Revisiting the role of the Jacknife methodology in the estimation of a positive tail index*

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Abstract. In this paper, and in a context of regularly varying tails, we analyse a generalization of the classical Hill estimator of a positive tail index. The members of this general class of estimators are not asymptotically more efficient than the original one, and we thus propose a Generalized Jackknife estimator based on two members of such a class. The Generalized Jackknife estimator is compared with the Hill estimator, both asymptotically and for finite samples.

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1 Introduction and preliminaries

Let $X_1, X_2, \ldots, X_n$ be independent random variables (r.v.'s) with common distribution function (d.f.) F, with a heavy upper tail, i.e. for large x,

$$1 - F(x) = x^{-1/\gamma} L(x), \tag{1.1}$$

where $L(x)$ is a slowly varying function, i.e., for every $x > 0$, $L(tx)/L(t) \to 1$ as $t \to \infty$. Consequently, $1 - F \in RV_{-1/\gamma}$, where RV_α stands for the class of regularly varying functions at infinity with index of regular variation equal to α, i.e., functions g with infinite right endpoint, and such that $\lim_{t\to\infty} g(tx)/g(t) = x^\alpha$, for all $x > 0$. F is thus in the max-domain of attraction of an *Extreme Value* (EV) d.f.,

$$EV_\gamma(x) := exp\left\{-(1+\gamma x)^{-1/\gamma}\right\},\ 1 + \gamma x > 0,\ \gamma \in \mathbb{R}, \tag{1.2}$$

with $\gamma > 0$. We denote such a fact by $F \in \mathcal{D}(G_\gamma)$.

Then the log-spacings

$$V_{ik} = \ln X_{n-i+1:n} - \ln X_{n-k:n},\ 1 \le i \le k,$$

where $X_{i:n}$ denotes the i-th ascending order statistic (o.s.) associated to the sample $\underline{X}_n = (X_1, X_2, \ldots, X_n)$, are, for $k = k_n \to \infty$, $k = o(n)$, as $n \to \infty$, approximately distributed as the k o.s.'s associated to an exponential random sample with mean value γ. This leads to the well known Hill estimator for γ (Hill, 1975),

$$\widehat{\gamma}_n^H(k) := \frac{1}{k}\sum_{i=1}^{k} \left[\ln X_{n-i+1:n} - \ln X_{n-k:n}\right].$$

Here we shall more generally work with the class of estimators,

$$\widehat{\gamma}_n^{(\alpha)}(k) := \frac{\alpha^2}{k^\alpha}\sum_{i=1}^{k} i^{\alpha-1} \ln\left(\frac{X_{n-i+1:n}}{X_{n-k:n}}\right), \quad \alpha \ge 1, \quad \left[\widehat{\gamma}_n^{(1)} \equiv \widehat{\gamma}_n^H\right]. \tag{1.3}$$

Apart from the first order condition,

$$F \in D(G_\gamma) \quad \text{iff} \quad 1 - F \in RV_{-1/\gamma} \quad \text{iff} \quad U \in RV_\gamma, \tag{1.4}$$

where

$$U(t) := F^{\leftarrow}(1 - 1/t),\ t > 1, \qquad F^{\leftarrow}(u) = \inf\{x : F(x) \geq u\},$$

we shall assume, in order to be able to derive the asymptotic normality of the estimators under study, the validity of a second order condition. More specifically, we shall assume that there exists a function A and a parameter $\rho < 0$ such that

$$\lim_{t\to\infty} \frac{\ln U(tx) - \ln U(t) - \gamma \ln x}{A(t)} = \frac{x^\rho - 1}{\rho}, \tag{1.5}$$

for every $x > 0$.

In section 2 of this paper we deal with the asymptotic behaviour of the class of estimators in (1.3) and compare them asymptotically at their optimal levels. As expected, the value $\alpha = 1$ provides the optimal results within this class. This led us to the introduction, in section 3, of a *Generalized Jackknife* estimator of γ associated to any pair of estimators in (1.3). We first come to a general class of Generalized Jackknife r.v.'s, dependent on a *tuning* parameter α and on the second order parameter ρ in (1.5). Such a second order parameter is either mispecified at $\rho = -1$ or adequately estimated through $\widehat{\rho}$, explicited later on. The class of tail index estimators is given by

$$\widehat{\gamma}_{n,\alpha}^{GJ(\widehat{\rho})}(k) = -\frac{\alpha(1-\widehat{\rho})}{\widehat{\rho}(\alpha-1)} \left\{ \widehat{\gamma}_n^{(1)}(k) - \frac{\alpha - \widehat{\rho}}{\alpha(1-\widehat{\rho})}\ \widehat{\gamma}_n^{(\alpha)}(k) \right\}, \quad \alpha \neq 1, \tag{1.6}$$

with $\widehat{\gamma}_n^{(\alpha)}(k)$ given in (1.3). Indication on the choice of α is also provided. Mispecification of $\widehat{\rho} \equiv -1$ in (1.6) is also considered. Finally, in section 4, we proceed to a comparison of the proposed estimators, for finite samples, through the use of Monte Carlo techniques.

2 The asymptotic behaviour of the initial estimators

We first state the following lemmas

Lemma 2.1. *Let us denote*

$$W_k^{(\alpha)} := \frac{1}{k^\alpha} \sum_{i=1}^{k} i^{\alpha-1} E_{k-i+1:k}, \quad \alpha \geq 1, \tag{2.1}$$

where $E_{i:k}$, $1 \leq i \leq k$, are the k ascending o.s. associated to a standard exponential random sample of size k. As $k \to \infty$,

$$\mathbb{E}\left[W_k^{(\alpha)}\right] = \frac{1}{\alpha^2} + o\left(1/\sqrt{k}\right), \tag{2.2}$$

$$\mathbb{V}ar\left[W_k^{(\alpha)}\right] \sim \frac{1}{\alpha^2(2\alpha-1)\ k}, \tag{2.3}$$

i.e.,

$$\alpha\sqrt{(2\alpha-1)\ k}\left(W_k^{(\alpha)} - \frac{1}{\alpha^2}\right) =: Z_k^{(\alpha)} \tag{2.4}$$

is asymptotically a standard normal r.v.

Moreover, the second order structure between $W_k^{(\alpha)}$ and $W_k^{(\beta)}$ is such that

$$\mathbb{C}ov\left[W_k^{(\alpha)},\ W_k^{(\beta)}\right] \sim \frac{1}{\alpha\beta(\alpha+\beta-1)\ k}, \tag{2.5}$$

and consequently,

$$\mathbb{C}ov\left(Z_k^{(\alpha)}, Z_k^{(\beta)}\right) \sim \frac{\sqrt{(2\alpha-1)(2\beta-1)}}{\alpha+\beta-1}. \tag{2.6}$$

Proof. Since $\mathbb{E}\left[E_{k-i+1:k}\right] = \sum_{j=i}^{k} 1/j$,

$$\mathbb{E}\left[W_k^{(\alpha)}\right] \quad = \quad \frac{1}{k^\alpha}\sum_{i=1}^{k} i^{\alpha-1} \sum_{j=i}^{k}\frac{1}{j} = \frac{1}{k^\alpha}\sum_{j=1}^{k}\frac{1}{j}\sum_{i=1}^{j} i^{\alpha-1}.$$

From the fact that $\sum_{i=1}^k i^{\alpha-1} = k^\alpha/\alpha + O(k^{\alpha-1})$, (2.2) follows. Also, denoting $\sigma_{ij:k} := \mathbb{C}ov\left(E_{k-i+1:k}, E_{k-j+1:k}\right) = \mathbb{V}ar\left(E_{k-\max(i,j)+1:k}\right) =: \sigma^2_{\max(i,j):k}$, and writing $W_k^{(\alpha)}$ in (2.1) as

$$W_k^{(\alpha)} = \frac{1}{k}\sum_{i=1}^{k}\left(\frac{i}{k}\right)^{\alpha-1} E_{k-i+1:k},$$

we get for $k\ \mathbb{V}ar\left[W_k^{(\alpha)}\right]$:

$$
\begin{aligned}
& \frac{1}{k}\sum_{i=1}^{k}\sum_{j=1}^{k}\left(\frac{i}{k}\right)^{\alpha-1}\left(\frac{j}{k}\right)^{\alpha-1}\sigma_{ij:k} \\
= \ & \frac{1}{k}\sum_{i=1}^{k}\left(\sum_{j=i}^{k}\left(\frac{i}{k}\ \frac{j}{k}\right)^{\alpha-1}\sigma_{j:k}^2+\sum_{j=1}^{i-1}\left(\frac{i}{k}\ \frac{j}{k}\right)^{\alpha-1}\sigma_{i:k}^2\right) \\
= \ & \frac{1}{k}\sum_{i=1}^{k}\left(\sum_{j=i}^{k}\left(\frac{i}{k}\ \frac{j}{k}\right)^{\alpha-1}\sum_{l=j}^{k}\frac{1}{l^2}+\sum_{j=1}^{i-1}\left(\frac{i}{k}\ \frac{j}{k}\right)^{\alpha-1}\sum_{l=i}^{k}\frac{1}{l^2}\right) \\
= \ & \frac{1}{k}\sum_{l=1}^{k}\frac{1}{l}\sum_{j=1}^{l}\frac{1}{j}\sum_{i=1}^{j}\frac{j}{l}\left(\frac{i}{k}\frac{j}{k}\right)^{\alpha-1}+\frac{1}{k}\sum_{l=1}^{k}\frac{1}{l}\sum_{i=1}^{l}\frac{1}{i}\sum_{j=1}^{i-1}\frac{i}{l}\left(\frac{i}{k}\frac{j}{k}\right)^{\alpha-1} \\
\underset{n\to\infty}{\longrightarrow} \ & 2\iiint_{[0,1]^3} y(xy)^{\alpha-1}(xyz)^{\alpha-1}=\frac{1}{\alpha^2(2\alpha-1)},
\end{aligned}
$$

and both (2.3) and (2.4) follows. Similar computations lead us to (2.5). Indeed,

$$
\begin{aligned}
k\ Cov\left(W_k^{(\alpha)},W_k^{(\beta)}\right) \ & \underset{n\to\infty}{\longrightarrow} \iiint_{[0,1]^3} y(xy)^{\beta-1}(xyz)^{\alpha-1}+\iiint_{[0,1]^3} y(xy)^{\alpha-1}(xyz)^{\beta-1} \\
& = \frac{1}{(\alpha+\beta)(\alpha+\beta-1)}\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)=\frac{1}{\alpha\beta(\alpha+\beta-1)},
\end{aligned}
$$

and (2.6) follows straighforwardly. □

Lemma 2.2. *The following identities hold true:*

$$
\sum_{i=1}^{k}\frac{\Gamma(i-\rho)}{\Gamma(i)}=\frac{\Gamma(k+1-\rho)}{(1-\rho)\Gamma(k)}. \tag{2.7}
$$

$$
\sum_{i=1}^{k}\frac{i\Gamma(i-\rho)}{\Gamma(i)}=\frac{\Gamma(k+1-\rho)}{(2-\rho)\Gamma(k)}\left\{k+\frac{1}{1-\rho}\right\}. \tag{2.8}
$$

$$
\sum_{i=1}^{k}\frac{i^2\Gamma(i-\rho)}{\Gamma(i)}=\frac{\Gamma(k+1-\rho)}{(3-\rho)\Gamma(k)}\left\{k^2+\frac{3k}{(2-\rho)}+\frac{1+\rho}{(1-\rho)(2-\rho)}\right\}. \tag{2.9}
$$

More generally we may say that

$$
\frac{\Gamma(k)}{\Gamma(k-\rho+1)}\ \frac{1}{k^\alpha}\sum_{i=1}^{k}\frac{i^\alpha\Gamma(i-\rho)}{\Gamma(i)}=\frac{1}{\alpha+1-\rho}\ (1+o(1)). \tag{2.10}
$$

Proof. Let us denote $\{E_j\}_{j\geq 1}$ a sequence of i.i.d. standard exponential r.v.'s and the set of exponential o.s $E_{k-i+1:k},\ 1\leq i\leq k$. Rényi's representation theorem enables us to write

$$E_{k-i+1:k} \stackrel{d}{=} \sum_{j=1}^{k-i+1} \frac{E_j}{k-j+1} \stackrel{d}{=} \sum_{j=i}^{k} \frac{E_j}{j},\ 1\leq i\leq k. \tag{2.11}$$

Then, since

$$\frac{1}{k}\sum_{i=1}^{k} \frac{e^{\rho E_{k-i+1:k}}-1}{\rho} = \frac{1}{k}\sum_{i=1}^{k} \frac{e^{\rho E_i}-1}{\rho},$$

and $E\left[\frac{e^{\rho E}-1}{\rho}\right] = \frac{1}{1-\rho}$, we have $E\left[\frac{1}{k}\sum_{i=1}^{k}\frac{e^{\rho E_{k-i+1:k}}-1}{\rho}\right] = \frac{1}{1-\rho}$. The use of Rényi's representation (2.11), or the direct computation of $E\left[e^{\rho E_{k-i+1:k}}\right]$, which is equal to $\Gamma(k+1)\Gamma(i-\rho)/(\Gamma(k-\rho+1)\Gamma(i))$, enables us to obtain immediately (2.7). From (2.7), and writing artificially $i=(i-1)+1$, (2.8) follows. Similarly, from (2.7) and (2.8), and writing artificially $i^2=(i-1)(i-2)+3(i-1)+1$, we get (2.9), and asymptotically we derive (2.10).

□

The main result of this section is given in the following:

Theorem 2.1. *Under the first order condition in* (1.4)*, and for k intermediate, i.e., such that*

$$k=k_n\to\infty,\quad k=o(n),\ \textit{as } n\to\infty,$$

the statistics $\widehat{\gamma}_n^{(\alpha)}(k)$ in (1.3) *are consistent for the estimation of the tail index γ. Moreover, under the second order framework in* (1.5) *we have the validity of the asymptotic distributional reprepresentations*

$$\widehat{\gamma}_n^{(\alpha)}(k) \stackrel{d}{=} \gamma + \frac{\gamma\alpha}{\sqrt{(2\alpha-1)\ k}}\ Z_k^{(\alpha)} + \frac{\alpha}{\alpha-\rho}\ A(n/k)(1+o_p(1)),$$

where $Z_k^{(\alpha)}$ is the asymptotically standard normal random variable in (2.4)*.*

Proof. We may write

$$\frac{1}{k^\alpha}\sum_{i=1}^{k} i^{\alpha-1}\left\{\ln\frac{X_{n-i+1:n}}{X_{n-k:n}}\right\} = \frac{1}{k^\alpha}\sum_{i=1}^{k} i^{\alpha-1}\left\{\ln\frac{U(Y_{n-i+1:n})}{U(Y_{n-k:n}}\right\}$$

where $Y_{i:n}$, $1 \le i \le n$, are the ascending o.s. associated to an i.i.d. standard Pareto sample of size n, from a parent $F_Y(y) = 1 - y^{-1}$, $y \ge 1$.

Since $Y_{n-i+1:n}/Y_{n-k:n} \stackrel{d}{=} Y_{k-i+1:k}$, $1 \le i \le k$, $\ln Y_{i:k} \stackrel{d}{=} E_{i:k}$, and from (1.5), which enables us to write

$$\ln \frac{U(tx)}{U(t)} = \gamma \ln x + \frac{x^\rho - 1}{\rho} A(t)(1 + o(1)),$$

for every $x > 0$, and as $t \to \infty$,, we get

$$\frac{1}{k^\alpha} \sum_{i=1}^{k} i^{\alpha-1} \left\{ \ln \frac{X_{n-i+1:n}}{X_{n-k:n}} \right\} = \frac{1}{k^\alpha} \sum_{i=1}^{k} \imath^{\alpha-1} \left\{ \ln \frac{U(Y_{n-k:n}\, Y_{k-i+1:k})}{U(Y_{n-k:n})} \right\}$$

$$= \frac{1}{k^\alpha} \sum_{i=1}^{k} i^{\alpha-1} \left\{ \gamma E_{k-i+1:k} + \left(\frac{e^{\rho E_{k-i+1:k}} - 1}{\rho} \right) A(n/k)(1 + o_p(1)) \right\}$$

$$= \gamma\, W_k^{(\alpha)} + \frac{1}{k^\alpha} \sum_{i=1}^{k} i^{\alpha-1} \left(\frac{e^{\rho E_{k-i+1:k}} - 1}{\rho} \right) A(n/k)(1 + o_p(1)),$$

with $W_k^{(\alpha)}$ given in (2.1).

The law of large numbers, together with the result (2.10) in Lemma 2.2, enables us to say that

$$\frac{1}{k^\alpha} \sum_{i=1}^{k} i^{\alpha-1} \left(\frac{e^{\rho E_{k-i+1:k}} - 1}{\rho} \right) \underset{n\to\infty}{\overset{p}{\longrightarrow}} \frac{1}{\alpha(\alpha-\rho)}.$$

Now the use of this relation and of (2.4) enables us to write

$$\frac{1}{k^\alpha} \sum_{i=1}^{k} i^{\alpha-1} \left\{ \ln \frac{X_{n-i+1:n}}{X_{n-k:n}} \right\} = \frac{\gamma}{\alpha^2} + \frac{\gamma\, Z_k^{(\alpha)}}{\alpha\sqrt{(2\alpha-1)k}} + \frac{1}{\alpha(\alpha-\rho)} A(n/k)(1 + o_p(1)),$$

where $Z_k^{(\alpha)}$, given in (2.4), is asymptotically standard normal, and the result follows. □

We next proceed to an asymptotic comparison of the estimators at their optimal levels in the lines of de Haan and Peng (1998), and also Gomes et al. (2000b, 2002b) for a set of Generalized Jackknife statistics, Gomes and

Martins (2001) and Caeiro and Gomes (2002) for specifically built "asymptotically unbiased" estimators of the tail index, and Gomes and Martins (2002a) for "Maximum Likelihood" and "Least Squares" estimators of γ, based on the scaled log-spacings. Suppose $\widehat{\gamma}_n^\bullet(k)$ is a general semi-parametric estimator of the tail index, for which the distributional representation

$$\widehat{\gamma}_n^\bullet(k) = \gamma + \frac{\sigma_\bullet}{\sqrt{k}} Z_n^\bullet + b_\bullet\ A(n/k) + o_p(A(n/k)) \tag{2.12}$$

holds for any intermediate k, and where $Z_n^\bullet$ is an asymptotically standard normal r.v.; then we have

$$\sqrt{k}\,[\widehat{\gamma}_n^\bullet(k) - \gamma] \stackrel{d}{\to} N(\lambda b_\bullet, \sigma_\bullet^2),\ \text{ as } n \to \infty,$$

provided k is such that $\sqrt{k}\ A(n/k) \to \lambda$, finite, as $n \to \infty$. In this situation we write $Bias_\infty\,[\widehat{\gamma}_n^\bullet(k)] := b_\bullet\ A(n/k)$ and $Var_\infty\,[\widehat{\gamma}_n^\bullet(k)] := \sigma_\bullet^2/k$. The so-called Asymptotic Mean Squared Error ($AMSE$) is then given by

$$AMSE\,[\widehat{\gamma}_n^\bullet(k)] := \frac{\sigma_\bullet^2}{k} + b_\bullet^2\ A^2(n/k).$$

Using regular variation theory it may be proved that, whenever $b_\bullet \neq 0$, there exists a function $\varphi(n)$, dependent only on the underlying model, and not on the estimator, such that

$$\lim_{n\to\infty} \varphi(n)\ AMSE\,[\widehat{\gamma}_{n0}^\bullet] = \frac{2\rho - 1}{2\rho}\left(\sigma_\bullet^2\right)^{-\frac{2\rho}{1-2\rho}}\left(b_\bullet^2\right)^{\frac{1}{1-2\rho}} =: LMSE\,[\widehat{\gamma}_{n0}^\bullet]\,,$$

where $\widehat{\gamma}_{n0}^\bullet := \widehat{\gamma}_n^\bullet\,(k_0^\bullet(n))$ and $k_0^\bullet(n) := \arg\inf_k AMSE\,[\widehat{\gamma}_n^\bullet(k)]$.

It is then sensible to consider the following:

Definition 2.1. *Given two biased estimators $\widehat{\gamma}_n^{(1)}(k)$ and $\widehat{\gamma}_n^{(2)}(k)$, for which distributional representations of the type* (2.12) *hold, with constants (σ_1, b_1) and (σ_2, b_2), $b_1, b_2 \neq 0$, respectively, both computed at their optimal levels, the Asymptotic Root Efficiency (AREFF) of $\widehat{\gamma}_{n0}^{(1)}$ relatively to $\widehat{\gamma}_{n0}^{(2)}$ is*

$$AREFF_{1|2} \equiv AREFF_{\widehat{\gamma}_{n0}^{(1)}|\widehat{\gamma}_{n0}^{(2)}} := \sqrt{\frac{LMSE\left[\widehat{\gamma}_{n0}^{(2)}\right]}{LMSE\left[\widehat{\gamma}_{n0}^{(1)}\right]}}.$$

Then we have:

Proposition 2.1. *For every* $\alpha > 1$,

$$AREFF_{\alpha|1} = \left(\frac{(\alpha-\rho)(2\alpha-1)^{-\rho}}{(1-\rho)\ \alpha^{1-2\rho}} \right)^{\frac{2}{1-2\rho}} < 1.$$

More than this: both the asymptotic bias and the asymptotic variance of $\widehat{\gamma}_n^{(\alpha)}(k)$, $\alpha > 1$, *are bigger than the asymptotic bias and the asymptotic variance, respectively, of the Hill estimator.*

This leads us to the consideration of a Generalized Jackknife estimator associated to any pair $(\alpha, \beta) \in [1, \infty)^2$. For details on the Generalized Jackknife estimation see Gray and Schucany (1972).

3 Generalized Jackknife estimators of the tail index

Let us think on two of the estimators in (1.3),

$$\widehat{\gamma}_n^{(\alpha)}(k) \text{ and } \widehat{\gamma}_n^{(\beta)}(k), \quad \alpha \neq \beta, \quad \alpha,\ \beta \geq 1.$$

The quotient between the dominant component of bias ot these estimators is given by

$$q_{\alpha,\beta}(\rho) = \frac{\alpha(\beta-\rho)}{\beta(\alpha-\rho)},$$

dependent on ρ, unknown. We may thus mispecify ρ, for instance in -1, a central and proeminent value of this second order parameter, as done before in several papers, among which we mention Feuerverger and Hall (1999), Gomes et al. (2000b, 2002b), Caeiro and Gomes (2002), Gomes and Martins (2002a). We may also estimate ρ adequately, either internally as in Beirlant et al. (1999) and Feuerverger and Hall (1999) or externally, as done successfully in Gomes and Martins (2002b), through any of the ρ-estimators available in the literature, like the ones in Gomes at al. (2002a) and Fraga Alves et al. (2003).

3.1 Mispecification of ρ

If we mispecify $\rho = -1$ we have the bias' quotient

$$q_{\alpha,\beta} \equiv q_{\alpha,\beta}(-1) = \frac{\alpha(\beta+1)}{\beta(\alpha+1)}.$$

We thus get the Generalized Jackknife class of estimators

$$\begin{aligned}\widehat{\gamma}^{GJ(-1)}_{n,\alpha,\beta}(k) &:= \frac{\widehat{\gamma}^{(\alpha)}_n(k) - q_{\alpha,\beta}\widehat{\gamma}^{(\beta)}_n(k)}{1-q_{\alpha,\beta}} \\ &= \frac{1}{\beta-\alpha}\left(\beta(\alpha+1)\widehat{\gamma}^{(\alpha)}_n(k) - \alpha(\beta+1)\widehat{\gamma}^{(\beta)}_n(k)\right), \qquad (3.1)\end{aligned}$$

dependent on the tuning parameters α and β ($\alpha,\ \beta \geq 1$).

On the basis of Theorem 2.1 and Lemma 2.1, we get straightforwardly the general distributional behaviour of the new estimators $\widehat{\gamma}^{GJ(-1)}_{n,\alpha,\beta}(k)$ in (3.1):

Theorem 3.1. *Under the conditions of Theorem 2.1 we get the following asymptotic distributional representation for the Generalized Jackknife estimators in (3.1),*

$$\widehat{\gamma}^{GJ(-1)}_{n,\alpha,\beta}(k) \stackrel{d}{=} \gamma + \frac{\gamma\sigma^{GJ(-1)}_{\alpha,\beta}}{\sqrt{k}}\ Z^{GJ(-1)}_{k,\alpha,\beta} + \frac{\alpha\beta(1+\rho)}{(\alpha-\rho)(\beta-\rho)}\ A(n/k)(1+o_p(1)) \qquad (3.2)$$

with $Z^{GJ(-1)}_{k,\alpha,\beta}$ asymptotically standard normal, and

$$\begin{aligned}\sigma^{GJ(-1)}_{\alpha,\beta} &= \frac{\alpha\beta(\alpha+1)}{|\beta-\alpha|}\sqrt{\frac{1}{2\alpha-1} + \frac{(\beta+1)^2}{(\alpha+1)^2(2\beta-1)} - \frac{2(\beta+1)}{(\alpha+1)(\alpha+\beta-1)}} \\ &= \alpha\beta\ \sqrt{\frac{2\alpha\beta-\beta-\alpha+5}{(2\alpha-1)(2\beta-1)(\alpha+\beta-1)}}, \quad \alpha \neq \beta. \qquad (3.3)\end{aligned}$$

For the particular case $\beta = 1$, we are dealing with the class of estimators

$$\widehat{\gamma}^{GJ(-1)}_{n,\alpha} := \frac{1}{\alpha-1}\left(2\alpha\widehat{\gamma}^{(1)}_n(k) - (\alpha+1)\widehat{\gamma}^{(\alpha)}_n(k)\right), \quad \alpha > 1, \qquad (3.4)$$

and we get, also with $Z^{GJ(-1)}_{k,\alpha}$ asymptotically standard normal,

$$\widehat{\gamma}^{GJ(-1)}_{n,\alpha} \stackrel{d}{=} \gamma + \frac{\gamma}{\sqrt{k}}\sqrt{\frac{\alpha(\alpha+4)}{2\alpha-1}}Z^{GJ(-1)}_{k,\alpha} + \frac{\alpha(1+\rho)}{(1-\rho)(\alpha-\rho)}\ A(n/k)(1+o_p(1)). \qquad (3.5)$$

Remark 3.1. *Notice that among the values* $(\alpha, \beta) \in [1, \infty)^2$, $\alpha \neq \beta$, *the pair providing the smallest value for* $\sigma_{\alpha,\beta}^{GJ(-1)}$ *in* (3.3) *is* $(\alpha, \beta) = (1, 2)$, *or equivalently* $(\alpha, \beta) = (2, 1)$. *In Figure 1 we present the asymptotic characteristics of the estimator* $\widehat{\gamma}_{n,\alpha}^{GJ(-1)}$, *i.e., of*

$$\sigma_{\alpha}^{GJ(-1)} = \sqrt{\alpha(\alpha + 4)/(2\alpha - 1)}$$

as a function of α, *and the squared bias term*

$$b_{\alpha}^{2}(\rho) = (\alpha(1 + \rho)/((1 - \rho)(\alpha - \rho)))^2$$

as a function of $|\rho|$, *for* $\alpha = 1.5, 2, 4$.

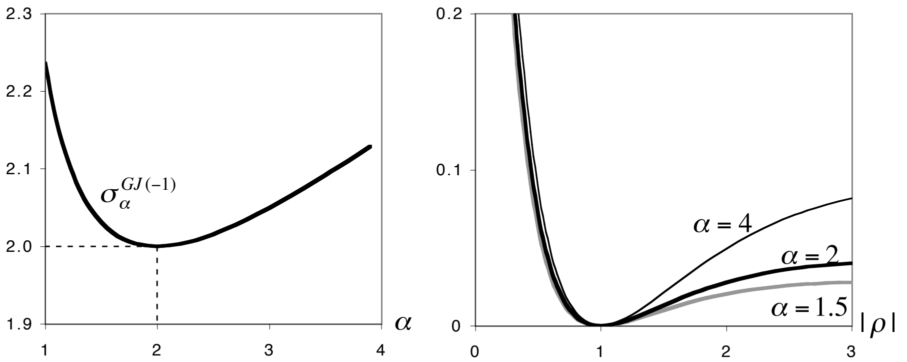

Figure 1: Asymptotic characteristics of $\widehat{\gamma}_{n,\alpha}^{GJ(-1)}$ — standard deviations (*left*) and squared bias (*right*).

Remark 3.2. *The estimator with smallest asymptotic variance in the class of estimators herewith considered, i.e.,*

$$\widehat{\gamma}_{n,2}^{GJ(-1)}(k) := 4\widehat{\gamma}_{n}^{(1)}(k) - 3\widehat{\gamma}_{n}^{(2)}(k) \tag{3.6}$$

is asymptotically equivalent to the "Maximum Likelihood" estimator studied in Gomes and Martins (2002a), and given by

$$\widehat{\gamma}_{n}^{ML}(k) := \frac{1}{k}\sum_{i=1}^{k} U_i - \left(\frac{1}{k}\sum_{i=1}^{k} iU_i\right) \frac{\sum\limits_{i=1}^{k}(2i-k-1)U_i}{\sum\limits_{i=1}^{k} i(2i-k-1)U_i},$$

where $U_i = i\left[\ln X_{n-i+1:n} - \ln X_{n-i:n}\right],\ 1 \le i \le k,$ *are the scaled log-spacings.*

An asymptotic comparison of $\widehat{\gamma}_{n,\alpha}^{GJ(-1)}$ and the Hill estimator $\widehat{\gamma}_n^H$ both computed at their optimal levels, enable us to state the following result:

Theorem 3.2. *For* $\rho \neq -1$*, we get the following asymptotic efficiency relatively to the Hill estimator:*

$$AREFF_{GJ_\alpha|H} = \left(\left(\frac{\alpha(\alpha+4)}{2\alpha-1}\right)^{\rho}\frac{\alpha-\rho}{\alpha|1+\rho|}\right)^{\frac{1}{1-2\rho}}.$$

The $AREFF$ measure is presented in Figure 2.

ρ \ α	1.00	1.05	1.10	1.15	1.20	1.25	1.30	1.35	1.40	1.45	1.50	1.55	1.60	1.65	1.70	1.75	1.80	1.85	1.90	1.95	2.00	2.05	2.10	2.15	2.20	2.25	2.30	2.35	2.40	2.45	2.50
	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
-0.05	1.02	1.02	1.02	1.02	1.02	1.02	1.01	1.01	1.01	1.01	1.01	1.01	1.01	1.01	1.01	1.01	1.01	1.01	1.01	1.01	1.01	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
-0.10	1.03	1.03	1.03	1.03	1.03	1.03	1.03	1.03	1.03	1.02	1.02	1.02	1.02	1.02	1.02	1.02	1.02	1.02	1.02	1.01	1.01	1.01	1.01	1.01	1.01	1.01	1.01	1.01	1.01	1.00	1.00
-0.15	1.05	1.05	1.05	1.05	1.04	1.04	1.04	1.04	1.04	1.04	1.04	1.03	1.03	1.03	1.03	1.03	1.03	1.03	1.02	1.02	1.02	1.02	1.02	1.02	1.02	1.01	1.01	1.01	1.01	1.01	1.01
-0.20	1.06	1.06	1.06	1.06	1.06	1.06	1.05	1.05	1.05	1.05	1.05	1.05	1.04	1.04	1.04	1.04	1.04	1.03	1.03	1.03	1.03	1.03	1.03	1.02	1.02	1.02	1.02	1.02	1.02	1.02	1.01
-0.25	1.07	1.07	1.07	1.07	1.07	1.07	1.07	1.07	1.06	1.06	1.06	1.06	1.06	1.05	1.05	1.05	1.05	1.05	1.04	1.04	1.04	1.04	1.04	1.03	1.03	1.03	1.03	1.03	1.03	1.02	1.02
-0.30	1.09	1.09	1.09	1.09	1.09	1.08	1.08	1.08	1.08	1.08	1.07	1.07	1.07	1.07	1.06	1.06	1.06	1.06	1.06	1.05	1.05	1.05	1.05	1.05	1.04	1.04	1.04	1.04	1.04	1.03	1.03
-0.35	1.10	1.10	1.10	1.10	1.10	1.10	1.10	1.10	1.09	1.09	1.09	1.09	1.08	1.08	1.08	1.08	1.07	1.07	1.07	1.07	1.06	1.06	1.06	1.06	1.06	1.05	1.05	1.05	1.05	1.04	1.04
-0.40	1.12	1.12	1.12	1.12	1.12	1.12	1.11	1.11	1.11	1.11	1.11	1.10	1.10	1.10	1.10	1.09	1.09	1.09	1.09	1.08	1.08	1.08	1.08	1.07	1.07	1.07	1.07	1.06	1.06	1.06	1.06
-0.45	1.14	1.14	1.14	1.14	1.14	1.14	1.13	1.13	1.13	1.13	1.12	1.12	1.12	1.12	1.11	1.11	1.11	1.11	1.10	1.10	1.10	1.09	1.09	1.09	1.09	1.08	1.08	1.08	1.08	1.07	1.07
-0.50	1.16	1.16	1.16	1.16	1.16	1.16	1.16	1.15	1.15	1.15	1.15	1.14	1.14	1.14	1.13	1.13	1.13	1.13	1.12	1.12	1.12	1.12	1.11	1.11	1.11	1.10	1.10	1.10	1.10	1.09	1.09
-0.55	1.18	1.18	1.18	1.18	1.18	1.18	1.18	1.18	1.18	1.17	1.17	1.17	1.17	1.16	1.16	1.16	1.15	1.15	1.15	1.15	1.14	1.14	1.14	1.13	1.13	1.13	1.12	1.12	1.12	1.12	1.11
-0.60	1.21	1.21	1.21	1.21	1.21	1.21	1.21	1.21	1.21	1.20	1.20	1.20	1.20	1.19	1.19	1.19	1.18	1.18	1.18	1.17	1.17	1.17	1.16	1.16	1.16	1.16	1.15	1.15	1.15	1.14	1.14
-0.65	1.25	1.25	1.25	1.25	1.25	1.25	1.25	1.24	1.24	1.24	1.24	1.23	1.23	1.23	1.22	1.22	1.22	1.22	1.21	1.21	1.21	1.20	1.20	1.20	1.19	1.19	1.19	1.18	1.18	1.18	1.17
-0.70	1.29	1.29	1.29	1.29	1.29	1.29	1.29	1.29	1.29	1.28	1.28	1.28	1.28	1.27	1.27	1.27	1.26	1.26	1.26	1.25	1.25	1.25	1.24	1.24	1.24	1.23	1.23	1.23	1.22	1.22	1.22
-0.75	1.34	1.35	1.35	1.35	1.35	1.35	1.35	1.35	1.34	1.34	1.34	1.34	1.33	1.33	1.33	1.32	1.32	1.32	1.31	1.31	1.30	1.30	1.30	1.29	1.29	1.29	1.28	1.28	1.28	1.27	1.27
-0.80	1.42	1.42	1.43	1.43	1.43	1.43	1.43	1.42	1.42	1.42	1.42	1.41	1.41	1.41	1.40	1.40	1.39	1.39	1.39	1.38	1.38	1.38	1.37	1.37	1.36	1.36	1.36	1.35	1.35	1.35	1.34
-0.85	1.53	1.53	1.54	1.54	1.54	1.54	1.54	1.53	1.53	1.53	1.53	1.52	1.52	1.52	1.51	1.51	1.50	1.50	1.50	1.49	1.49	1.48	1.48	1.48	1.47	1.47	1.46	1.46	1.46	1.45	1.45
-0.90	1.71	1.71	1.72	1.72	1.72	1.72	1.72	1.72	1.71	1.71	1.71	1.70	1.70	1.70	1.69	1.69	1.68	1.68	1.67	1.67	1.66	1.66	1.66	1.65	1.65	1.64	1.64	1.63	1.63	1.62	1.62
-0.95	2.09	2.10	2.10	2.10	2.11	2.11	2.10	2.10	2.10	2.10	2.09	2.09	2.08	2.08	2.07	2.07	2.06	2.06	2.05	2.05	2.04	2.03	2.03	2.02	2.02	2.01	2.01	2.00	1.99	1.99	1.98
-1.00	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞
-1.05	1.92	1.93	1.94	1.94	1.94	1.94	1.94	1.94	1.94	1.93	1.93	1.93	1.92	1.92	1.91	1.91	1.90	1.90	1.89	1.89	1.88	1.88	1.87	1.87	1.86	1.86	1.85	1.85	1.84	1.84	1.83
-1.10	1.49	1.50	1.50	1.50	1.51	1.51	1.51	1.50	1.50	1.50	1.50	1.50	1.49	1.49	1.49	1.48	1.48	1.47	1.47	1.47	1.46	1.46	1.45	1.45	1.45	1.44	1.44	1.43	1.43	1.43	1.42
-1.15	1.28	1.29	1.29	1.29	1.29	1.29	1.29	1.29	1.29	1.29	1.29	1.29	1.28	1.28	1.28	1.27	1.27	1.27	1.26	1.26	1.26	1.25	1.25	1.25	1.24	1.24	1.24	1.23	1.23	1.23	1.22
-1.20	1.15	1.15	1.16	1.16	1.16	1.16	1.16	1.16	1.16	1.16	1.16	1.16	1.15	1.15	1.15	1.15	1.14	1.14	1.14	1.13	1.13	1.13	1.12	1.12	1.12	1.11	1.11	1.11	1.10	1.10	1.10
-1.25	1.05	1.06	1.06	1.07	1.07	1.07	1.07	1.07	1.07	1.07	1.07	1.06	1.06	1.06	1.06	1.05	1.05	1.05	1.05	1.04	1.04	1.04	1.03	1.03	1.03	1.03	1.02	1.02	1.02	1.01	1.01
-1.30	0.98	0.99	0.99	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	0.99	0.99	0.99	0.99	0.99	0.98	0.98	0.98	0.98	0.97	0.97	0.97	0.97	0.96	0.96	0.96	0.95	0.95	0.95	0.95
-1.35	0.93	0.94	0.94	0.94	0.94	0.94	0.95	0.94	0.94	0.94	0.94	0.94	0.94	0.94	0.93	0.93	0.93	0.93	0.93	0.92	0.92	0.92	0.92	0.91	0.91	0.91	0.91	0.90	0.90	0.90	0.89
-1.40	0.89	0.89	0.89	0.90	0.90	0.90	0.90	0.90	0.90	0.90	0.90	0.90	0.90	0.89	0.89	0.89	0.89	0.89	0.88	0.88	0.88	0.88	0.87	0.87	0.87	0.87	0.86	0.86	0.86	0.86	0.85
-1.45	0.85	0.85	0.86	0.86	0.86	0.86	0.86	0.86	0.86	0.86	0.86	0.86	0.86	0.86	0.86	0.85	0.85	0.85	0.85	0.85	0.84	0.84	0.84	0.84	0.83	0.83	0.83	0.83	0.82	0.82	0.82
-1.50	0.82	0.82	0.83	0.83	0.83	0.83	0.83	0.83	0.83	0.83	0.83	0.83	0.83	0.83	0.83	0.82	0.82	0.82	0.82	0.82	0.81	0.81	0.81	0.81	0.80	0.80	0.80	0.80	0.80	0.79	0.79

$AREFF_{\alpha|H} \geq 2$ $1.5 \leq AREFF_{\alpha|H} < 2$ $1.1 \leq AREFF_{\alpha|H} < 1.5$ $1 < AREFF_{\alpha|H} < 1.1$ $AREFF_{\alpha|H} = 1$

Figure 2: Asymptotic relative efficiency [plane (α, ρ)].

3.2 Estimation of ρ

Let us assume next that we estimate ρ through an estimator $\widehat{\rho}$, adequately chosen so that $\widehat{\rho} - \rho = o_p(1)$ at any of the levels on which we are going to base the estimation of the tail index γ.

Theorem 3.3. *Under the conditions of Theorem 2.1,*

$$\widehat{\gamma}_{n,\alpha}^{GJ(\rho)}(k) := -\frac{1}{\rho(\alpha-1)}\left(\alpha(1-\rho)\widehat{\gamma}_n^{(1)}(k) - (\alpha-\rho)\widehat{\gamma}_n^{(\alpha)}(k)\right)$$
$$\stackrel{d}{=} \gamma + \frac{\gamma\sqrt{\alpha(\alpha-2\rho(1-\rho))}}{|\rho|\sqrt{(2\alpha-1)\ k}}\ Z_k^{GJ(\rho)} + o_p(A(n/k)), \quad (3.7)$$

where $Z_k^{(GJ(\rho))}$ *is asymptotically standard normal. The same distributional result* (3.7) *holds for the tail index estimator in* (1.6)*, i.e., if we replace* ρ *by an estimator* $\widehat{\rho}$ *such that* $\widehat{\rho}-\rho = o_p(1)$ *for levels* k *such that* $\sqrt{k}\ A(n/k) \to \lambda$*, finite.*

Proof. The proof of (3.7) follows the lines of the proof of Theorem 2.1. The second part of the theorem follows from the fact that

$$\varphi_\rho(k) := \frac{\partial}{\partial\rho}\widehat{\gamma}_{n,\alpha}^{GJ(\rho)}(k) = \frac{\alpha}{\rho^2(\alpha-1)}\left(\widehat{\gamma}_n^{(1)}(k) - \widehat{\gamma}_n^{(\alpha)}(k)\right) = O_p\left(1/\sqrt{k}\right),$$

and

$$\widehat{\gamma}_{n,\alpha}^{GJ(\widehat{\rho})}(k) = \widehat{\gamma}_{n,\alpha}^{GJ(\rho)}(k) + (\widehat{\rho}-\rho)\ \ \varphi_\rho(k)(1+o_p(1)).$$

Then, whenever $\sqrt{k}\ A(n/k) \to \lambda$, finite, i.e., $A(n/k) = O_p(1/\sqrt{k})$, the conditions in the theorem enable us to guarantee that $(\widehat{\rho}-\rho)\ \ \varphi_\rho(k) = o_p(A(n/k))$. □

Remark 3.3. *For* $\widehat{\rho}$ *we may choose the estimator considered in Gomes and Martins (2002b), which is based on the class of estimators in Fraga Alves et al. (2003). More specifically, we shall here consider the estimator*

$$\widehat{\rho} := \min\left(0, \frac{3(T_n^{(1)}(k_1)-1)}{(T_n^{(1)}(k_1)-3)}\right), \quad k_1 = \min\left(n-1, \left[\frac{2n}{\ln\ln n}\right]\right)$$

where

$$T_n^{(1)}(k) := \frac{\left(M_n^{(1)}(k)\right) - \left(M_n^{(2)}(k)/2\right)^{1/2}}{\left(M_n^{(2)}(k)/2\right)^{1/2} - \left(M_n^{(3)}(k)/6\right)^{1/3}},$$

with

$$M_n^{(j)}(k) := \frac{1}{k}\sum_{i=1}^{k}\left(\ln\frac{X_{n-i+1:n}}{X_{n-k:n}}\right)^j, \quad j=1,2,3.$$

We have $\widehat\rho-\rho = O_p\left(1/\left(\sqrt{k_1}\ A(n/k_1)\right)\right) = o_p(1)$ *for any level* k *such that* $\sqrt{k}\ A(n/k)\to\lambda$*, finite.*

Remark 3.4. *The minimization of the variance term in* (3.7)*, i.e., the value*

$$\alpha_{min} := \arg\min_\alpha \frac{\alpha\ (\alpha-2\rho(1-\rho))}{\rho^2(2\alpha-1)} \quad\Longleftrightarrow\quad \alpha_{min}=\frac{1+\sqrt{1-4\rho(1-\rho)}}{2}.$$

This suggests the consideration of the estimator

$$\widehat\gamma_{n,\widehat\alpha}^{GJ(\widehat\rho)}(k) = -\frac{\widehat\alpha(1-\widehat\rho)}{\widehat\rho(\widehat\alpha-1)}\left\{\widehat\gamma_n^{(1)}(k)-\frac{\widehat\alpha-\widehat\rho}{\widehat\alpha(1-\widehat\rho)}\widehat\gamma_n^{(\alpha)}(k)\right\},$$
$$\widehat\alpha=\frac{1+\sqrt{1-4\widehat\rho(1-\widehat\rho)}}{2}, \quad (3.8)$$

for which the distributional representation in Theorem 3.3 also holds.

4 Simulated distributional behaviour

In Figures 3 and 4 we present the mean value and mean squared error patterns, as functions of k, the number of top o.s. used, of the estimator (3.8) (denoted $\widehat\rho$ in the figures), together with the estimator in (3.6), $\alpha=2$ (denoted (-1) in the figures), for two parents with $\rho=-1$: the Fréchet, $F(x)=\exp\left(-x^{-1/\gamma}\right)$, $x\ge 0$, with $\gamma=1$, and the Student-t with $\nu=2$ degrees of freedom, for which $\rho=-2/\nu=-1$. The behaviour of the Hill estimator is pictured for reference, and the simulation is based on 5000 runs.

Figures 5 and 6 are equivalent to the previous two figures, but now for populations with $\rho\neq-1$: the Generalized Pareto, $F(x)=1-(1+\gamma\ x)^{-1/\gamma}$, $x>0$, with $\rho=-1/\gamma=-0.5$ (and consequently, with $\gamma=0.5$) and the Student with 8 degrees of freedom, for which $\gamma=1/8=0.125$ and $\rho=-2/8=-0.25$.

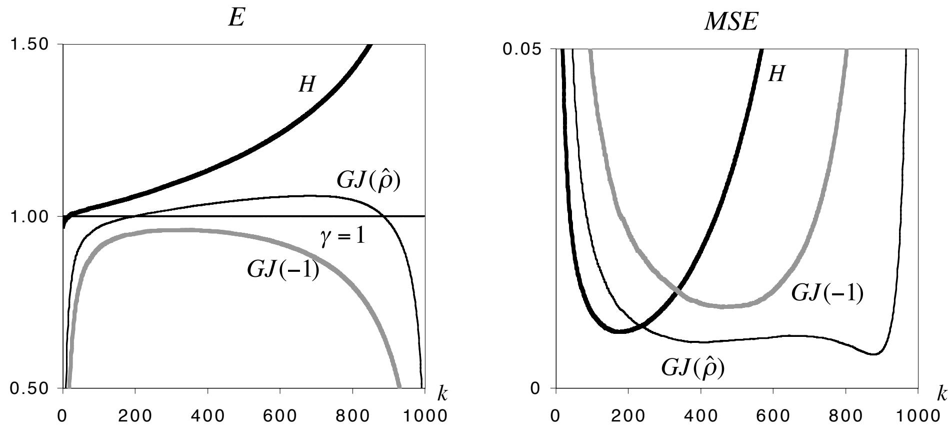

Figure 3: Fréchet parent with $(\gamma, \rho) = (1, -1)$

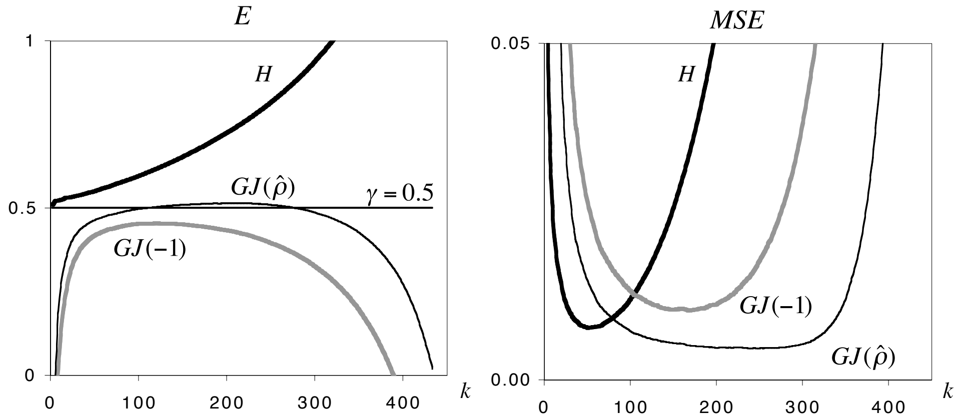

Figure 4: Student(2) parent with $(\gamma, \rho) = (0.5, -1)$

In Table 1 we present for different simulated models a simulated measure of efficiency of the new estimator relatively to the Hill estimator, both computed at their optimal levels, i.e.,

$$REFF_{GJ(\widehat{\rho})|H} = \sqrt{\frac{MSE\left[\widehat{\gamma}^{H}_{n0}\right]}{MSE\left[\widehat{\gamma}^{GJ(\widehat{\rho})}_{n0}\right]}}. \tag{4.1}$$

This measure was computed on the basis of a multi-sample simulation of size 5000×10. For details on multi-sample simulation see Gomes and Oliveira (2001).

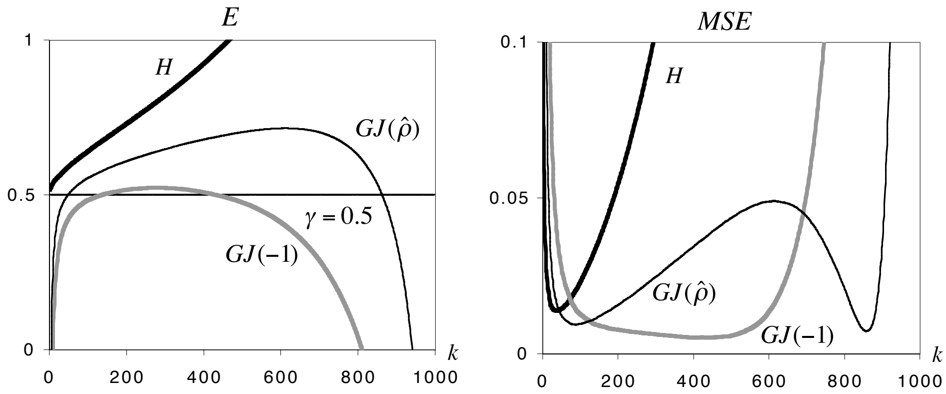

Figure 5: Generalized Pareto parent with $(\gamma, \rho) = (0.5, -0.5)$

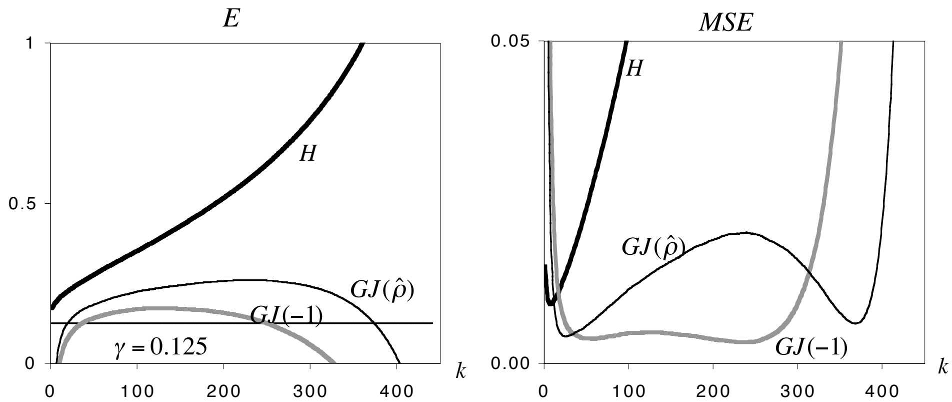

Figure 6: Student(8) parent with $(\gamma, \rho) = (0.125, -0.25)$

Some overall remarks:

1. For Fréchet models, one of the typical heavy tail parents, the new estimator in (3.8), at its optimal level, overpasses the Hill estimator, also at its optimal level, if we estimate ρ through $\widehat{\rho}$. If we consider the actual $\rho = -1$ the reduction of bias is too big, and the new estimator has a negative bias for all k. A similar conclusion may be drawn for all simulated models with $\rho = -1$.

2. As ρ approaches 0, the new estimators considered, computed at their optimal levels, compare both favourably with the Hill estimator. This is

Table 1: Relative efficiencies $REFF_{GJ(\widehat{\rho})|H}$ in (4.1).

n	100	200	500	1000	2000
$STU(\rho=-.25)$	1.8864	1.7541	1.6029	1.4994	1.4306
$STU(\rho=-.5)$	1.4002	1.4905	1.3702	1.3130	1.3385
$GP(\rho=-.5)$	1.3131	1.2825	1.2714	1.4182	1.6768
$FRE\ (\rho=-1)$	0.9858	1.1034	1.1834	1.3014	1.2840
$STU(\rho=-1)$	0.7135	1.1776	1.2609	1.2831	1.3318
$GP\rho=(-1)$	1.0752	1.2100	1.2999	1.4128	1.5210
$STU(\rho=-2)$	0.3529	0.6082	0.7945	0.8039	0.9185
$GP(\rho=-2)$	0.7774	0.8732	0.9845	1.0640	1.0947

indeed a kind of behaviour shared by all the "asymptotically unbiased" estimators considered before, and it is essentially due to the high bias of the Hill estimator in this region of ρ-values.

3. For small values of ρ it has been difficult to find competitors to the Hill estimator. However, with this class we are indeed able to overpass the Hill estimator for large n, again through the use $\widehat{\rho}$. An illustration of this is presented in Figure 7.

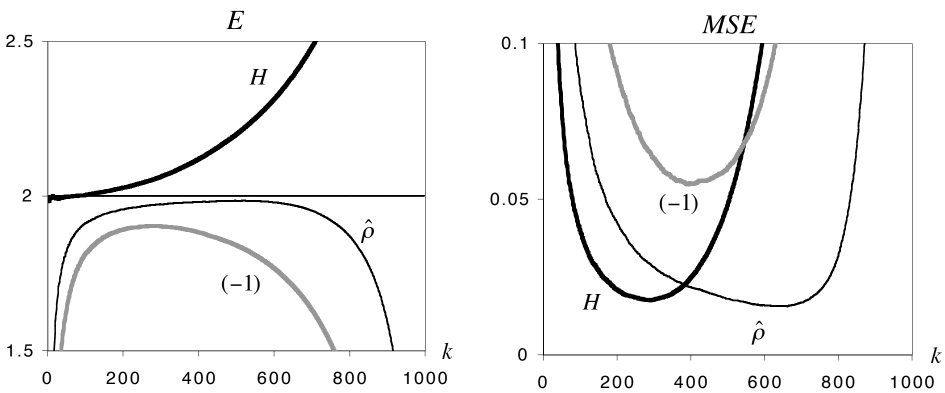

Figure 7: Generalized Pareto parent with $(\gamma, \rho) = (2, -2)$

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