process parameters. Simulation studies developed in Figueiredo and Gomes (2000), for a few symmetric and

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asymmetric distributions related to the normal, allow us to suggest the total median, defined in section 2, as a robust location estimator, and to advance with its use in *Statistical Quality Control* (*SQC*).

In section 2 of this paper we shall present some definitions related with the total median, some lemmas and their proofs. In section 3 we shall analyse the efficiency and robustness of the total median. In section 4 we deal with the robustness of the total median chart comparatively to the sample mean chart, drawing a comparison of this new chart with the classical Shewhart $\overline{X}$-chart in an out-of-control context. Finally, in section 5, we shall draw some overall conclusions.

2 The total median

Let $(X_1, X_2, \cdots, X_n)$ be a random sample of size n from a parent F. Let us denote $X_{i:n}$, $1 \le i \le n$, the random sample of the associated ascending order statistics (o.s.), and $(X_1^*, X_2^*, \cdots, X_n^*)$ the random bootstrap sample associated to an observed sample $(x_1, x_2, \cdots, x_n)$:

Definition 2.1. *Given an observed sample, $(x_1, x_2, \cdots, x_n)$, the random bootstrap sample X_i^*, $1 \le i \le n$, is a random sample of independent, identically distributed (i.i.d.) replicates from a random variable (r.v.) X^*, with distribution function (d.f.)*

$$F_n^*(x) = \frac{1}{n} \sum_{i=1}^{n} I_{\{x_i \le x\}},$$

the empirical d.f. of our observed sample. The notation $I_A = \begin{cases} 1 & \textit{if } A \textit{ occurs} \\ 0 & \textit{otherwise} \end{cases}$ *is used for the indicator function of the set A.*

Definition 2.2. *The Bootstrap Median, denoted BMd, is the median of the bootstrap sample $(X_1^*, X_2^*, \cdots, X_n^*)$. More precisely, and with $X_{i:n}^*$ denoting the i-th ascending order statistic, $1 \le i \le n$, associated with the bootstrap sample X_i^*, $1 \le i \le n$,*

$$BMd := \begin{cases} X_{m:n}^* & \textit{if} \quad n = 2m-1 \\ \frac{X_{m:n}^* + X_{m+1:n}^*}{2} & \textit{if} \quad n = 2m \end{cases}, \quad m = 1, 2, 3, \cdots. \tag{2.1}$$

We shall denote Md, the *Median* of the original random sample $(X_1, X_2, \cdots, X_n)$, i.e., the Md statistic is given by an expression similar to (2.1), but with X^* replaced by X.

Remark 2.1. *Note that, given the observed ordered sample $x_{i:n}$, $1 \leq i \leq n$, the support of the Bootstrap Median is the set:*

$$\mathcal{S} = \left\{ \frac{x_{i:n} + x_{j:n}}{2}, \ 1 \leq i \leq j \leq n \right\}.$$

Let us denote α_{ij} the probability that the Bootstrap Median assumes the value $(x_{i:n} + x_{j:n})/2$, $1 \leq i \leq j \leq n$, i.e.,

$$\alpha_{ij} := P\left(BMd = \frac{x_{i:n} + x_{j:n}}{2} \right), \quad 1 \leq i \leq j \leq n, \tag{2.2}$$

with $P(A)$ denoting the probability of the event A.

Following closely the arguments in Efron (1979) and Efron and Tibshirani (1993) we may obtain the probabilities α_{ij} in terms of binomial distributions.

Lemma 2.1. *Let $N_j^* = \sum_{l=1}^{n} I_{\{X_l^* = x_j\}}$, $1 \leq j \leq n$. The random vector $(N_1^*, \ N_2^*, \ \cdots, \ N_n^*)$ follows a multinomial scheme, with all parameters equal to $1/n$. For every integer i, $1 \leq i < n$,*

$$P(X_{m:n}^* > x_{i:n}) \quad = \quad \frac{1}{n^n} \sum_{j=0}^{m-1} \binom{n}{j} \, i^j \ (n-i)^{n-j}. \tag{2.3}$$

Proof. The proof follows closely the arguments in Efron (1979). Since the bootstrap sample comes from a resampling with replacement of the observed sample, and any of the elements in the original sample may be selected with probability $1/n$, we get the multinomial distribution for the random vector $(N_1^*, \cdots, N_n^*)$. Let $N_{(j)}^*$ denote the values of N_j^* induced by the o.s. of our sample, $1 \leq j \leq n$. Denoting $\mathrm{Bi}(n,p)$ a Binomial r.v. with parameters (n,p), we have:

$$\begin{aligned} P(X_{m:n}^* > x_{i:n}) &= P(N_{(1)}^* + \cdots + N_{(i)}^* \leq m-1) \\ &= P\left(\mathrm{Bi}\left(n, \tfrac{i}{n}\right) \leq m-1\right) \\ &= \sum_{j=0}^{m-1} \binom{n}{j} \left(\frac{i}{n}\right)^j \left(1 - \frac{i}{n}\right)^{n-j}, \end{aligned}$$

and consequently (2.3) follows. □

On the basis of direct probabilistic arguments or a reasoning of the type of the one used in Lemma 2.1, we may compute the probabilities α_{ij} in (2.2):

Lemma 2.2.

(*a*) *For $i = j$ we have for all i from 1 till n,*

$$\alpha_{ii} = \frac{1}{n^n} \sum_{k=0}^{[(n-1)/2]} \binom{n}{k} (i-1)^k \sum_{r=[n/2]-k+1}^{n-k} \binom{n-k}{r} (n-i)^{n-k-r}, \quad (2.4)$$

where $[x]$ denotes the integer part of x.
The particular cases $i = 1$ and $i = n$ may thus be written as

$$\alpha_{11} = \alpha_{nn} = \frac{1}{n^n} \sum_{k=0}^{[(n-1)/2]} \binom{n}{k} (n-1)^k.$$

(*b*) *For $i \neq j$, $1 \leq i < j \leq n$,*

$$\alpha_{ij} = 0, \quad \text{if } n \text{ is odd.} \quad (2.5)$$

(*c*) *For n even, and for $1 \leq i < j \leq n$, we obtain*

$$\alpha_{ij} = \frac{n!}{n^n \; ((n/2)!)^2} \left\{ i^{\frac{n}{2}} - (i-1)^{\frac{n}{2}} \right\} \left\{ (n-j+1)^{\frac{n}{2}} - (n-j)^{\frac{n}{2}} \right\}, \quad (2.6)$$

and $\alpha_{i,j} = \alpha_{n-j+1,n-i+1}$.
For the particular case $i = 1$, $j = n > 1$ we have

$$\alpha_{1n} = \frac{n!}{n^n \; ((n/2)!)^2}.$$

Proof. (a) We begin with the case $i = j$ and $n = 2m - 1$, odd. Then, with $U_{i:n}$, $1 \leq i \leq n$ denoting the ascending o.s.'s associated with a random sample of size n from a uniform model in $(0, 1)$,

$$\begin{aligned} \alpha_{ii} &= P\left(X^*_{m:n} = x_{i:n}\right) = P\left(X^*_{m:n} > x_{i-1:n}\right) - P\left(X^*_{m:n} > x_{i:n}\right) \\ &= P\left(\mathrm{Bi}\left(n, \tfrac{i-1}{n}\right) \leq m-1\right) - P\left(\mathrm{Bi}\left(n, \tfrac{i}{n}\right) \leq m-1\right) \\ &= P\left(1 - \frac{i}{n} \leq U_{m:n} \leq 1 - \frac{i-1}{n}\right) = P\left(\frac{i-1}{n} \leq U_{m:n} \leq \frac{i}{n}\right) \\ &= \sum_{k=0}^{m-1} \sum_{r=m-k}^{n-k} \left(\frac{i-1}{n}\right)^k \left(\frac{i}{n} - \frac{i-1}{n}\right)^r \left(1 - \frac{i}{n}\right)^{n-k-r} \end{aligned}$$

and (2.4) follows. The same result follows also through a direct argument and may be obtained for n even and $i = j$.

(b) For n odd, the sample median is always the central observation. Hence, $\alpha_{ij} = 0$ if $i \neq j$.

(c) For $i \neq j$, a direct combinatorial argument leads us to:

$$\begin{aligned}
\alpha_{ij} &= \frac{n!}{((n/2)!)^2} \sum_{k=0}^{\frac{n}{2}-1} \binom{\frac{n}{2}}{k} \left(\frac{i-1}{n}\right)^k \sum_{r=1}^{\frac{n}{2}} \binom{\frac{n}{2}}{r} \left(\frac{1}{n}\right)^{\frac{n}{2}-k+r} \left(\frac{n-j}{n}\right)^{\frac{n}{2}-r} \\
&= \frac{n!}{n^n\left((n/2)!\right)^2} \left\{i^{n/2} - (i-1)^{n/2}\right\} \left\{(n-j+1)^{n/2} - (n-j)^{n/2}\right\} \\
&= \alpha_{n-j+1,n-i+1}.
\end{aligned}$$

This completes the proof of the lemma. □

Definition 2.3. *The total median is the statistic*

$$TMd = \sum_{i=1}^{n} \sum_{j=i}^{n} \alpha_{ij} \left(\frac{X_{i:n} + X_{j:n}}{2}\right), \tag{2.7}$$

with α_{ij} given in (2.2), and made explicit in Lemma 2.2

From Definition 2.3 and Lemma 2.2 we get:

Proposition 2.1. *The total median may be expressed as a linear combination of the sample order statistics, i.e.,*

$$TMd = \sum_{i=1}^{n} a_i \, X_{i:n}, \tag{2.8}$$

where the coefficients a_i are related with the previous coefficients α_{ij} in (2.4), (2.5) and (2.6), through the relation

$$a_i = \frac{1}{2}\left(\sum_{j=i}^{n} \alpha_{ij} + \sum_{j=1}^{i} \alpha_{ji}\right).$$

Denoting $U_{i:n}$, $1 \leq i \leq n$, the ascending o.s.'s associated with a random sample of size n from a uniform model in $(0,1)$, we have for n odd,

$$a_i = \alpha_{ii} = P\left(\frac{i-1}{n} \leq U_{[n/2]+1:n} \leq \frac{i}{n}\right).$$

For n even, and with $\beta_{in} = P\left(Bi\left(n, \frac{i-1}{n}\right) = \frac{n}{2}\right) - P\left(Bi\left(n, \frac{i}{n}\right) = \frac{n}{2}\right)$, we have

$$a_i = P\left(\frac{i-1}{n} \leq U_{n/2:n} \leq \frac{i}{n}\right) + \frac{\beta_{in}}{2}.$$

Due to the way the total median was built, we may also state the following:

Proposition 2.2. *Both the Bootstrap Median, BMd, in* (2.1) *and the Total Median, TMd, in* (2.7) *converge in probability, as $n \to \infty$, towards $\chi_{1/2} = F^{\leftarrow}(1/2)$, where $F^{\leftarrow}(x) := \inf\{y : F(y) \geq x\}$ is the generalized inverse function of F, i.e., they are consistent estimators of the population median.*

Remark 2.2. *Note that the coefficients a_i, $1 \leq i \leq n$, in* (2.8) *are "distribution-free", i.e., they are independent of the underlying model F, and depend only on the sample size n.*

In Table 1 we present the values a_i, with 3 decimal figures, for the most usual rational subgroups n in SQC. The missing coefficients in the table are either zero or obtained by symmetrization, i.e., through the condition

$$a_i = a_{n-i+1}, \quad 1 \leq i \leq n, \quad 0 < a_1 \leq a_2 \leq \cdots \leq a_{[n/2]}, \quad \sum_{i=1}^{n} a_i = 1.$$

Table 1: Values of the coefficients a_i, for sample sizes $n \leq 20$.

i	1	2	3	4	5	6	7	8	9	10	15	20
1	1.000	0.500	0.259	0.156	0.058	0.035	0.010	0.007	0.001	0.001	0.000	0.000
2			0.482	0.344	0.259	0.174	0.098	0.064	0.029	0.019	0.000	0.000
3					0.366	0.291	0.239	0.172	0.115	0.078	0.040	0.000
4							0.306	0.257	0.221	0.168	0.021	0.001
5									0.268	0.234	0.063	0.070
6											0.125	0.023
7											0.183	0.055
8											0.208	0.099
9												0.143
10												0.172

3 Efficiency and robustness of the total median

Definition 3.1. *The mean squared error of T_n, a consistent estimator of a parameter θ, is the quantity*

$$MSE(T_n) = \mathbb{E}\,(T_n - \theta)^2\,, \tag{3.1}$$

where $\mathbb{E}$ denotes the expected value operator.

Definition 3.2. *Given a model F dependent of an unknown parameter θ, let $\left\{T_n^{(i)},\ i \in \mathcal{I}\right\}$ be a set of consistent estimators of θ. The efficiency of $T_n^{(i)}$ relatively to $T_n^{(j)}$, denoted $REFF_{i|j}^F$, is the quotient*

$$REFF_{i|j}^F := MSE\left(T_n^{(j)}|F\right)/MSE\left(T_n^{(i)}|F\right),\ i,j \in \mathcal{I}. \tag{3.2}$$

Definition 3.3. *Given a model F dependent on an unknown parameter θ, the most efficient estimator of θ, among the set, $\left\{T_n^{(i)},\ i \in \mathcal{I}\right\}$, of consistent estimators of θ, it is the one with the smallest mean squared error, i.e., it is*

$$T_n^{(i_0)}|F, \quad \textit{such that} \quad i_0 := \arg\min_{i \in \mathcal{I}} MSE\left(T_n^{(i)}|F\right). \tag{3.3}$$

3.1 The simulation study

A large scale multi-sample Monte Carlo simulation study of size $r \times m$, with $r = 25$ replicates of size $m = 2,500$, has been undertaken to evaluate the comparative efficiency of the location estimators $\overline{X}$ versus TMd; such evaluation was done in terms of their simulated *Mean Squared Error* (MSE); as a by-product, we have also evaluated the performance of the sample median Md and of the bootstrap median BMd.

The multi-sample simulation or replication is a common practice in Monte Carlo procedures whenever we do not have an easy way to estimate measures of dispersion of a statistic, like the MSE (see Fishman (1972), for details). The idea is reasonably simple: in a multi-sample simulation of size $r \times m$, instead of generating a sample of a very large size (let us say $N = r \times m$) of observed values of a statistic, we collect m observations of the statistic on each of r independent replications of the experiment. The value of m also needs to be large enough to reduce bias, and provide asymptotic normality. We next take as an overall estimate of the population parameter of interest (in this case the MSE of a statistic, generally denoted T_n) the average of the r corresponding estimates computed on the independent replications. Then, under very broad conditions, that overall estimator (which is a sample mean) converges to normality as r increases. Moreover we may estimate the standard error of this overall estimate, even if r is small. For small values of r, and whenever we may guarantee the asymptotic normality of the estimator of the characteristic under study, we may use the t-distribution with $r-1$ degrees of freedom to approximate its true distribution, and to derive a confidence interval for the parameter of interest, here $MSE(T_n)$, the Mean Squared Error of T_n.

In our simulation study, for any model F, dependent on an unknown parameter θ, for any sample size n, for any estimator (generally denoted T_n) of θ, and for each replicate $i = 1, 2, \cdots r = 25$, $m = 2,500$ independent samples of size n have been randomly generated and $m = 2,500$ observed values of T_n have been obtained, $\left(t_{n,i}^{(1)}, \cdots, t_{n,i}^{(m)}\right)$. For each of the r replicates, an estimate of the mean squared error of T_n has been obtained,

$$\widehat{MSE}_i(T_n) = \frac{1}{m}\sum_{j=1}^{m}\left(t_{n,i}^{(j)} - \theta\right)^2, \quad 1 \le i \le r.$$

The overall estimate of the MSE of our statistic is then the sample mean

$$\widehat{MSE}(T_n) = \frac{1}{r}\sum_{i=1}^{r}\widehat{MSE}_i(T_n),$$

with an associated standard error given by

$$\sqrt{\frac{1}{r(r-1)}\sum_{i=1}^{r}\left(\widehat{MSE}_i(T_n) - \widehat{MSE}(T_n)\right)^2}.$$

To obtain what we shall call the "robust estimator" we have applied a $MaxMin$ criterion:

Definition 3.4. *Given a class $\mathcal{F}$ of non-degenerate models F dependent on an unknown parameter θ, and a class $\mathcal{T} = \left\{T_n^{(i)},\ i \in \mathcal{I}\right\}$ of consistent estimators of θ, consider*

1. *for any value of n and for any model $F \in \mathcal{F}$, the most efficient estimator $T_n^{(i_0)}|F$ defined in (3.3), i.e., the one with the smallest mean squared error, among the set $\mathcal{T}$;*
2. *next, compute the efficiency of all the estimators relatively to the best one, selected in 1., through the measure $REFF_{i|i_0}^F$ given in (3.2).*

The "degree of robustness" of the estimator $T_n^{(i)} \in \mathcal{T}$ is given by its minimum relative efficiency among the models $F \in \mathcal{F}$, i.e.,

$$dr\left(T_n^{(i)}\right) := \min_{F\in\mathcal{F}}\left\{REFF_{i|i_0}^F\right\}. \tag{3.4}$$

The "robust estimator" is the one with the highest minimum efficiency, i.e.,

$$T_n^{(i_r)} \quad \textit{such that} \quad i_r := \arg\max_{i\in\mathcal{I}}\left\{dr\left(T_n^{(i)}\right)\right\}. \tag{3.5}$$

For details on robust estimators, see Hampel (1971) and Hampel *et al.* (1986).

3.2 The set of models

In this simulation we have considered a reasonably large set of parent distributions, in order to have different skewness and tail-weight coefficients. The skewness and tail-weight coefficients herewith considered are:

Definition 3.5. *The skewness coefficient of the model F (r.v. X) is the parameter*

$$\gamma_F = \frac{\mu_3}{\mu_2^{3/2}},$$

where μ_r denotes the r-th central moment of F (r.v. X), i.e., $\mu_r = \mathbb{E}\left(X - \mathbb{E}X\right)^r$.

Definition 3.6. *The tail-weight coefficient of the model F is the parameter*

$$\tau_F = \frac{\frac{1}{2}\left(\frac{F^{-1}(0.99)-F^{-1}(0.5)}{F^{-1}(0.75)-F^{-1}(0.5)} + \frac{F^{-1}(0.5)-F^{-1}(0.01)}{F^{-1}(0.5)-F^{-1}(0.25)}\right)}{\left(\frac{\Phi^{-1}(0.99)-\Phi^{-1}(0.5)}{\Phi^{-1}(0.75)-\Phi^{-1}(0.5)}\right)},$$

where F^{-1} and Φ^{-1} denote the inverse functions of F and of the standard normal distribution function Φ, respectively.

Remark 3.1. *Hoaglin et al.* (1983) *consider a different definition for the tail-weight τ. Here, we have adjusted such a coefficient in order to consider the quantiles associated to each tail, left and right, which are different for asymmetric distributions.*

The global set of distributions considered includes well-known symmetric distributions, with the following standard types, parameterised so that they all have zero mean and unit variance:

1. the Normal, $N(0,1)$, with probability density function (p.d.f.)

$$\varphi(x) = \frac{1}{\sqrt{2\pi}}\ \exp\left(-\frac{x^2}{2}\right),\ x \in \mathbb{R};$$

2. the Logistic, $L(0,1)$, with d.f.

$$F_L(x) = \left(1 + \exp\left(-\frac{\pi x}{\sqrt{3}}\right)\right)^{-1},\ x \in \mathbb{R},$$

   with tail-weight 1.212;

3. the Student-t_ν, denoted $t(\nu)$, with p.d.f.

$$f_{t_\nu}(t) = \frac{\Gamma((\nu+1)/2)}{\sqrt{\pi(\nu-2)}\;\Gamma(\nu/2))}\left[1+\frac{t^2}{\nu-2}\right]^{-(\nu+1)/2},\ t\in\mathbb{R}\quad(\nu>0)$$

(with ν ranging from 20 till 3 so that the tail-weights range from 1.067 till 1.721); and

4. several Contaminated Normal models, $CN(m,k,a)$, with p.d.f.

$$f_{CN}(x;m,k,a) = \tfrac{ca}{k}\left(\varphi\left(\frac{cx+m}{k}\right)+\varphi\left(\frac{cx-m}{k}\right)\right)+c(1-2a)\ \varphi(cx),$$

with $0<a<1/2,\ m\ge 0,\ k>0,\ c=\sqrt{1+2a(k^2+m^2-1)}$, and where $\varphi(x)=\varphi(x;0,1)$ denotes the standard normal p.d.f. The values of (m,k,a) were chosen in order to have tail-weights ranging from 1 till 2.978.

The asymmetric distributions considered were:

5. the Lognormal, $LN(\delta)$, $\delta>0$, with null mean, unit variance and a normal generating model with variance equal to δ^2, i.e., with $\xi = -1/\sqrt{\exp(\delta^2)-1}$

$$\ln\left(LN(\delta)-\xi\right) = N\left(-\ln\sqrt{\exp(\delta^2)-1}-\delta^2/2,\delta\right);$$

and

6. the chi-square with ν degrees of freedom, denoted $\chi^2(\nu)$, with p.d.f.

$$f_{\chi^2_\nu}(x) = \frac{1}{2^{\nu/2}\Gamma(\nu/2)}\ x^{\nu/2-1}\exp\left[-x/2\right],\ x\ge 0\quad(\nu>0).$$

For more details on all these models, see Johnson *et al.* (1994, 1995). In this set of distributions we have got values γ_F between 0 and 33.468, and values τ_F between 1 and 3. Notice that for all symmetric models $\gamma_F=0$ and that for the normal d.f. $\tau_F=1$.

3.3 Results and comments

In Figure 1 we present the most efficient estimator (the one with smallest simulated mean squared error) among the ones considered (the mean, the median, the bootstrap median and the total median), for the estimation of the mean of a symmetric distribution. Figure 2 is equivalent to Figure 1, but for the estimation of the median of an asymmetric distribution, and whenever we consider the median, the bootstrap median and the total median as possible estimators of the population median $\chi_{1/2} := F^{\leftarrow}(1/2)$. The symmetric

τ_F	F	Sample size 3	5	7	9	11	13	15	17	19	21	23	25	27	29
1.000	$N(0,1)$	M	M	M	M	M	M	M	M	M	M	M	M	M	M
1.015	$CN(1,1,0.05)$	M	M	M	M	M	M	M	M	M	M	M	M	M	M
1.021	$CN(1,1,0.1)$	M	M	M	M	M	M	M	M	M	M	M	M	M	M
1.067	$t(20)$	M	M	M	M	M	M	M	M	M	M	M	M	M	M
1.091	$t(15)$	TMd	M	M	M	M	M	M	M	M	M	M	M	M	M
1.145	$t(10)$	TMd	TMd	M	M	M	M	M	M	M	M	M	M	M	M
1.148	$CN(0,2,0.05)$	TMd	TMd	TMd	M	M	M	M	M	M	M	M	M	M	M
1.164	$t(9)$	TMd	TMd	TMd	M	M	M	M	M	M	M	M	M	M	M
1.189	$t(8)$	TMd	TMd	TMd	TMd	M	M	M	M	M	M	M	M	M	M
1.212	$L(0,1)$	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	M	M	M	M	M	M
1.222	$t(7)$	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	M	M	M	M	M	M
1.269	$t(6)$	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd
1.321	$CN(0,2,0.15)$	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd
1.342	$t(5)$	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd
1.355	$CN(2,3,0.025)$	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd
1.466	$t(4)$	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd
1.624	$CN(1,3,0.05)$	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd
1.721	$t(3)$	Md	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd
1.802	$CN(0,3,0.1)$	Md	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd
1.923	$CN(1,3,0.1)$	Md	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd
2.015	$CN(0,4,0.05)$	Md	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd
2.087	$CN(1,4,0.05)$	Md	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd
2.236	$CN(2,3,0.1)$	Md	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd
2.302	$CN(2,4,0.05)$	Md	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd
2.439	$CN(1,4,0.1)$	Md	Md	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd
2.560	$CN(1,5,0.05)$	Md	Md	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd
2.705	$CN(2,4,0.1)$	Md	Md	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd
2.841	$CN(0,5,0.15)$	Md	Md	Md	Md	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd
2.936	$CN(1,5,0.15)$	Md	Md	Md	Md	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd
2.978	$CN(1,5,0.125)$	Md	Md	Md	Md	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd

M Mean   TMd Total Median   Md Median

Figure 1: Most efficient estimator for the mean of a symmetric distribution.

γ_F	τ_F	F	Sample size 3	5	7	9	11	13	15	17	19	21	23	25	27	29
0.302	1.005	$LN(0.1)$	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd
0.614	1.022	$LN(0.2)$	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd
0.632	1.001	$\chi^2(20)$	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd
0.730	1.002	$\chi^2(15)$	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd
0.894	1.004	$\chi^2(10)$	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd
0.943	1.005	$\chi^2(9)$	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd
0.950	1.050	$LN(0.3)$	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd
1.000	1.006	$\chi^2(8)$	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd
1.069	1.007	$\chi^2(7)$	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd
1.155	1.009	$\chi^2(6)$	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd
1.265	1.012	$\chi^2(5)$	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd
1.322	1.090	$LN(0.4)$	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd
1.414	1.018	$\chi^2(4)$	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd
1.633	1.030	$\chi^2(3)$	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd
1.750	1.143	$LN(0.5)$	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd
2.000	1.062	$\chi^2(2)$	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd
2.260	1.210	$LN(0.6)$	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd
2.828	1.218	$\chi^2(1)$	Md	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd
2.888	1.293	$LN(0.7)$	Md	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd
3.689	1.394	$LN(0.8)$	Md	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd
4.745	1.514	$LN(0.9)$	Md	Md	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd
6.185	1.657	$LN(1.0)$	Md	Md	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd
8.213	1.826	$LN(1.1)$	Md	Md	Md	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd
11.164	2.025	$LN(1.2)$	Md	Md	Md	Md	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd
15.598	2.258	$LN(1.3)$	Md	Md	Md	Md	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd
22.472	2.531	$LN(1.4)$	Md	Md	Md	Md	Md	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd
33.468	2.851	$LN(1.5)$	Md	Md	Md	Md	Md	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd	TMd

TMd Total Median   Md Median

Figure 2: Most efficient estimator for the median of an asymmetric distribution.

distributions are ordered according to their tail-weight coefficient τ_F and the asymmetric ones are first ordered according to their skewness coefficient γ_F and next according to their tail-weight coefficient τ_F.

The conclusions regarding the "robustness" of the total median are summarized graphically in Figures 3 and 4. In Figure 3 we represent the "degree of robustness" in (3.4) of the location estimators under study, whenever we take into account all the symmetric distributions in Figure 1. Figure 4 is analogous to Figure 3, but related to the estimation of the median of the asymmetric distributions under consideration.

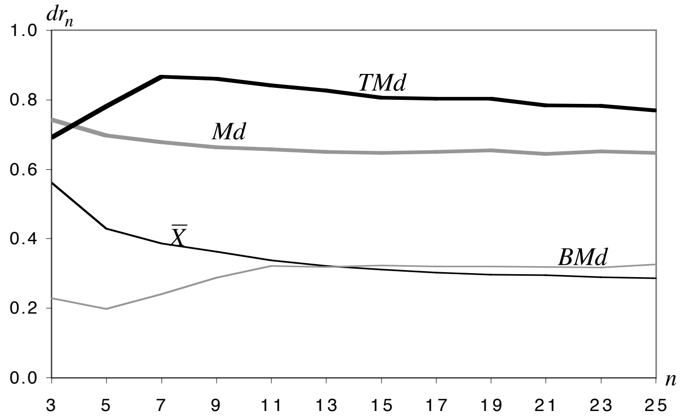

Figure 3: "Degree of robustness" of the location estimators under study, for symmetric parents.

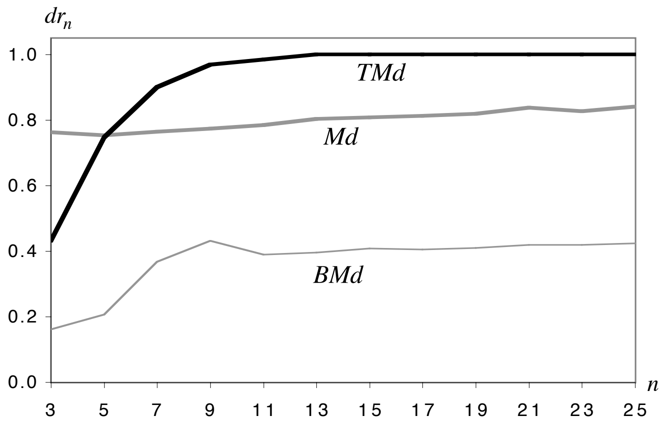

Figure 4: "Degree of robustness" of the median estimators under study, for asymmetric parents.

For the same coverage probability, the total median also enables us to obtain non-parametric confidence intervals for the quantiles equal or contained within those based on the sample median. Details on robust estimators may be found in Hoaglin *et al.* (1983), Lax (1985) and Tatum (1997). Cox and Iguzquiza (2001) provide an application of the total median in metrology, in the area of intra-laboratory comparisons, providing a robust estimate of a reference value, which enables the comparison of different laboratories' nominal measures of the same quantity.

4 Robustness and power of the $\overline{X}$ and TMd control charts

Whenever we want to control the mean value of a normal process or quality characteristic X at a target $\mu_0 = \mathbb{E}(X|IN)$, the mean value of the process under control (state IN) (with standard deviation $\sigma_0 = \sigma(X|IN)$ fixed), we often use in practice a 3-sigma control chart based on the statistic $\overline{X}_n$, the average of sequentially collected rational subgroups of size n. A control chart for the mean value μ of a process may thus be regarded as a test of the hypothesis $H_0 : \mu = \mu_0$ ($\sigma = \sigma_0$), in a model F, against the alternative $H_1 : \mu \neq \mu_0$ ($\sigma = \sigma_0$), for the same underlying model F. Such a test is performed along time. Without loss of generality we shall assume $\mu_0 = 0$ and $\sigma_0 = 1$.

Definition 4.1. *A 3-sigma control chart, based on a consistent estimator W_n of the parameter μ_0 has a lower control limit placed at $LCL := \mu_0(W_n) - 3\sigma_0(W_n)$ and an upper control limit placed at $UCL := \mu_0(W_n) + 3\sigma_0(W_n)$, where $\mu_0(W_n)$ and $\sigma_0(W_n)$ denote the mean and the standard deviation of W_n, respectively, given that the process is in control.*

Remark 4.1. *If the statistic W_n is the sample mean $\overline{X}_n$, then $\mu_0(W_n) = \mu_0$ and $\sigma_0(W_n) = \sigma_0/\sqrt{n}$.*

Definition 4.2. *The (false) alarm rate of the 3-sigma control chart based on the statistic W_n, also called the α-risk of the chart, is given by the probability*

$$\alpha_{F} := P\left(W_n \notin (LCL, UCL)|\mu_0; F\right), \tag{4.1}$$

where F is the model underlying the process, with a mean equal to μ_0, whenever the process is in control.

Remark 4.2. *If the process is normal, i.e., $F \equiv \Phi$, and $W_n \equiv \overline{X}_n$ then $\alpha_{\Phi} = 2\left(1 - \Phi(3)\right) = 0.0027$.*

Definition 4.3. *A 3-sigma control chart for the mean value μ_0, based on a statistic W_n, is said to be "robust" if the (false) alarm rate is kept close to the expected α-risk, α_Φ, whenever the model changes but the mean is kept at the target μ_0.*

For details on the robustness of control charts see, for instance, Rocke (1989), Borror *et al.* (1999), Stoumbos and Reynolds (2000), Ryan and Faddy (2001) and Shore (2001).

Definition 4.4. *A quantile$-W_n-$chart with risk equal to α, based on a consistent estimator W_n of the parameter μ_0 has a lower control limit placed at $LCL := \chi_{\alpha/2}^{W_n}$ and an upper control limit placed at $UCL := \chi_{1-\alpha/2}^{W_n}$, where $\chi_p^{W_n}$ is the p-quantile of W_n, given that the process is in control at a model F, i.e., $P\left(W_n \le \chi_p^{W_n}|F\right) = p$.*

The ability of a W_n-chart to detect shifts in process quality is described either by its operating characteristic curve, or equivalently, by its power function:

Definition 4.5. *Consider a W_n-chart devised to control the mean value μ of a process at the target μ_0, with the standard deviation σ_0 known and constant. If the mean of the process changes from the in-control mean value μ_0 to another value μ, the power function $\pi_F(\mu)$ of the chart is the probability of detecting this shift on the first subsequent rational subgroup, i.e., given the lower and upper control limits of the chart, LCL and UCL,*

$$\pi_F(\mu) = \mathbb{P}\left(W_n \notin (LCL, UCL)\,|\mu;\ F\right) \quad \left[\pi_F(\mu_0) = \alpha_F\right]. \tag{4.2}$$

4.1 The simulation study

To investigate the robustness of the 3-sigma $\overline{X}$ and TMd control charts we have thus computed through a Monte Carlo simulation of size $1 \times 1,000,000$, the alarm rates of both charts given that there are deviations from normality, always maintaining the mean at $\mu_0 = 0$, without loss of generality. In this study we have considered 3-*sigma* control charts for $\overline{X}$ and TMd, devised under a normal regime. For the standard normal parent we have $E[TMd] = 0$ and $Var[TMd]$ is given in Table 2, for sample sizes $n \le 20$.

The power functions of these 3-sigma control charts have been obtained for some of the symmetric models considered in section 3. Since we are however aware that it would have been "more fair" for both control statistics to use control limits based on their sampling distributions, given the model F, we have also peformed the comparison of the quantile TMd and $\overline{X}$ charts, with an alarm rate equal to 0.002, i.e., with the control limits placed at the quantiles

Table 2: Variance of the total median in a standard normal model.

n	$Var(TMd)$	n	$Var(TMd)$	n	$Var(TMd)$	n	$Var(TMd)$
1	—-	6	0.18062	11	0.10757	16	0.07647
2	0.50000	7	0.16038	12	0.09881	17	0.07286
3	0.33907	8	0.14066	13	0.09267	18	0.06886
4	0.25681	9	0.12853	14	0.08628	19	0.06592
5	0.21476	10	0.11592	15	0.08157	20	0.06274

$\chi_{0.001}$ and $\chi_{0.999}$ of the control statistics, given the model F.

4.2 Results and comments

In Tables 3 and 4 we present, for sub-rational groups of size $n = 3, 4, 5, 6, 7$, 10, 15 and 20, the alarm rates and the control limits of the 3-sigma $\overline{X}$ and TMd charts, respectively, whenever we consider standardized data from some of the symmetric distributions considered before, splitting them in two sets, one ($\mathcal{S}_1$) containing the Normal, the Logistic and the Student's-t models, and the other ($\mathcal{S}_2$) containing the contaminated normal models.

Table 3: Alarm rates of the 3-sigma $\overline{X}$-chart.

Distribution	τ	$n = 3$	$n = 4$	$n = 5$	$n = 6$	$n = 7$	$n = 10$	$n = 15$	$n = 20$
$N(0, 1)$	1	0.0027	0.0028	0.0028	0.0028	0.0027	0.0028	0.0027	0.0028
$t(20)$	1.067	0.0034	0.0032	0.0032	0.0031	0.0030	0.0030	0.0030	0.0028
$t(15)$	1.091	0.0038	0.0035	0.0034	0.0033	0.0033	0.0030	0.0029	0.0029
$t(10)$	1.145	0.0046	0.0043	0.0039	0.0037	0.0037	0.0034	0.0032	0.0031
$L(0, 1)$	1.212	0.0051	0.0046	0.0041	0.0040	0.0039	0.0036	0.0033	0.0031
$t(5)$	1.342	0.0079	0.0072	0.0065	0.0060	0.0058	0.0051	0.0045	0.0041
$t(3)$	1.721	0.0117	0.0112	0.0107	0.0104	0.0100	0.0091	0.0084	0.0079
$CN(0, 2, 0.05)$	1.148	0.0207	0.0200	0.0193	0.0190	0.0189	0.0182	0.0177	0.0173
$CN(0, 2, 0.15)$	1.321	0.0765	0.0757	0.0760	0.0760	0.0755	0.0751	0.0750	0.0745
$CN(0, 3, 0.1)$	1.802	0.0932	0.0935	0.0933	0.0934	0.0929	0.0923	0.0916	0.0919
$CN(0, 5, 0.05)$	2.501	0.0969	0.1044	0.1086	0.1108	0.1120	0.1132	0.1133	0.1133
$CN(0, 5, 0.15)$	2.841	0.1691	0.1706	0.1716	0.1727	0.1725	0.1743	0.1764	0.1779
Control limits		± 1.732	± 1.500	± 1.342	± 1.225	± 1.134	± 0.949	± 0.775	± 0.671

For the most usual rational subgroup in SQC, $n = 5$, we present in Figure 5 the alarm rates of the 3-sigma $\overline{X}$ and TMd charts. We have again ordered the distributions by the tail-weight coefficient τ, within each of the sets $\mathcal{S}_1$ and $\mathcal{S}_2$. It is clear from Tables 3 and 4, partially pictured in Figure 5, that there is a reasonably high variability of alarm rates for both charts when the model is no longer normal (but much more evident for the $\overline{X}$-chart). Neverthless, for the usual rational subgroup size, $n = 5$, the differences to the normal-case are really smaller whenever we consider the TMd-chart, even for the contaminated normal models with a high tail-weight.

Table 4: Alarm rates of the 3-sigma TMd-chart.

Distribution	τ	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$	$n=10$	$n=15$	$n=20$
$N(0,1)$	1	0.0027	0.0026	0.0028	0.0028	0.0027	0.0028	0.0027	0.0028
$t(20)$	1.067	0.0033	0.0029	0.0027	0.0025	0.0025	0.0024	0.0021	0.0021
$t(15)$	1.091	0.0035	0.0031	0.0027	0.0025	0.0024	0.0021	0.0018	0.0019
$t(10)$	1.145	0.0040	0.0034	0.0026	0.0024	0.0021	0.0018	0.0016	0.0014
$L(0,1)$	1.212	0.0045	0.0036	0.0027	0.0025	0.0021	0.0016	0.0013	0.0011
$t(5)$	1.342	0.0059	0.0043	0.0024	0.0019	0.0014	0.0009	0.0006	0.0005
$t(3)$	1.721	0.0081	0.0055	0.0020	0.0011	0.0006	0.0002	0.0001	0.0000
$CN(0,2,0.05)$	1.148	0.0178	0.0161	0.0133	0.0126	0.0117	0.0109	0.0104	0.0102
$CN(0,2,0.15)$	1.321	0.0684	0.0639	0.0558	0.0538	0.0500	0.0455	0.0421	0.0405
$CN(0,3,0.1)$	1.802	0.0756	0.0649	0.0475	0.0436	0.0358	0.0289	0.0243	0.0220
$CN(0,5,0.05)$	2.501	0.0782	0.0650	0.0341	0.0284	0.0196	0.0133	0.0100	0.0092
$CN(0,5,0.15)$	2.841	0.1298	0.1073	0.0737	0.0640	0.0452	0.0276	0.0137	0.0096
Control limits		±1.747	±1.520	±1.390	±1.275	±1.201	±1.021	±0.857	±0.751

Remark 4.3. *Note that the irregular pattern in Figure 5 (right) may be misleading, due to the small number of contaminated models considered, which were ordered merely according to the tail-weight coefficient. Indeed, α_F increases both with τ and k, as well as with τ and a.*

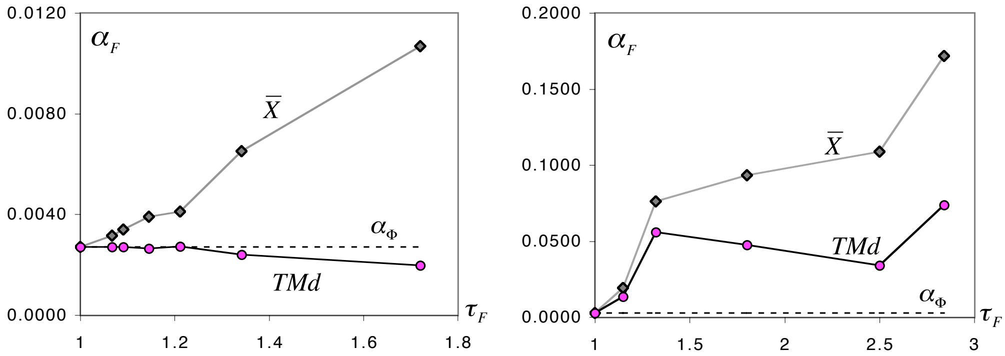

Figure 5: Alarm rates of the 3-sigma $\overline{X}$ and TMd control charts for $n=5$, and for the two sets of models, S_1 (*left*) and S_2 (*right*).

The way the 3-sigma TMd-chart has been devised, with the main objective of providing a false alarm risk close to $\alpha_\Phi = 0.0027$ whenever the mean of the process is close to the target value μ_0, together with the detection of the large false alarm rates associated to the 3-sigma $\overline{X}$-chart, when the model is no longer normal, led us to think that the 3-sigma TMd-chart would have, in an out-of-control situation, a much worse performance than the $\overline{X}$-chart. In Table 5 we present the power functions of both charts for processes from the same families of models. Astonishingly, as may be seen from Table 5, the TMd-chart is even able to overpass the $\overline{X}$-chart, for some of the models herewith considered, providing a faster alarm signal when the process is out of control. This obviously happens only when the deviations from the target are large, and the underlying model has reasonably heavy tail-weights. A graphical visualization of such an assertion is provided in Figure 6, where we picture, for Logistic and Student's t with 5 degrees of freedom underlying processes, the power functions of the two charts, $\overline{X}$ and TMd, based on 3-sigma control limits, built under a normal regime.

Table 5: Power functions of the 3-sigma $\overline{X}$ and the TMd charts for detecting changes in the mean value (μ) of the process ($n = 5$).

μ	-3.0	-2.0	-1.5	-1.0	0.0	1.0	1.5	2.0	3.0
				$Normal$					
$\overline{X}$	0.9999	0.9298	0.6380	0.2221	0.0027	0.2231	0.6382	0.9298	0.9999
TMd	0.9998	0.9064	0.5937	0.1997	0.0027	0.2003	0.5937	0.9062	0.9997
				$Logistic$					
$\overline{X}$	0.9997	0.9316	0.6421	0.2166	0.0042	0.2173	0.6419	0.9315	0.9997
TMd	0.9997	0.9205	0.6018	0.1812	0.0028	0.1817	0.6014	0.9207	0.9996
				$t(10)$					
$\overline{X}$	0.9997	0.9307	0.6412	0.2181	0.0038	0.2189	0.6410	0.9308	0.9998
TMd	0.9997	0.9160	0.5989	0.1864	0.0026	0.1877	0.5994	0.9163	0.9997
				$t(5)$					
$\overline{X}$	0.9992	0.9346	0.6470	0.2099	0.0065	0.2093	0.6480	0.9344	0.9991
TMd	0.9996	0.9305	0.6079	0.1680	0.0024	0.1673	0.6082	0.9304	0.9996
				$t(3)$					
$\overline{X}$	0.9976	0.9484	0.6708	0.1783	0.0107	0.1793	0.6695	0.9481	0.9977
TMd	0.9996	0.9576	0.6326	0.1224	0.0019	0.1233	0.6317	0.9578	0.9996
				$CN(0, 2, 0.05)$					
$\overline{X}$	0.9976	0.8842	0.6146	0.2654	0.0193	0.2658	0.6151	0.8843	0.9975
TMd	0.9976	0.8671	0.5794	0.2371	0.0133	0.2381	0.5798	0.8672	0.9976
				$CN(0, 2, 0.15)$					
$\overline{X}$	0.9844	0.8163	0.5872	0.3199	0.0760	0.3212	0.6049	0.8168	0.9843
TMd	0.9854	0.8075	0.5626	0.2894	0.0558	0.2899	0.5764	0.8077	0.9853
				$CN(0, 5, 0.05)$					
$\overline{X}$	0.9661	0.8379	0.6055	0.3047	0.0193	0.3051	0.6055	0.8378	0.9665
TMd	0.9905	0.8519	0.5769	0.2510	0.0133	0.2510	0.5769	0.8514	0.9906
				$CN(0, 5, 0.15)$					
$\overline{X}$	0.9487	0.7791	0.5864	0.3510	0.1755	0.3511	0.5855	0.7782	0.9487
TMd	0.9752	0.8308	0.5747	0.2699	0.0737	0.2700	0.5744	0.8305	0.9751
				$CN(0, 3, 0.1)$					
$\overline{X}$	0.9770	0.8160	0.5907	0.3188	0.0933	0.3190	0.5897	0.8176	0.9769
TMd	0.9870	0.8252	0.5690	0.2731	0.0475	0.2737	0.5681	0.8266	0.9869

We shall next compare the power functions of the quantile$-\overline{X}$ and $-TMd$ control charts, whenever monitoring the mean value of non-normal processes, for rational subgroups of size $n = 5$ and control limits placed at the quantiles $\chi_{0.001}$ and $\chi_{0.999}$. This is a more fair comparison situation, possible in a simulation, but implying, in practice, an *a priori* identification of the underlying model and/or the need to estimate the quantiles of the control statistic, with a high precision. In Table 6 we present the power functions of both charts for processes from the same families of models. A graphic visualization of part of Table 6 is given in Figure 7. There we picture the power functions of the two charts, $\overline{X}$ and TMd, now based on quantile control limits, built according to the specific model underlying the process. As may be seen, the

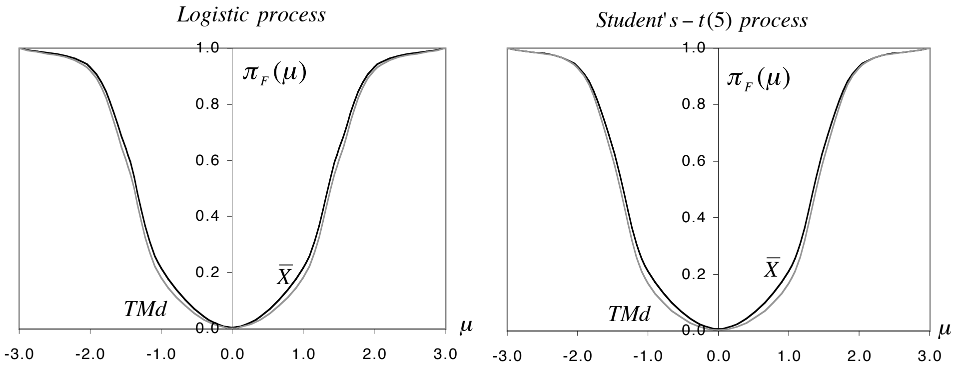

Figure 6: Power functions of the 3-sigma $\overline{X}$ and TMd control charts for $n = 5$, and for Logistic and Student's $t(5)$ processes.

quantile$-TMD-$chart is able to clearly overpass the quantile$-\overline{X}-$chart when the tails of the underlying model become heavier and heavier.

Table 6: Power functions of the $\overline{X}$ and the TMd charts with control limits at $\chi_{0.001}$ and $\chi_{0.999}$, whenever detecting changes in the mean value (μ) of the process ($n = 5$).

μ	-3.0	-2.0	-1.5	-1.0	0.0	1.0	1.5	2.0	3.0	LCL	UCL
					$Normal(\mu,1)$						
$\overline{X}$	1.0000	1.0000	0.9999	0.9165	0.0020	0.9171	0.9999	1.0000	1.0000	-1.382	1.382
TMd	1.0000	1.0000	0.9996	0.8863	0.0020	0.8934	0.9997	1.0000	1.0000	-1.442	1.425
					$Logistic(\mu,1)$						
$\overline{X}$	0.9995	0.8828	0.5232	0.1398	0.0020	0.1504	0.5420	0.8919	0.9994	-1.475	1.454
TMd	0.9993	0.8973	0.5450	0.1466	0.0020	0.1561	0.5619	0.9045	0.9996	-1.452	1.434
					$t(20)$						
$\overline{X}$	0.9997	0.9045	0.5739	0.1733	0.0020	0.1848	0.5904	0.9115	0.9998	-1.418	1.399
TMd	0.9995	0.8897	0.5489	0.1622	0.0020	0.1734	0.5666	0.8978	0.9996	-1.445	1.425
					$t(10)$						
$\overline{X}$	0.9994	0.8883	0.5364	0.1478	0.0020	0.1559	0.5483	0.8946	0.9994	-1.461	1.447
TMd	0.9995	0.8964	0.5518	0.1562	0.0020	0.1659	0.5662	0.9033	0.9996	-1.444	1.427
					$t(5)$						
$\overline{X}$	0.9968	0.7977	0.3618	0.0691	0.0020	0.0771	0.3884	0.8166	0.9973	-1.647	1.618
TMd	0.9995	0.9146	0.5616	0.1415	0.0020	0.1500	0.5797	0.9207	0.9996	-1.437	1.420
					$t(3)$						
$\overline{X}$	0.9665	0.2648	0.0401	0.0076	0.0020	0.0092	0.0531	0.3342	0.9746	-2.228	2.154
TMd	0.9996	0.9574	0.6291	0.1218	0.0020	0.1358	0.6567	0.9622	0.9996	-1.392	1.369
					$CN(0,5,0.1)$						
$\overline{X}$	0.1413	0.0341	0.0148	0.0062	0.0020	0.0068	0.0162	0.0366	0.1521	-3.944	3.891
TMd	0.2555	0.0295	0.0121	0.0053	0.0020	0.0058	0.0130	0.0325	0.2797	-3.408	3.360
					$CN(0,3,0.1)$						
$\overline{X}$	0.5525	0.1135	0.0403	0.0127	0.0020	0.0140	0.0441	0.1213	0.5753	-2.897	2.861
TMd	0.7221	0.1691	0.0493	0.0126	0.0020	0.0142	0.0555	0.1869	0.7430	-2.622	2.576

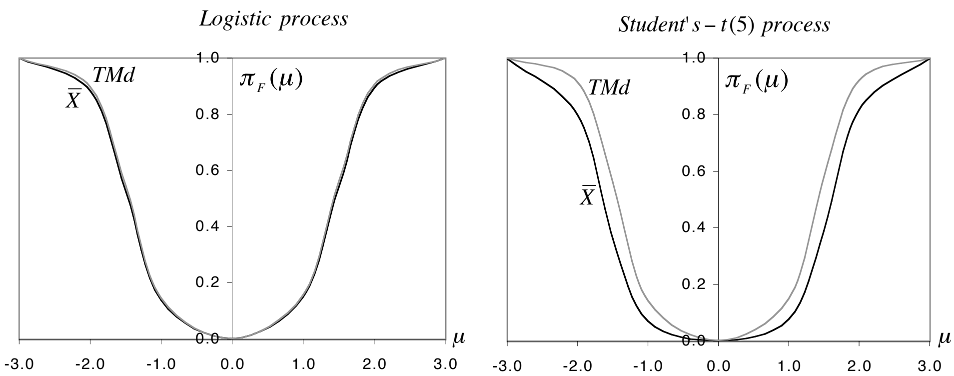

Figure 7: Power functions of the quantile$-\overline{X}$ and TMd control charts for $n = 5$, and for Logistic and Student's $t(5)$ processes.

5 Some overall conclusions

We finally summarize the main conclusions:

1. The total median TMd is a location estimator, highly efficient and robust for small-to-moderate rational subgroup sizes, the most usual ones in SQC.

2. The bootstrap median, which might appear as a serious alternative to the sample median shows the worst performance among the location estimators herewith considered.

3. The sample mean $\overline{X}$ is an efficient estimator of the mean value of a symmetric distribution with moderate tails, although it is not at all robust; for symmetric parents the sample median turns out to be the most robust estimator (among the ones considered) only for very small sample sizes, being then the total median the most robust one. If the parent model has not too heavy tails, i.e. if we compute the "degree of robustness" only on the basis of a smaller set of symmetric distributions, choosing a threshold in the tail-weight, the total median works better than the sample median, from a point of view of robustness, for sample sizes larger than a small value n which decreases as the threshold decreases.

4. For the estimation of the median of an asymmetric distribution we suggest the total median as a robust and efficient location estimator for small-to-moderate samples. Indeed, the TMd clearly overpasses all the other estimators for sample sizes $n > 5$.

5. The TMd statistic must however be carefully chosen, because in non-normal situations the alarm rate of such a control chart can be much smaller than expected, particularly if n is large. Together with the consideration of the TMd control statistic, the use of a rational subgroup size $n = 5$ is highly advisable in practice.

6. As expected, for normal data the TMd chart is less efficient than the equivalent $\overline{X}$-chart to monitor the location of a symmetric process; however, for large magnitudes of shift both charts are approximately equivalent, even for normal data. For other distributions, like for instance the Student's $t(3)$, the 3-sigma TMd chart performs better for large deviations from the target μ_0, being thus advisable in practice due to its robustness comparatively to the common $\overline{X}$ control chart, which provides a very high false alarm rate. For the same type of heavy-tailed distributions, the quantile$-TMd-$chart performs much better than the $\overline{X}$ chart.

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