Mixed Moment Estimator*

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Abstract. A new class of estimators of the extreme value index is developed. It has a simple form and is very close to the maximum likelihood estimator for a wide class of heavy-tailed models. We also propose an alternative class of estimators, dependent on a tuning parameter $p \in (0, 1)$ and invariant for changes in both scale and/or location.

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1 Introduction

The Fisher and Tippett theorem of extreme values (Fisher and Tippett, 1928) states that all possible non-degenerate weak limit distributions of normalized partial maxima of independent, identically distributed (i.i.d.) random variables (r.v.'s) $X_1, X_2, \cdots, X_n \cdots$ are (generalized) extreme value distributions. That is, if there are normalizing constants $a_n > 0$ and b_n such that, for all x,

$$\lim_{n\to\infty} P\left\{a_n^{-1}\left(\max\left(X_1, \cdots, X_n\right) - b_n\right) \le x\right\} = G(x), \tag{1.1}$$

where G is some non-degenerate distribution function (d.f.), we can redefine the constants in such a way that the limit G is one of a one-parameter family of distributions,

$$G_\gamma(x) := \begin{cases} \exp\left(-(1+\gamma x)^{-1/\gamma}\right), \ 1+\gamma x > 0 & \text{if} \quad \gamma \neq 0 \\ \exp(-\exp(-x)), \ x \in \mathbb{R} & \text{if} \quad \gamma = 0 \end{cases}, \tag{1.2}$$

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the (generalized) *extreme value* distributions, given here in the von Mises-Jenkinson form (von Mises, 1928; Jenkinson, 1955). We say that the d.f. F of the r.v.'s $X_1, X_2, \cdots$ is in the domain of attraction of G_γ if (**??**) holds with $G = G_\gamma$, and use the notation $F \in \mathcal{D}_{\mathcal{M}}(G_\gamma)$.

For the application of extreme value theory it is necessary to estimate the parameter γ. Several estimators for γ have been proposed in the literature. We mention Hill's estimator (Hill, 1975), valid in the range $\gamma > 0$, Pickand's estimator (Pickands, 1975), the so-called "maximum likelihood" or ML estimator (Smith, 1987), based on the approximate Paretian behaviour of the log-excesses over a high random threshold and valid for $\gamma > -1/2$, the probability weighted moment estimator (Hosking and Wallis, 1987), valid for $\gamma < 1/2$ and the moment estimator (Dekkers *et al.*, 1989).

In order to develop a new estimator, consider a combination of Theorems 2.6.1 and 2.6.2 of de Haan (1970): a d.f. F, with right endpoint

$$x^F := \sup\{t : F(t) < 1\} \in (0, +\infty],$$

is in the domain of attraction of the extreme value d.f. G_γ, i.e., (**??**) holds with $G = G_\gamma$ and $X_1, X_2, \cdots$ have d.f. F, if and only if

$$\lim_{t \to x^F} \frac{(1 - F(t)) \int_t^{x^F} \int_y^{x^F} (1 - F(x)) x^{-2} dx\, dy}{t^2 \left(\int_t^{x^F} (1 - F(x)) x^{-2} dx \right)^2} = \varphi(\gamma) := \begin{cases} 1 + \gamma & \text{if} \quad \gamma > 0 \\ \frac{1-\gamma}{1-2\gamma} & \text{if} \quad \gamma \le 0 \end{cases}. \tag{1.3}$$

Let $X_i,\ 1 \le i \le n$, be n i.i.d. r.v.'s with d.f. $F \in \mathcal{D}_{\mathcal{M}}(G_\gamma)$, and let $X_{i,n},\ 1 \le i \le n$, denote the associated ascending order statistics. We can build a statistic starting from the left hand-side of (**??**), noticing that it can be written as

$$\frac{(1 - F(t))^{-1} \left\{ \int_t^\infty \ln \frac{x}{t} dF(x) - \int_t^\infty \left(1 - \frac{t}{x}\right) dF(x) \right\}}{\left\{ (1 - F(t))^{-1} \int_t^\infty \left(1 - \frac{t}{x}\right) dF(x) \right\}^2} = \frac{E\left(\ln(X/t) | X > t\right) - E\left(1 - t/X | X > t\right)}{E^2\left(1 - t/X | X > t\right)},$$

and replacing F and t by the empirical d.f. F_n and a high random threshold $X_{n-k,n}$ with $k < n$, respectively. The result is

$$\widehat{\varphi}_n(k) := \frac{M_n^{(1)}(k) - L_n^{(1)}(k)}{\left(L_n^{(1)}(k)\right)^2}, \tag{1.4}$$

where, we define for all $j \ge 1$,

$$L_n^{(j)}(k) := \frac{1}{k} \sum_{i=1}^{k} \left(1 - \frac{X_{n-k,n}}{X_{n-i+1,n}}\right)^j, \qquad M_n^{(j)}(k) := \frac{1}{k} \sum_{i=1}^{k} \left(\ln \frac{X_{n-i+1,n}}{X_{n-k,n}}\right)^j. \tag{1.5}$$

The statistic in (**??**) is easily transformed into what we call the *Mixed Moment* (MM) estimator for the *extreme value index* $\gamma \in \mathbb{R}$:

$$\widehat{\gamma}_n^{MM}(k) \equiv \widehat{\gamma}_n^{MM}(k; X_{n-j+1,n}, 1 \le j \le k+1) := \frac{\widehat{\varphi}_n(k) - 1}{1 + 2\min\left(\widehat{\varphi}_n(k) - 1, 0\right)}. \tag{1.6}$$

Since the estimator in (**??**) is not location invariant, we also propose an alternative class of extreme value index estimators, invariant for changes in location, and dependent on a *tuning parameter* p, $0 < p < 1$. Such an estimator has the same functional expression of the estimator in (**??**), but the original sample, $X_i,\ 1 \le i \le n$, is replaced everywhere by

$$X_i^* := X_i - X_{[np]+1,n},\ 0 < p < 1,\ 1 \le i \le n.$$

We shall thus define for any $p \in (0, 1)$,

$$\widehat{\gamma}_n^{MM}(k; p) := \widehat{\gamma}_n^{MM}(k; X_{n-j+1,n} - X_{[np]+1,n}, 1 \le j \le k+1). \tag{1.7}$$

A similar procedure can be applied to any of the classical estimators, like the Hill, the Pickands and the Moment estimator, and has already been used for quantile estimation in Araújo Santos *et al.* (2006).

The estimator in (**??**) seems an interesting alternative to the most popular *extreme value index* estimators for a general $\gamma \in \mathbb{R}$. The most attractive features of this new estimator are:

- It is valid for any $\gamma \in \mathbb{R}$ and, contrary to the ML estimator (valid only for $\gamma > -1/2$), and it has a simple explicit functional form, similar to the ones of Pickands and Moment estimators, also valid for all $\gamma \in \mathbb{R}$.
- It is very close to the ML estimator for $\gamma \ge 0$.
- Its variance falls off rapidly when γ becomes negative. See Figure **??**, where we use the obvious notations σ_H^2, σ_P^2, σ_M^2, σ_{ML}^2, σ_{WM}^2 and σ_{MM}^2 for the asymptotic variances of the Hill, the Pickands, the moment, the maximum-likelihood, the weighted-moment and the mixed moment estimators, respectively. Note that for all values of $\gamma < 0$, the new estimator attains the minimal asymptotic variance, among the above mentioned estimators. For $\gamma > 0$, its asymptotic variance is equal to that of the ML extreme value index estimator.
- A shift invariant version with similar properties is available, the one provided in (**??**). The asymptotic variance of the estimators in (**??**) is the same and the dominant component of asymptotic bias is never bigger, provided we keep to adequate k-values and choose an adequate *tuning parameter* p whenever $\gamma \le 0$.
- There are accompanying shift and scale estimators that make e.g. high quantile estimation straightforward.

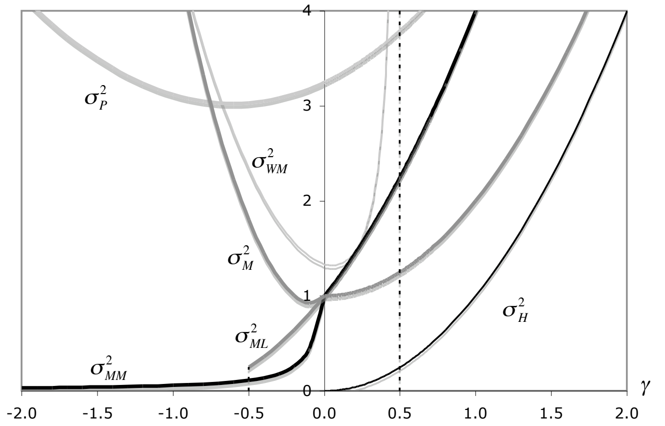

Figure 1: Asymptotic variances of different extreme value index estimators as a function of the extreme value index γ.

The scope of this paper is as follows. In section **??**, we state the main results leading to weak consistency and asymptotic normality of the estimator in (**??**) and its location invariant version in (**??**). Section **??** is devoted to a small-scale Monte Carlo simulation, that illustrates the performance of the new proposed estimators. In section **??** we state and prove a few auxiliary results needed in section **??**, where we provide the proofs of the theorems in section **??**.

2 Main results

The following *extended regular variation* property (de Haan, 1984) is a well-known necessary and sufficient condition for $F \in \mathcal{D}_{\mathcal{M}}(G_\gamma)$:

$$\lim_{t\to\infty} \frac{U(tx)-U(t)}{a(t)} = \begin{cases} \frac{x^\gamma-1}{\gamma} & \text{if} \quad \gamma \neq 0 \\ \ln x & \text{if} \quad \gamma = 0 \end{cases}, \tag{2.1}$$

for every $x > 0$ and some positive measurable function a, with U standing for a quantile type function associated to F and defined by

$$U(t) := \left(\frac{1}{1-F}\right)^{\leftarrow} = \inf\left\{x : F(x) \geq 1 - \tfrac{1}{t}\right\}.$$

Theorem 2.1. *Suppose that F has a right endpoint $x^F = U(\infty) > 0$ and (**??**) holds for some $\gamma \in \mathbb{R}$. Let $k = k_n$ be an intermediate sequence, i.e., a sequence of positive integers k_n such that $k_n \to \infty$ and $k_n = o(n)$, as $n \to \infty$. Then,*

$$\lim_{n\to\infty} \widehat{\varphi}_n(k) = \varphi(\gamma)$$

*in probability, with $\varphi(\gamma)$ defined in (**??**).*

Corollary 2.1. *Under the conditions of Theorem* **??**, *the mixed moment estimator* $\widehat{\gamma}_n^{MM}(k)$ *is a consistent estimator of the extreme value index* $\gamma \in \mathbb{R}$, *i.e., for any intermediate sequence* $k = k_n$,

$$\widehat{\gamma}_n^{MM}(k) \underset{n\to\infty}{\overset{P}{\longrightarrow}} \gamma.$$

Apart from the first order condition in (**??**), we shall need a second order condition, specifying the rate of convergence in (**??**). We shall assume the existence of a function A, possibly not changing in sign and tending to zero as $t \to \infty$, such that

$$\lim_{t\to\infty} \frac{\frac{U(tx)-U(t)}{a(t)} - \frac{x^\gamma - 1}{\gamma}}{A(t)} = H_{\gamma,\rho}(x) := \frac{1}{\rho}\left(\frac{x^{\gamma+\rho}-1}{\gamma+\rho} - \frac{x^\gamma - 1}{\gamma}\right) \tag{2.2}$$

for all $x > 0$, where $\rho \le 0$ is a *second order* parameter controlling the speed of convergence of maximum values, linearly normalized, towards the limit law in (**??**). Under these circumstances, we say that the function U is of *second order extended regular variation*, and use the notation $U \in 2ERV(\gamma, \rho)$. We remark that $\lim_{t\to\infty} A(tx)/A(t) = x^\rho$, for every $x > 0$, i.e., $|A|$ is regularly varying with an index of regular variation equal to ρ. We denote this by $|A| \in RV_\rho$.

From Theorem A in Draisma *et al.* (1999), together with the modifications in Ferreira *et al.* (2003), we know that if $x^F > 0$ and there exist $a(\cdot)$ and $A(\cdot)$ such that (**??**) holds, with $\rho \le 0$, $\gamma \ne \rho$, then, with

$$\overline{A}(t) := \left(\frac{a(t)}{U(t)} - \gamma_+\right), \qquad \gamma_+ := \max(0,\ \gamma), \tag{2.3}$$

and

$$l := \lim_{t\to\infty}\left(U(t) - \frac{a(t)}{\gamma}\right), \quad \text{for } \gamma+\rho<0, \tag{2.4}$$

$$\frac{\overline{A}(t)}{A(t)} \underset{t\to\infty}{\longrightarrow} c = \begin{cases} 0 & \text{if} \quad \gamma < \rho \le 0 \\ \frac{\gamma}{\gamma+\rho} & \text{if} \quad 0 \le -\rho < \gamma \text{ or } \ (0<\gamma<-\rho \text{ and } l=0) \\ \pm\infty & \text{if} \quad \gamma+\rho = 0 \text{ or } \ (0<\gamma<-\rho \text{ and } l \ne 0) \ \text{ or } \rho<\gamma\le 0 \end{cases} . \tag{2.5}$$

Theorem 2.2. *Assume that* U *satisfies* (**??**), *with the restriction* $\gamma \ne \rho$, *and that* $x^F > 0$. *Let* $k = k_n$ *be an intermediate sequence, let* $\overline{A}$ *be the function in* (**??**) *and let* c *be the limit in* (**??**). *If we further assume that*

$$\lambda := \lim_{n\to\infty} \begin{cases} \sqrt{k}\ \overline{A}(n/k) & \textit{if} \quad c = \pm\infty \\ \sqrt{k}\ A(n/k) & \qquad \textit{otherwise} \end{cases} \tag{2.6}$$

is finite, we may guarantee that

$$\sqrt{k}\left(\widehat{\varphi}_n(k) - \varphi(\gamma)\right) \underset{n\to\infty}{\overset{d}{\longrightarrow}} N\left(\lambda\ b_\varphi, \sigma_\varphi^2\right),$$

where

$$b_\varphi = b_\varphi(\gamma,\rho) := \begin{cases} \frac{1-\gamma}{(1-2\gamma)(1-\gamma-\rho)(1-2\gamma-\rho)} & \textit{if} \quad \gamma < \rho \le 0 \\ \frac{-4\gamma(1-\gamma)}{(1-2\gamma)^2(1-3\gamma)} & \textit{if} \quad \rho < \gamma \le 0 \\ \frac{1+\gamma}{(1-\rho)(1+\gamma-\rho)} & \textit{if} \quad c = \frac{\gamma}{\gamma+\rho} \\ 0 & \quad \textit{otherwise} \end{cases}$$

and

$$\sigma_\varphi^2 = \sigma_\varphi^2(\gamma) := \begin{cases} (1+\gamma)^2 & \textit{if} \quad \gamma \ge 0 \\ \frac{(1-\gamma)^2(6\gamma^2-\gamma+1)}{(1-2\gamma)^3(1-3\gamma)(1-4\gamma)} & \textit{if} \quad \gamma < 0 \end{cases}.$$

Corollary 2.2. *Under the conditions of Theorem* **??**

$$\sqrt{k}\left(\widehat{\gamma}_n^{MM}(k) - \gamma\right) \underset{n\to\infty}{\overset{d}{\longrightarrow}} N\left(\lambda\ b, \sigma^2\right),$$

where

$$b = b_{MM}(\gamma,\rho) := \begin{cases} b_\varphi(\gamma,\ \rho) & \textit{if} \quad \gamma \ge 0 \\ \frac{b_\varphi(\gamma,\rho)}{(1-2\gamma)^2} & \textit{if} \quad \gamma < 0 \end{cases}, \qquad \sigma^2 = \sigma^2_{MM}(\gamma) := \begin{cases} \sigma^2_\varphi(\gamma) & \textit{if} \quad \gamma \ge 0 \\ \frac{\sigma^2_\varphi(\gamma)}{(1-2\gamma)^4} & \textit{if} \quad \gamma < 0 \end{cases}.$$

We may further specify a non-null asymptotic bias in the region of the (γ,ρ)-plane where $b_\varphi = 0$, i.e., the region $\mathcal{R} := \{(\gamma,\rho) : \rho < \gamma = 0 \text{ or } (0 < \gamma < -\rho \text{ and } l \ne 0) \text{ or } \gamma = -\rho\}$. It is then necessary to split $\mathcal{R}$ in three regions, $\mathcal{R} = \mathcal{R}_0 \cup \mathcal{R}_1 \cup \mathcal{R}_2$, with $\mathcal{R}_0 := \{(\gamma,\rho) : \gamma = -\rho/2\}$,

$$\begin{aligned} \mathcal{R}_1 &:= \{(\gamma,\rho) : \rho < \gamma = 0 \text{ or } (0 < \gamma < -\rho/2,\ l \ne 0)\}, \\ \mathcal{R}_2 &:= \{(\gamma,\rho) : (-\rho/2 < \gamma < -\rho,\ l \ne 0) \text{ or } \gamma = -\rho\}. \end{aligned}$$

Note that in $\mathcal{R}_1$, $A = o\left(\overline{A}^2\right)$ and in $\mathcal{R}_2$, $\overline{A}^2 = o(A)$.

Proposition 2.1. *Under the conditions in Theorem* **??**, *if we now further assume that in the region* $\mathcal{R}_1 \cup \mathcal{R}_2$,

$$\lambda := \lim_{n\to\infty} \begin{cases} \sqrt{k}\ \overline{A}^2(n/k) & \textit{if} \quad (\gamma,\rho) \in \mathcal{R}_1 \\ \sqrt{k}\ A(n/k) & \textit{if} \quad (\gamma,\rho) \in \mathcal{R}_2 \end{cases}$$

is finite, we may guarantee that

$$\sqrt{k}\left(\widehat{\varphi}_n(k) - \varphi(\gamma)\right) \underset{n\to\infty}{\overset{d}{\longrightarrow}} N\left(\lambda\ b_0, \sigma_0^2\right),$$

where

$$b_0 = b_0(\gamma,\rho) := \begin{cases} \frac{2(1+\gamma)}{(1+2\gamma)^2(1+3\gamma)} & \textit{if} \quad (\gamma,\rho) \in \mathcal{R}_1 \\ \frac{1+\gamma}{(1-\rho)(1+\gamma-\rho)} \equiv b_\varphi & \textit{if} \quad (\gamma,\rho) \in \mathcal{R}_2 \end{cases}$$

and $\sigma_0^2 = \sigma_\varphi^2 = (1+\gamma)^2$.

Remark 2.1. *Corollary* **??** *remains valid under the conditions of Proposition* **??**, *with* b_φ *replaced by* b_0.

In order to tackle the extreme quantile estimation problem, we shall next introduce an estimator for the scale $a(n/k)$, on the basis of the fact that not only

$$\frac{X_{n-k,n}\; L_n^{(1)}(k)\,(1+|\gamma|)}{a(n/k)} \underset{n\to\infty}{\overset{P}{\longrightarrow}} 1$$

(see Proposition **??**), but also that it is possible to combine $L_n^{(1)}(k)$ and $L_n^{(2)}(k)$, both given in (**??**), in order to obtain a consistent estimator for $1+|\gamma|$ (see Proposition **??**, again). Specifically,

$$\frac{1}{2}\left(1-\frac{\left(L_n^{(1)}(k)\right)^2}{L_n^{(2)}(k)}\right)^{-1} \underset{n\to\infty}{\overset{P}{\longrightarrow}} 1+|\gamma|,$$

and this enables us to introduce the following estimator for $a(n/k)$:

$$\widehat{a}\left(\frac{n}{k}\right) := \frac{X_{n-k,n}}{2} \times \frac{L_n^{(1)}(k)\; L_n^{(2)}(k)}{L_n^{(2)}(k) - \left[L_n^{(1)}(k)\right]^2}. \tag{2.7}$$

Theorem 2.3. *Suppose* F *satisfies* (**??**) *for some* $\gamma \in \mathbb{R}$ *and* $x^F > 0$. *Let* $k = k_n$ *be an intermediate sequence. Then, with* $\widehat{a}(n/k)$ *given in* (**??**),

$$\frac{\widehat{a}(n/k)}{a(n/k)} \underset{n\to\infty}{\overset{P}{\longrightarrow}} 1. \tag{2.8}$$

Let us further assume that the second order condition (**??**) *holds with* $\gamma \neq \rho$, *and that, with* $\overline{A}$ *and* c *given in* (**??**) *and* (**??**), *respectively,* (**??**) *holds, for* λ *finite. Then,*

$$\sqrt{k}\left(\frac{\widehat{a}(n/k)}{a(n/k)}-1\right) \underset{n\to\infty}{\overset{d}{\longrightarrow}} N\left(\lambda b_*, \sigma_*^2\right),$$

where

$$b_* = b_*(\gamma,\rho) := \begin{cases} -\frac{\rho}{(1-\gamma-\rho)(1-2\gamma-\rho)} & \textit{if} \;\; \gamma < \rho \le 0 \\ \frac{2\,\gamma}{(1-2\gamma)(1-3\gamma)} & \textit{if} \;\; \rho < \gamma \le 0 \\ \frac{(1+\gamma)(1+2\gamma-2\rho)}{(\gamma+\rho)(1+\gamma-\rho)(1+2\gamma-\rho)} & \textit{if} \;\; c = \frac{\gamma}{\gamma+\rho} \\ -\frac{2\gamma}{(1+2\gamma)(1+3\gamma)} & \textit{otherwise} \end{cases},$$

and

$$\sigma_*^2 = \sigma_*^2(\gamma) := \left(\gamma_+\right)^2 + \frac{2(1+|\gamma|)^2\left(1+6|\gamma|+12\gamma^2\right)}{(1+2|\gamma|)(1+3|\gamma|)(1+4|\gamma|)}, \quad \gamma \in \mathbb{R}.$$

Proposition 2.2. *Assume that the second order condition* (**??**) *holds, with* $\gamma \neq \rho$, *and that* $x^F > 0$. *Let* $k = k_n$ *be an intermediate sequence such that* (**??**) *holds, for* λ *finite. Then*

$$\sqrt{k}\left(\widehat{\gamma}_n^{MM}(k) - \gamma,\ \frac{\widehat{a}(n/k)}{a(n/k)} - 1,\ \frac{X_{n-k,n} - U(n/k)}{a(n/k)}\right) \underset{n\to\infty}{\overset{d}{\longrightarrow}} (\Gamma,\ \Lambda,\ B),$$

where $(\Gamma,\ \Lambda,\ B)$ *has a joint normal distribution with mean vector* $(\lambda b, \lambda b_*, 0)$, b *and* b_* *provided in Corollary* **??** *and Theorem* **??**, *respectively, and covariances given by*

$$\begin{aligned}
Cov\left(\sqrt{k}\left(\widehat{\gamma}_n^{MM}(k) - \gamma\right), \sqrt{k}\left(\frac{\widehat{a}(n/k)}{a(n/k)} - 1\right)\right) &= \begin{cases} -\frac{(1-\gamma)^2(12\gamma^2-4\gamma+1)}{(1-2\gamma)^4(1-3\gamma)(1-4\gamma)} & \text{if } \gamma \le 0 \\ -\frac{(1+\gamma)^2}{1+3\gamma} & \text{if } \gamma > 0 \end{cases};\\
Cov\left(\sqrt{k}\left(\widehat{\gamma}_n^{MM}(k) - \gamma\right), \sqrt{k}\left(\frac{X_{n-k,n} - U(n/k)}{a(n/k)}\right)\right) &= 0;\\
Cov\left(\sqrt{k}\left(\frac{\widehat{a}(n/k)}{a(n/k)} - 1\right), \sqrt{k}\left(\frac{X_{n-k,n} - U(n/k)}{a(n/k)}\right)\right) &= \begin{cases} 0 & \text{if } \gamma \le 0 \\ -\gamma & \text{if } \gamma > 0 \end{cases}.
\end{aligned}$$

We shall next compare the estimator in (**??**) with the so-called *Maximum Likelihood* (*ML*) estimator, $\widehat{\gamma}_n^{ML}(k)$, obtained on the basis of the excesses $X_{n-i+1,n} - X_{n-k,n}$, $1 \le i \le k$. These excesses are approximately distributed as the k top order statistics associated to a sample of size k from a *Generalized Pareto* (*GP*) d.f., given by

$$GP_\gamma(x;\sigma) = \begin{cases} 1 - (1+\gamma x/\sigma)^{-1/\gamma},\ x \ge 0,\ 1 + \gamma\, x/\sigma > 0 & \text{if } \gamma \neq 0 \\ 1 - \exp(-x/\sigma),\ x \ge 0 & \text{if } \gamma = 0 \end{cases},$$

with $\sigma > 0$. The asymptotic behaviour of $\widehat{\gamma}_n^{ML}(k)$ has been worked out by Smith (1987) and Drees *et al.* (2004). We shall prove the following result:

Theorem 2.4. *Let* $\widehat{\gamma}_n^{ML}(k)$ *be a sequence of solutions of the maximum likelihood equations associated to the above mentioned set-up. Assume also that* $F \in \mathcal{D}_{\mathcal{M}}(G_\gamma)$ *with* $\gamma \ge 0$, (**??**) *holds and* $k = k_n$ *is an intermediate sequence. If* $\sqrt{k}\ \overline{A}(n/k) = O(1)$, *as* $n \to \infty$, *then*

$$\sqrt{k}\left(\widehat{\gamma}_n^{MM}(k) - \widehat{\gamma}_n^{ML}(k)\right) \underset{n\to\infty}{\overset{P}{\longrightarrow}} 0. \tag{2.9}$$

If $\sqrt{k}\ A(n/k) = O(1)$, *as* $n \to \infty$, *and* $\gamma > -\rho/2$, (**??**) *still holds.*

If we consider the shift invariant version of the extreme value index estimator in (**??**), i.e., the estimator in (**??**), the asymptotic variance of $\widehat{\gamma}_n^{MM}(k;p)$ is kept at the same level and the dominant component of bias changes only in a few cases, as may be seen in the following:

Theorem 2.5. *Assume that all the requirements in Theorem* **??** *hold, except possibly the requirement* $x^F > 0$*. Assume further that* U *has a positive derivative at the point* $1/(1-p)$*. Then, for the same levels* k *as in Theorem* **??***, i.e., intermediate levels* k *such that* (**??**) *holds, for* λ *finite,*

$$\sqrt{k}\left(\widehat{\gamma}_n^{MM}(k;\ p) - \gamma\right) \underset{n\to\infty}{\overset{d}{\longrightarrow}} N\left(\lambda b^*, \sigma^2\right),$$

with

$$b^* = b^*(\gamma,\rho,p) := \begin{cases} b & \text{if} \quad \gamma > 0 \ \text{or}\ \gamma < \rho \le 0 \\ \frac{b\ x^F}{x^F - U(1/(1-p))} & \text{if} \quad \rho < \gamma \le 0 \end{cases},$$

where b *and* σ^2 *are defined in Corollary* **??** *and* $x^F/\left(x^F - U(1/(1-p))\right)$ *is interpreted as 1 when* $x^F = \infty$*.*

Remark 2.2. *Recall that in the region* $0 < \gamma \le -\rho/2,\ l \ne 0$*, in order to get a possibly non null and finite value for the asymptotic bias, like the one provided in Proposition* **??***, we had to consider* k *values such that* $\sqrt{k}\ \overline{A}^2(n/k) \to \lambda$*, finite, as* $n \to \infty$*. For these values of* k*, the changes in the asymptotic bias are then significant, when we consider the shift invariant estimator. Everything depends on the relative behaviour of the three regularly varying functions,* $\overline{A}^2$*,* $\overline{A}/U$ *and* $1/U^2$*, all with an index of regular variation equal to* -2γ*. This is the main reason why we think sensible to restrict ourselves to levels* k *such that* $\sqrt{k}\ \overline{A}(n/k) \to \lambda$*, getting then a null value for* b*. If in this region of the* $(\gamma,\ \rho)$*-plane, we consider levels* k *such that* $\sqrt{k}\ A(n/k) \to \lambda$*, finite, we get an infinite asymptotic bias. Consequently, in this particular region, the* ML *estimator, also shift invariant, is expected to perform better than the* MM *estimator.*

Remark 2.3. *Note however that for heavy tails, and regarding bias, the new estimator compares favourably with the classical Hill estimator,* $\widehat{\gamma}_n^H(k) \equiv M_n^{(1)}(k)$*, with* $M_n^{(1)}(k)$ *given in* (**??**)*, in the whole* $(\gamma,\ \rho)$*-plane. Indeed, for levels* k *such that* $\sqrt{k}\ \overline{A}(n/k) \to \lambda \ne 0$*, the ones that are usually considered under a heavy-tailed second order framework, the asymptotic bias of* $\sqrt{k}\left(\widehat{\gamma}_n^H(k) - \gamma\right)$ *is always greater than zero and equal to* $\lambda/(1-\rho)$*, whereas we are now able to get a null bias in the region* $\gamma + \rho \le 0$*. A similar remark applies to the Moment's estimator,* $\widehat{\gamma}_n^M(k) := M_n^{(1)}(k) + \frac{1}{2}\left\{1 - (M_n^{(2)}(k)/[M_n^{(1)}(k)]^2 - 1)^{-1}\right\}$*, with* $M_n^{(j)}$ *given in* (**??**)*, as well as to Pickands' estimator,* $\widehat{\gamma}^P(k) := \ln\left(\left(X_{n-[k/4],n} - X_{n-[k/2],n}\right) / \left(X_{n-[k/2],n} - X_{n-k,n}\right)\right) / \ln 2$*.*

3 Finite sample properties

We have run a small-scale Monte Carlo simulation, on the basis of $N = 1000$ runs, for underlying Fréchet parents, with d.f. $F(x) = \exp(-x^{-1/\gamma}),\ x \ge 0\ (\gamma = 1)$, for a sample size $n = 1000$. In Figure **??**, we picture the mean values (E) and the mean squared errors (MSE) of the mixed moment estimator in (**??**) and its location invariant versions in (**??**) associated to $p = 0.1$ and

$p = 0.01$. We use for these three estimators the obvious notation MM, $MM^{(0.1)}$ and $MM^{(0.001)}$, respectively. For comparison, we also picture the same characteristics for the Hill, the Moment and the POT-maximum likelihood estimators, denoted H, M and ML, respectively.

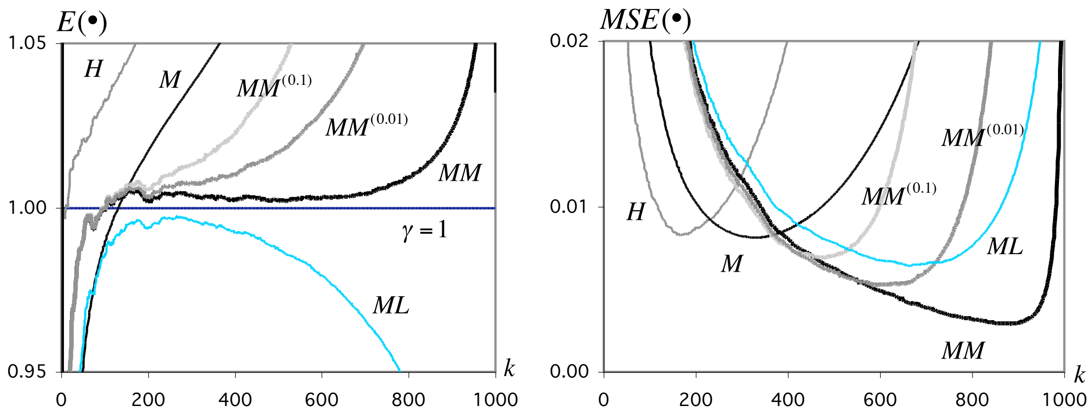

Figure 2: Mean values (*left*) and mean squared errors (*right*) of the MM estimators, together with the alternative estimators H, M and ML, for samples of size $n = 1000$ from a Fréchet model with $\gamma = 1$.

This same type of graph appears for other models, and makes clear the importance of the new class of estimators. Note also that the class of estimators in (**??**) is location invariant, and such a nice property has been obtained practically without any kind of payment: the variance of the new class of estimators is the same, and the bias may even be smaller, as may be seen in Figure **??**, equivalent to Figure **??**, but for underlying extreme value G_γ parents, with $\gamma = 0.5$.

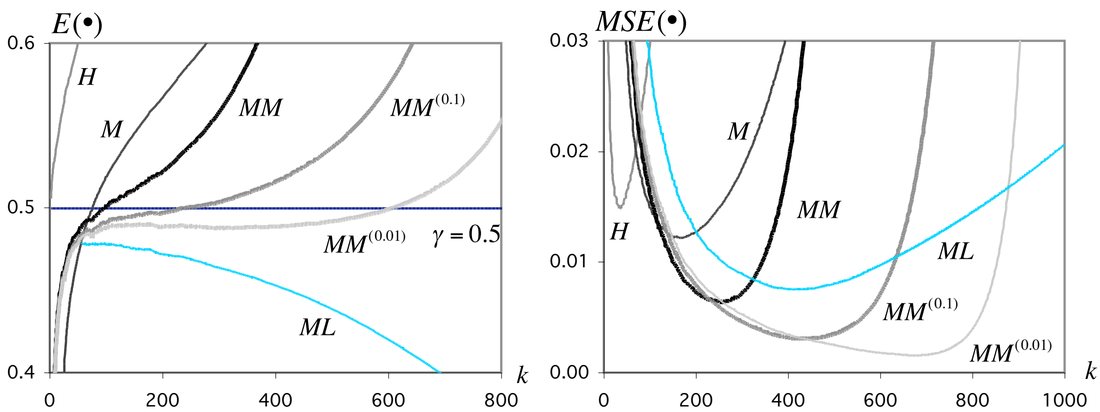

Figure 3: Mean values (*left*) and mean squared errors (*right*) of the MM estimators, together with the alternative estimators H, M and ML, for samples of size $n = 1000$ from an extreme value G_γ model with $\gamma = 0.5$.

We find particularly interesting the consideration of the tuning parameter p, which may help us on the choice of the "best" estimate, on the basis of any adequate stability criterion. Note however that, as illustrated in Araújo Santos *et al.* (2006), a shift induced by small values of p

when the underlying model, unknown, has an infinite left endpoint, may lead to stable sample paths around a target not close to the true value of γ. We illustrate such a fact with an underlying standard normal model. Figure **??** is thus equivalent to the previous figures, but for a standard normal underlying parent.

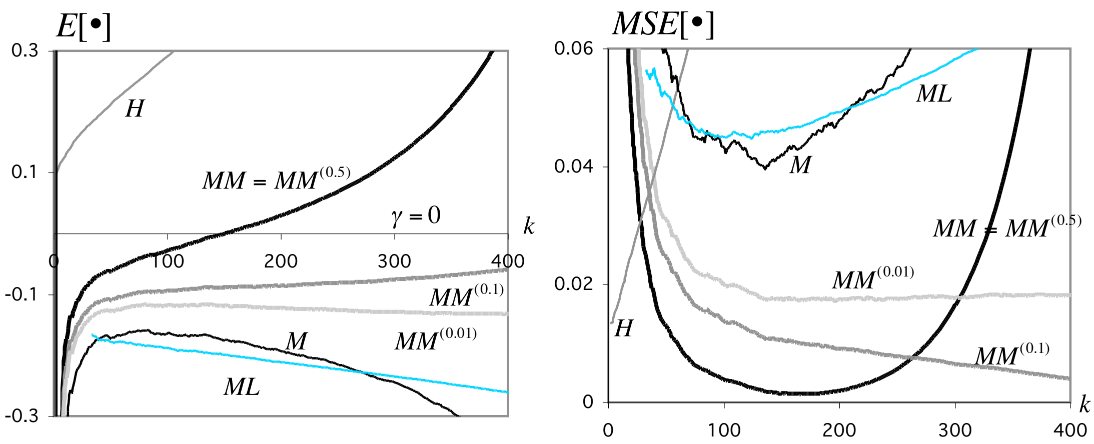

Figure 4: Mean values (*left*) and mean squared errors (*right*) of the $MM \equiv MM$ estimators, together with the alternative estimators H, M and ML, for samples of size $n = 1000$ from a standard normal model with $(\gamma = 0, \rho = 0)$.

Finally, in Figure **??** we exhibit the behavior of the different estimators for an extreme value model with $\gamma = -0.1$.

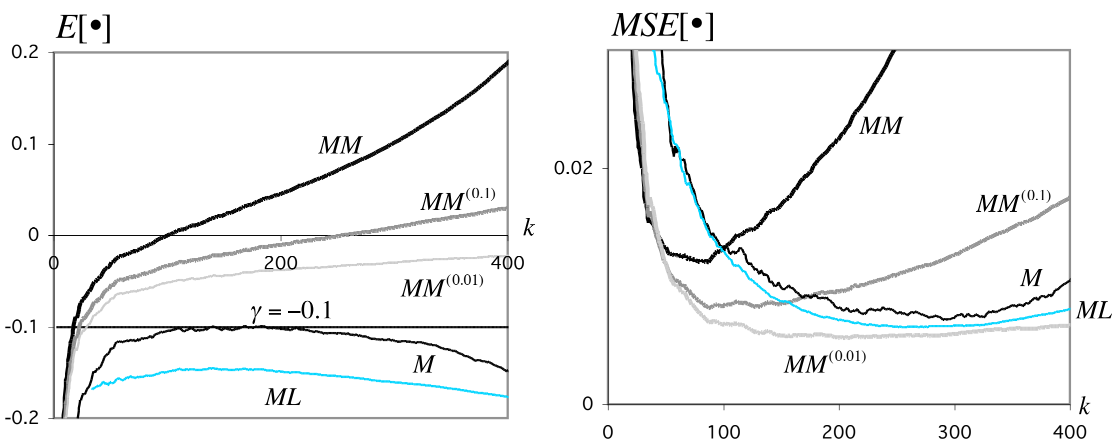

Figure 5: Mean values (*left*) and mean squared errors (*right*) of the $MM \equiv MM$ estimators, together with the alternative estimators H, M and ML, for samples of size $n = 1000$ from an extreme value G_γ model with $\gamma = -0.1$.

Note the fact that as p decreases we get sample paths closer and closer to the target value γ. Note also the fact that despite the much larger bias of the mixed moment estimators, the statistic

$MM^{(0.001)}$ clearly overpasses all other estimators considered, regarding mean squared error.

4 Auxiliary results

Lemma 4.1. *If (??) holds for some $\gamma \in \mathbb{R}$, then the auxiliary function $a(t)$ in (??) is of regular variation at infinity with index γ, i.e., $a \in RV_\gamma$ and*

$$\lim_{t\to\infty} \frac{a(t)}{U(t)} = \gamma_+ := \max(0,\gamma).$$

Moreover,

- *if $\gamma > 0$, both functions a and U belong to RV_γ;*
- *if $\gamma < 0$, then $x^F := \lim_{t\to\infty} U(t)$ exists, $\lim_{t\to\infty} a(t)/(x^F - U(t)) = -\gamma$ and $x^F - U(t) \in RV_\gamma$.*

Furthermore, with $\gamma_- := \min(\gamma, 0)$, and provided that $x^F = U(\infty) > 0$,

$$\lim_{t\to\infty} \frac{\ln U(tx) - \ln U(t)}{a(t)/U(t)} = \frac{x^{\gamma_-} - 1}{\gamma_-}, \quad \textit{for every } x > 0. \tag{4.1}$$

Proof. The first part of the lemma comes from Theorems 1.9 and 1.10 in Geluk and de Haan (1987). Equation (??) follows easily when we distinguish between the cases $\gamma > 0$ and $\gamma \le 0$. ■

Lemma ?? above together with Drees inequality (Drees, 1998) and Proposition 1.7 in Geluk and de Haan (1987) yield the following uniform bounds:

Lemma 4.2. *If (??) holds for some $\gamma \in \mathbb{R}$ then, for any $\epsilon > 0$, there exists $t_0 = t_0(\epsilon)$ such that for $t \ge t_0$ and $x \ge 1$,*

$$\left| \frac{U(tx) - U(t)}{a(t)} - \frac{x^\gamma - 1}{\gamma} \right| \le \epsilon\ x^{\gamma+\epsilon}, \qquad \left| \frac{\ln U(tx) - \ln U(t)}{a(t)/U(t)} - \frac{x^{\gamma_-} - 1}{\gamma_-} \right| \le \epsilon\ x^{\gamma_- + \epsilon}$$

and

$$(1-\epsilon)\ x^{-\gamma_+ - \epsilon} < \frac{U(t)}{U(tx)} < (1+\epsilon)\ x^{-\gamma_+ + \epsilon}.$$

Lemma 4.3. *Assume that (??) holds, i.e., $U \in 2RV(\gamma, \rho)$. Then,*

$$\frac{U(tx)}{U(t)} = \begin{cases} 1 + \overline{A}(t) \left(\ln x + \frac{\ln^2 x}{2}\ A(t) \right) (1+o(1)) & \textit{if} \ \ \gamma = \rho = 0 \\ 1 + \overline{A}(t) \left\{ \frac{x^\gamma - 1}{\gamma} + \frac{A(t)}{\gamma} \left(x^\gamma \ln x - \frac{x^\gamma - 1}{\gamma} \right) (1+o(1)) \right\} & \textit{if} \ \ \gamma < \rho = 0 \\ 1 + \overline{A}(t) \left\{ \frac{x^\gamma - 1}{\gamma} + \frac{A(t)}{\rho} \left(\frac{x^{\gamma+\rho} - 1}{\gamma+\rho} - \frac{x^\gamma - 1}{\gamma} \right) (1+o(1)) \right\} & \textit{if} \ \ \gamma \le 0,\ \rho < 0 \\ x^\gamma + \overline{A}(t) \left(\frac{x^\gamma - 1}{\gamma} \right) + \frac{\gamma\ A(t)}{\rho} \left(\frac{x^{\gamma+\rho} - 1}{\gamma+\rho} - \frac{x^\gamma - 1}{\gamma} \right) (1+o(1)) & \textit{if} \ \ \gamma > 0,\ \rho < 0 \\ x^\gamma + \overline{A}(t) \left(\frac{x^\gamma - 1}{\gamma} \right) + A(t) \left(x^\gamma \ln x - \frac{x^\gamma - 1}{\gamma} \right) (1+o(1)) & \textit{if} \ \ \rho = 0 < \gamma \end{cases}. \tag{4.2}$$

Proof. Directly from (**??**), we get

$$\frac{U(tx)}{U(t)} - 1 = \frac{a(t)}{U(t)} \left\{ \frac{x^\gamma - 1}{\gamma} + \frac{A(t)}{\rho} \left(\frac{x^{\gamma+\rho} - 1}{\gamma + \rho} - \frac{x^\gamma - 1}{\gamma} \right) (1 + o(1)) \right\}$$

With the notation in (**??**), i.e., $a(t)/U(t) = \gamma_+ + \overline{A}(t)$, we may write

$$\frac{U(tx)}{U(t)} - 1 = \gamma_+ \left(\frac{x^\gamma - 1}{\gamma} \right) + \overline{A}(t) \left(\frac{x^\gamma - 1}{\gamma} \right) + \frac{A(t)}{\rho} \left(\frac{x^{\gamma+\rho} - 1}{\gamma + \rho} - \frac{x^\gamma - 1}{\gamma} \right) \left(\gamma_+ + \overline{A}(t) \right) (1 + o(1))$$

and (**??**) follows for any $\rho < 0$.

If $\rho = 0$ and $\gamma \neq 0$, then, also directly from (**??**), and by continuity arguments,

$$\frac{U(tx)}{U(t)} - 1 = \frac{a(t)}{U(t)} \left\{ \frac{x^\gamma - 1}{\gamma} + \frac{A(t)}{\gamma} \left(x^\gamma \ln x - \frac{x^\gamma - 1}{\gamma} \right) (1 + o(1)) \right\},$$

and things work as before, with $A(t)/\rho$ replaced by $A(t)/\gamma$ and $\frac{x^{\gamma+\rho}-1}{\gamma+\rho}$ replaced by $x^\gamma \ln x$. The case $\gamma = \rho = 0$ comes again directly from (**??**) and by continuity arguments. ■

Proposition 4.1. *Let $U \in 2ERV(\gamma, \rho)$ as introduced in* (**??**)*. Let c be the limit in* (**??**)*.*

(i) If $\gamma > 0$,

$$\lim_{t\to\infty} \frac{\frac{U(t)}{U(tx)} - x^{-\gamma}}{\widetilde{A}(t)} = K_{\gamma,\rho}(x) := \begin{cases} -x^{-\gamma} \, \frac{x^\rho - 1}{\rho} & \text{if} \quad c = \frac{\gamma}{\gamma+\rho} \\ -x^{-\gamma} \, \frac{x^{-\gamma}-1}{-\gamma} & \text{if} \quad c = \pm\infty \end{cases}, \tag{4.3}$$

for all $x > 0$, where, with $\overline{A}(t)$ given in (**??**)*,*

$$\widetilde{A}(t) := \begin{cases} \frac{\gamma \ A(t)}{\gamma+\rho} & \text{if} \quad c = \frac{\gamma}{\gamma+\rho} \\ \overline{A}(t) & \text{if} \quad c = \pm\infty \end{cases}, \tag{4.4}$$

Necessarily, $\left|\widetilde{A}(t)\right| \in RV_{\widetilde{\rho}}$, with

$$\widetilde{\rho} = \begin{cases} \rho & \text{if} \quad c = \frac{\gamma}{\gamma+\rho} \\ -\gamma & \text{if} \quad c = \pm\infty \end{cases}. \tag{4.5}$$

(ii) If $\gamma \leq 0$, we need further to assume that $\gamma \neq \rho$. Then,

$$\lim_{t\to\infty} \frac{\frac{U(t)}{a^*(t)} \left(1 - \frac{U(t)}{U(tx)}\right) - \frac{x^\gamma - 1}{\gamma}}{A^*(t)} = K^*_{\gamma,\,\rho}(x) = \begin{cases} x^\gamma \ln x & \text{if} \quad \gamma < \rho = 0 \\ \frac{x^{\gamma+\rho}-1}{\gamma+\rho} & \text{if} \quad \gamma < \rho < 0 \\ \frac{x^{2\gamma}-1}{2\gamma} & \text{if} \quad \rho < \gamma < 0 \\ \ln^2 x & \text{if} \quad \rho < \gamma = 0 \end{cases}, \tag{4.6}$$

where

$$A^*(t)=\begin{cases}\frac{A(t)}{\gamma} & \textit{if} \ \ \gamma<\rho=0\\ \frac{A(t)}{\rho} & \textit{if} \ \ \gamma<\rho<0\\ -\frac{2\ \overline{A}(t)}{\gamma} & \textit{if} \ \ \rho<\gamma<0\\ -\overline{A}(t) & \textit{if} \ \ \rho<\gamma=0\end{cases}, \qquad a^*(t)=\begin{cases}a(t)\left(1-\frac{A(t)}{\gamma}\right) & \textit{if} \ \ \gamma<\rho=0\\ a(t)\left(1-\frac{A(t)}{\rho}\right) & \textit{if} \ \ \gamma<\rho<0\\ a(t)\left(1+\frac{2\ \overline{A}(t)}{\gamma}\right) & \textit{if} \ \ \rho<\gamma<0\\ a(t) & \textit{if} \ \ \rho<\gamma=0\end{cases}. \tag{4.7}$$

Necessarily, $|A^*(t)|\in RV_{\rho^*}$*, with*

$$\rho^*=\begin{cases}\rho & \textit{if} \ \ \gamma<\rho\le 0\\ \gamma & \textit{if} \ \ \rho<\gamma\le 0\end{cases}. \tag{4.8}$$

Proof. We shall consider the cases (i) and (ii) separately.
Case $(i):\gamma>0$

- If $\gamma>0$, $\rho<0$ and (**??**) holds, we have, from (**??**),

$$\frac{U(tx)}{U(t)}-x^\gamma=\overline{A}(t)\left(\frac{x^\gamma-1}{\gamma}\right)+\frac{\gamma\ A(t)}{\rho}\left(\frac{x^{\gamma+\rho}-1}{\gamma+\rho}-\frac{x^\gamma-1}{\gamma}\right)+o(A(t))$$

If $c=\pm\infty$, then $A(t)=o\left(\overline{A}(t)\right)$ and

$$\frac{U(tx)}{U(t)}-x^\gamma=x^\gamma\left(\frac{x^{-\gamma}-1}{-\gamma}\right)\ \overline{A}(t)+o\left(\overline{A}(t)\right).$$

If $c=\gamma/(\gamma+\rho)$, we get $\overline{A}(t)=\frac{\gamma\ A(t)}{\gamma+\rho}(1+o(1))$. Since in this region $\gamma\neq-\rho$, we may further write

$$\begin{aligned}\frac{U(tx)}{U(t)}-x^\gamma &= x^\gamma\left(\overline{A}(t)\left(\frac{x^{-\gamma}-1}{-\gamma}\right)+\frac{\gamma\ A(t)}{\gamma+\rho}\left(\frac{x^\rho-1}{\rho}-\frac{x^{-\gamma}-1}{-\gamma}\right)+o(A(t))\right)\\ &= \frac{\gamma\ A(t)}{\gamma+\rho}\ x^\gamma\left(\frac{x^\rho-1}{\rho}\right)+o(A(t)).\end{aligned}$$

Consequently,

$$\lim_{t\to\infty}\frac{\frac{U(tx)}{U(t)}-x^\gamma}{\widetilde{A}(t)}=\begin{cases}x^\gamma\ \frac{x^\rho-1}{\rho} & \text{if} \ \ c=\frac{\gamma}{\gamma+\rho}\\ x^\gamma\ \frac{x^{-\gamma}-1}{-\gamma} & \text{if} \ \ c=\pm\infty\end{cases}=-x^{2\gamma}K_{\gamma,\rho}(x), \tag{4.9}$$

with $K_{\gamma,\rho}(x)$ and $\widetilde{A}(t)$ defined in (**??**) and (**??**), respectively. Finally, (**??**), together with the fact that

$$\frac{U(tx)}{U(t)}-x^\gamma=-x^\gamma\ \frac{U(tx)}{U(t)}\left(\frac{U(t)}{U(tx)}-x^{-\gamma}\right)=-x^{2\gamma}\left(\frac{U(t)}{U(tx)}-x^{-\gamma}\right)(1+o(1)),$$

leads us to the limit in (**??**), with $\widetilde{A}(t)$ and $\widetilde{\rho}$ given in (**??**) and (**??**), respectively.

- If $\gamma > 0$ and $\rho = 0$, we get, again from (**??**),

$$\begin{aligned}\frac{U(tx)}{U(t)} - x^\gamma &= \overline{A}(t)\ \left(\frac{x^\gamma - 1}{\gamma}\right) + A(t)\left(x^\gamma\ \ln x - \frac{x^\gamma - 1}{\gamma}\right) + o(A(t)) \\ &= x^\gamma\left(\overline{A}(t)\ \left(\frac{x^{-\gamma} - 1}{-\gamma}\right) + A(t)\left(\ln x - \frac{x^{-\gamma} - 1}{-\gamma}\right) + o(A(t))\right).\end{aligned}$$

But if $\gamma > 0$ and $\rho = 0$, then $c = \gamma/(\gamma + \rho) = 1$, $\overline{A}(t) = A(t) + o(A(t))$, and

$$\frac{U(tx)}{U(t)} - x^\gamma = A(t)\ x^\gamma \ln x + o(A(t)).$$

Consequently, (**??**) holds, with $\widetilde{A}(t) = A(t) \equiv \gamma\ A(t)/(\gamma + \rho)$ and $\widetilde{\rho} = \rho = 0$.

Case $(ii) : \gamma \leq 0$

- If $\gamma < \rho = 0$, we get, again from (**??**),

$$\begin{aligned}\frac{U(t)}{a(t)}\left(1 - \frac{U(t)}{U(tx)}\right) &= \frac{x^\gamma - 1}{\gamma} + \frac{A(t)}{\gamma}\left(x^\gamma \ln x - \frac{x^\gamma - 1}{\gamma}\right) + o(A(t)) \\ &= \frac{x^\gamma - 1}{\gamma}\left(1 - \frac{A(t)}{\gamma}\right) + \frac{A(t)}{\gamma}\ x^\gamma \ln x + o(A(t))\end{aligned}$$

and with $a^*(t) = a(t)\left(1 - \frac{A(t)}{\gamma}\right)$,

$$\frac{U(t)}{a^*(t)}\left(1 - \frac{U(t)}{U(tx)}\right) = \frac{x^\gamma - 1}{\gamma} + \frac{A(t)}{\gamma}\ x^\gamma \ln x + o(A(t)).$$

Consequently, (**??**), (**??**) and (**??**) follow in this region of the (γ, ρ)-plane.

- If $\gamma < \rho < 0$, $a(t)/U(t) \equiv \overline{A}(t) = o(A(t))$, and again from (**??**),

$$\begin{aligned}\frac{U(t)}{a(t)}\left(1 - \frac{U(t)}{U(tx)}\right) &= \frac{x^\gamma - 1}{\gamma} + \frac{A(t)}{\rho}\left(\frac{x^{\gamma+\rho} - 1}{\gamma + \rho} - \frac{x^\gamma - 1}{\gamma}\right) + o(A(t)) \\ &= \frac{x^\gamma - 1}{\gamma}\left(1 - \frac{A(t)}{\rho}\right) + \frac{A(t)}{\rho}\left(\frac{x^{\gamma+\rho} - 1}{\gamma + \rho}\right) + o(A(t))\end{aligned}$$

and with $a^*(t) = a(t)\left(1 - \frac{A(t)}{\rho}\right)$,

$$\frac{U(t)}{a^*(t)}\left(1 - \frac{U(t)}{U(tx)}\right) = \frac{x^\gamma - 1}{\gamma} + \frac{A(t)}{\rho}\left(\frac{x^{\gamma+\rho} - 1}{\gamma + \rho}\right) + o(A(t)),$$

and the results in the proposition hold.

- If $\rho < \gamma \leq 0$, $A(t) = o\left(\overline{A}(t)\right)$, and also from (**??**), we get

$$\frac{U(t)}{U(tx)} = 1 - \overline{A}(t)\left(\frac{x^\gamma - 1}{\gamma}\right) + \overline{A}^2(t)\left(\frac{x^\gamma - 1}{\gamma}\right)^2 (1 + o(1)). \tag{4.10}$$

– Consequently, for $\gamma < 0$, since $\left(\frac{x^\gamma-1}{\gamma}\right)^2 = \frac{2}{\gamma}\left(\frac{x^{2\gamma}-1}{2\gamma} - \frac{x^\gamma-1}{\gamma}\right)$

$$\begin{aligned}\frac{U(t)}{a(t)}\left(1-\frac{U(t)}{U(tx)}\right) &= \frac{x^\gamma-1}{\gamma} - \frac{2\,\overline{A}(t)}{\gamma}\left(\frac{x^{2\gamma}-1}{2\gamma} - \frac{x^\gamma-1}{\gamma}\right)(1+o(1))) \\ &= \frac{x^\gamma-1}{\gamma}\left(1+\frac{2\,\overline{A}(t)}{\gamma}\right) - \frac{2\,\overline{A}(t)}{\gamma}\left(\frac{x^{2\gamma}-1}{2\gamma}\right)(1+o(1)),\end{aligned}$$

and with $a^*(t) = a(t)\left(1+\frac{2\,\overline{A}(t)}{\gamma}\right)$,

$$\frac{U(t)}{a^*(t)}\left(1-\frac{U(t)}{U(tx)}\right) = \frac{x^\gamma-1}{\gamma} - \frac{2\,\overline{A}(t)}{\gamma}\left(\frac{x^{2\gamma}-1}{2\gamma}\right)(1+o(1)),$$

and the results in the proposition follow.

– If $\rho < \gamma = 0$, then from (**??**), we get

$$\frac{U(t)}{a(t)}\left(1-\frac{U(t)}{U(tx)}\right) = \ln x - \overline{A}(t)\;\ln^2 x(1+o(1)),$$

and the result in the proposition follows as well.

■

Corollary 4.1. *Under the conditions and notations of Proposition* **??**, *and for* $\gamma > 0$,

$$\lim_{t\to\infty} \frac{\ln U(tx) - \ln U(t) - \gamma \ln x}{\widetilde{A}(t)} = \widetilde{K}_{\gamma,\rho}(x) := \begin{cases} \frac{x^\rho-1}{\rho} & \text{if} \quad c = \frac{\gamma}{\gamma+\rho} \\ \frac{x^{-\gamma}-1}{-\gamma} & \text{if} \quad c = \pm\infty \end{cases}, \tag{4.11}$$

for every $x > 0$, *and with* $\widetilde{A}$ *provided in* (**??**).

Proof. The proof follows immediately from relation (**??**). ■

Remark 4.1. *Note that the second order condition in* (**??**) *is the usual second order condition for heavy tails, i.e., for* $\gamma > 0$. *Note also that, from* (**??**), $A(t) = o\left(\widetilde{A}(t)\right)$ *whenever* $c = \pm\infty$. *Consequently, and as* $n \to \infty$, *if* $\sqrt{k}\;\widetilde{A}(n/k) \to \lambda \neq 0$, *finite,* $\sqrt{k}\;A(n/k) \to \lambda(\gamma+\rho)/\gamma \neq 0$ *if* $c = \gamma/(\gamma+\rho)$, *but* $\sqrt{k}\;A(n/k) \to 0$ *if* $c = \pm\infty$.

Remark 4.2. *The most common heavy-tailed models with* $\widetilde{\rho} = -\gamma$ *and* $0 < \gamma < -\rho$ *(then necessarily with* $l \neq 0$*), are such that*

$$U(t) = C\;t^\gamma\left(1 + A\;t^{-\gamma} + B\;t^{-2\gamma} + o\left(t^{-2\gamma}\right)\right), \quad A,\; B \neq 0,\; C > 0.$$

For these models,

$$U(tx) - U(t) = C\;\gamma\;t^\gamma\left(\frac{x^\gamma-1}{\gamma} - B\;t^{-2\gamma}\left(\frac{x^{-\gamma}-1}{-\gamma}\right) + o\left(t^{-2\gamma}\right)\right),$$

and

$$\frac{\frac{U(tx)-U(t)}{C\ \gamma\ t^{\gamma}} - \frac{x^{\gamma}-1}{\gamma}}{-B\ t^{-2\gamma}} \underset{t\to\infty}{\longrightarrow} \frac{x^{-\gamma}-1}{-\gamma},$$

i.e., $\rho+\gamma=-\gamma$*, or equivalently,* $\rho=-2\gamma$*. Then,* (**??**) *holds, provided that we choose*

$$a(t)=\frac{C\ \gamma\ t^{\gamma}}{1+B\ t^{-2\gamma}}, \qquad A(t)=2\ B\ \gamma\ t^{-2\gamma}$$

and

$$\frac{a(t)}{U(t)}=\gamma\left(1-A\ t^{-\gamma}-\left(2\ B-A^2\right)\ t^{-2\gamma}+o\left(t^{-2\gamma}\right)\right).$$

Consequently, with $\overline{A}(t)$*,* l *and* c *provided in* (**??**)*,* (**??**) *and* (**??**)*, respectively,*

$$\overline{A}(t)=-A\ \gamma\ t^{-\gamma}\left(1+O\left(t^{-\gamma}\right)\right), \quad \frac{\overline{A}(t)}{A(t)}=-\frac{A}{2\ B\ t^{-\gamma}}\left(1+O(t^{-\gamma})\right)\underset{t\to\infty}{\to}\pm\infty, \ \textit{i.e. } c=\pm\infty,$$

and

$$\begin{aligned} U(t)-\frac{a(t)}{\gamma} &= C\ t^{\gamma}\left(1+A\ t^{-\gamma}+B\ t^{-2\gamma}+o(t^{-2\gamma})\right)-C\ t^{\gamma}\left(1-2\ B\ t^{-2\gamma}+o(t^{-2\gamma})\right) \\ &= C\ t^{\gamma}\left(A\ t^{-\gamma}+3\ B\ t^{-2\gamma}+o(t^{-2\gamma})\right)\underset{t\to\infty}{\to} AC\neq 0, \ \textit{i.e.,} \quad l=AC\neq 0, \end{aligned}$$

as mentioned at the very beginning of this remark. Indeed, we could also have written

$$U(t)=l+C\ t^{\gamma}\left(1+B\ t^{-2\gamma}+o\left(t^{-2\gamma}\right)\right), \ \textit{as } t\to\infty.$$

We shall now provide in more detail the general behaviour of $\ln U(tx)-\ln U(t)$ and $U(t)/U(tx)$ in the region where the limiting value c in (**??**) is $\pm\infty$, i.e., the region where $A(t)=o\left(\overline{A}(t)\right)$. Recall that such a region is $\{(\gamma,\rho):\rho<\gamma\le 0 \text{ or } \gamma=-\rho \text{ or } (0<\gamma<-\rho,\ l\neq 0)\}$, with l given in (**??**). The results in the following lemma come straightforwardly from (**??**) and Taylor's expansion.

Lemma 4.4. *Assume that* (**??**) *holds. If* $\rho<\gamma\le 0$*, then*

$$\begin{aligned} \frac{U(t)}{U(tx)} = 1-\overline{A}(t)\left(\frac{x^{\gamma}-1}{\gamma}\right)+\overline{A}^2(t)\left(\frac{x^{\gamma}-1}{\gamma}\right)^2-\overline{A}^3(t)\left(\frac{x^{\gamma}-1}{\gamma}\right)^3(1+o(1)) \\ -\frac{A(t)\ \overline{A}(t)}{\rho}\left(\frac{x^{\gamma+\rho}-1}{\gamma+\rho}-\frac{x^{\gamma}-1}{\gamma}\right)(1+o(1)). \end{aligned} \quad (4.12)$$

If $0<\gamma\le-\rho$*, since* A *may dominate* $\overline{A}^2$ *as well as the other way round, we write*

$$\begin{aligned} \ln U(tx)-\ln U(t) = \gamma\ln x+\overline{A}(t)\left(\frac{x^{-\gamma}-1}{-\gamma}\right)+\frac{\gamma\ A(t)}{\gamma+\rho}\left(\frac{x^{\rho}-1}{\rho}-\frac{x^{-\gamma}-1}{-\gamma}\right)+o(A(t)) \\ -\frac{1}{2}\ \overline{A}^2(t)\left(\frac{x^{-\gamma}-1}{-\gamma}\right)^2+o\left(\overline{A}^2(t)\right) \end{aligned} \quad (4.13)$$

and

$$\frac{U(t)}{U(tx)} = x^{-\gamma}\left\{1 - \overline{A}(t)\left(\frac{x^{-\gamma}-1}{-\gamma}\right) - \frac{\gamma\ A(t)}{\gamma+\rho}\left(\frac{x^{\rho}-1}{\rho} - \frac{x^{-\gamma}-1}{-\gamma}\right) + o(A(t)) \right.$$
$$\left. + \overline{A}^2(t)\left(\frac{x^{-\gamma}-1}{-\gamma}\right)^2 + o\left(\overline{A}^2(t)\right)\right\}. \qquad (4.14)$$

Remark 4.3. *Note that for heavy-tailed models, the second order condition* (**??**) *implies a third order behaviour of the function* $\ln U(t)$*, whenever we are in the region* $0 < \gamma \le -\rho$*, and* $l \neq 0$*, a region where* $A(t) = o\left(\overline{A}(t)\right)$*. Also, since* $\left|\overline{A}\right| \in RV_{-\gamma}$*,* $|A| \in RV_\rho$ *and* $\overline{A}^2 \in RV_{-2\gamma}$*,* A *dominates* $\overline{A}^2$ *if* $\rho > -2\gamma$*, but if* $\rho < -2\gamma$*,* $\overline{A}^2$ *dominates* A*. The third order behaviour of* $\ln U(t)$ *may be written as*

$$\lim_{t\to\infty} \frac{\frac{\ln U(tx) - \ln U(t) - \gamma \ln x}{\overline{A}(t)} - \frac{x^{-\gamma}-1}{-\gamma}}{\widetilde{B}(t)} = K_{\widetilde{\rho},\,\widetilde{\eta}}(x),$$

where

$$\widetilde{B}(t) := \begin{cases} \overline{A}(t) & \text{if } \ 0<\gamma\le -\frac{\rho}{2} \\ \frac{A(t)}{\overline{A}(t)} & \text{if } \ -\frac{\rho}{2} < \gamma < -\rho \end{cases} \qquad K_{\widetilde{\rho},\,\widetilde{\eta}}(x) := \begin{cases} \frac{1}{\gamma}\left\{\frac{x^{-2\gamma}-1}{-2\gamma} - \frac{x^{-\gamma}-1}{-\gamma}\right\} & \text{if } \ 0<\gamma\le -\frac{\rho}{2} \\ \frac{\gamma}{\gamma+\rho}\left\{\frac{x^{\rho}-1}{\rho} - \frac{x^{-\gamma}-1}{-\gamma}\right\} & \text{if } \ -\frac{\rho}{2}<\gamma<-\rho \end{cases}.$$

Consequently, for this type of models, the second and third order parameters, $\widetilde{\rho}$ *and* $\widetilde{\eta}$*, respectively, are given by*

$$\widetilde{\rho} = -\gamma, \qquad \widetilde{\eta} = \begin{cases} -\gamma & \text{if } \ 0<\gamma\le -\frac{\rho}{2} \\ \gamma+\rho & \text{if } \ -\frac{\rho}{2}<\gamma<-\rho \end{cases}.$$

Note also that in the region $-\frac{\rho}{2} < \gamma < -\rho$ *we get* $\widetilde{\rho} \neq \widetilde{\eta}$*. And the situation* $\widetilde{\eta} = \widetilde{\rho}$ *is the one that most often happens in practice, for standard heavy-tailed models like the Fréchet, the Burr, the Generalized Pareto and the Student's t d.f.'d. For these d.f.'s* $\rho = -2\gamma$*. However, if we induce a shift* $l \neq 0$ *in the above mentioned models, this relation between* γ *and* ρ *no longer happens. Also, for the extreme value* G_γ *model,* $\rho = -1$*,* $\widetilde{\rho} = -\gamma$ *and* $\widetilde{\eta} = \begin{cases} \widetilde{\rho} & \text{if } \ 0\le\gamma\le 1/2 \\ \gamma-1 & \text{if } \ 1/2<\gamma<1 \end{cases}$*. For more details on the way the third order framework may be used in Statistics of Extremes, see Gomes et al.* (2002) *and Fraga Alves et al.* (2003)*, papers dealing with the estimation of the second order parameter* ρ*, and Gomes et al.* (2004)*, a paper dealing with reduced bias extreme value index estimation.*

After the statements in Proposition **??** and Corollary **??**, and again on the basis of Drees' inequality (Drees, 1998), we may now state the following:

Lemma 4.5. *If* (**??**) *holds for some* $\gamma > 0$ *then, for any* $\epsilon > 0$*, there exists* $t_0 = t_0(\epsilon)$ *such that* *for $t \geq t_0$ and $x \geq 1$,*

$$\left| \frac{\frac{U(t)}{U(tx)} - x^{-\gamma}}{\widetilde{A}(t)} - K_{\gamma,\rho}(x) \right| \leq \epsilon x^{-\gamma+\widetilde{\rho}+\epsilon}, \quad \left| \frac{\left(1 - \frac{U(t)}{U(tx)}\right)^2 - (1 - x^{-\gamma})}{\widetilde{A}(t)} - 2(x^{-\gamma} - 1)K_{\gamma,\rho}(x) \right| \leq \epsilon x^{\widetilde{\rho}+\epsilon},$$

$$\left| \frac{\ln U(tx) - \ln U(t) - \gamma \ln x}{\widetilde{A}(t)} - \widetilde{K}_{\gamma,\rho}(x) \right| \leq \epsilon x^{\widetilde{\rho}+\epsilon},$$

with $A(t)$, $K_{\gamma,\rho}(x)$, $\widetilde{A}(t)$, $\widetilde{\rho}$ and $\widetilde{K}_{\gamma,\rho}(x)$ given in (??), (??), (??), (??) and (??), respectively.

If (??) holds for some $\gamma \leq 0$ then also, for any $\epsilon > 0$, there exists $t_0 = t_0(\epsilon)$ such that for $t \geq t_0$ and $x \geq 1$,

$$\left| \frac{\frac{U(t)}{a^*(t)}\left(1 - \frac{U(t)}{U(tx)}\right) - \frac{x^\gamma - 1}{\gamma}}{A^*(t)} - K^*_{\gamma,\rho}(x) \right| \leq \epsilon \ x^{\gamma+\rho^*+\epsilon}$$

and

$$\left| \frac{\left(\frac{U(t)}{a^*(t)}\left(1 - \frac{U(t)}{U(tx)}\right)\right)^2 - \left(\frac{x^\gamma - 1}{\gamma}\right)^2}{A^*(t)} - 2\left(\frac{x^\gamma - 1}{\gamma}\right) K^*_{\gamma,\rho}(x) \right| \leq \epsilon x^{\gamma+\rho^*+\epsilon},$$

now with $K^*_{\gamma,\rho^*}(x)$, $(A^*(t), a^*(t))$ and ρ^* given in (??), (??) and (??), respectively.

Here and throughout the paper, let $\{Y_{i,n}\}_{i=1}^n$ be the ascending order statistics associated to the independent r.v.'s $\{Y_i\}_{i=1}^n$ with common standard Pareto d.f., $1 - y^{-1}$, for all $y \geq 1$. For the proofs, bear in mind the equality in distribution

$$\{X_{i,n}\}_{i=1}^n \stackrel{d}{=} \{U(Y_{i,n})\}_{i=1}^n .$$

Since $k = k_n$ is an intermediate sequence, $Y_{n-k,n} \xrightarrow[n\to\infty]{as} \infty$, Rényi's representation enables us to write

$$Q_{i,n} := \frac{Y_{n-i+1,n}}{Y_{n-k,n}} \stackrel{d}{=} Y_{k-i+1:k}, \ 1 \leq i \leq k, \tag{4.15}$$

and for any measurable function g,

$$\frac{1}{k} \sum_{i=1}^k g(Q_{i,n}) \stackrel{d}{=} \frac{1}{k} \sum_{i=1}^k g(Y_i). \tag{4.16}$$

Proposition 4.2. *Under the first order condition (??), for intermediate sequences $k = k_n$, with*

*$M_n^{(j)}(k)$ and $L_n^{(j)}(k)$, $j \geq 1$, given in (**??**), the following limits in probability hold:*

$$\left(\frac{X_{n-k,n}}{a(n/k)}\right) M_n^{(1)}(k) \underset{n\to\infty}{\overset{P}{\longrightarrow}} \frac{1}{1-\gamma_-},$$

$$\left(\frac{X_{n-k,n}}{a(n/k)}\right) L_n^{(1)}(k) \underset{n\to\infty}{\overset{P}{\longrightarrow}} \frac{1}{1+|\gamma|}, \quad \left(\frac{X_{n-k,n}}{a(n/k)}\right)^2 L_n^{(1)}(k) \underset{n\to\infty}{\overset{P}{\longrightarrow}} \frac{\gamma_+^{-1}}{1+|\gamma|},$$

$$\left(\frac{X_{n-k,n}}{a(n/k)}\right)^2 L_n^{(2)}(k) \underset{n\to\infty}{\overset{P}{\longrightarrow}} \frac{2}{(1+|\gamma|)(1+2|\gamma|)},$$

$$\left(\frac{X_{n-k,n}}{a(n/k)}\right)^3 L_n^{(3)}(k) \underset{n\to\infty}{\overset{P}{\longrightarrow}} \frac{6}{(1+|\gamma|)(1+2|\gamma|)(1+3|\gamma|)},$$

$$\left(\frac{X_{n-k,n}}{a(n/k)}\right)^4 L_n^{(4)}(k) \underset{n\to\infty}{\overset{P}{\longrightarrow}} \frac{24}{(1+|\gamma|)(1+2|\gamma|)(1+3|\gamma|)(1+4|\gamma|)}.$$

Proof. First define

$$L_{k,n} := \left(\frac{k}{n}\right) Y_{n-k,n}.$$

Since $k = k_n$ is an intermediate sequence,

$$L_{k,n} \underset{n\to\infty}{\overset{P}{\longrightarrow}} 1 \tag{4.17}$$

(cf. e.g. Smirnov, 1952). If (**??**) holds, then, as stated in Lemma **??**, $a \in RV_\gamma$. It thus follows from the uniform convergence theorem for regularly varying functions (see Geluk and de Haan, 1987, Theorem 1.3) that

$$\frac{a\left(Y_{n-k,n}\right)}{a\left(\frac{n}{k}\right)} = \frac{a\left(\frac{n}{k}\ L_{k,n}\right)}{a\left(\frac{n}{k}\right)} \underset{n\to\infty}{\overset{P}{\longrightarrow}} 1. \tag{4.18}$$

Concerning $M_n^{(1)}(k)$, we may write

$$\left(\frac{X_{n-k,n}}{a(n/k)}\right) M_n^{(1)}(k) \overset{d}{=} \frac{a\left(Y_{n-k,n}\right)}{a(n/k)}\ \frac{1}{k}\sum_{i=1}^{k} \frac{\ln U\left(Y_{n-i+1,n}\right) - \ln U\left(Y_{n-k,n}\right)}{a\left(Y_{n-k,n}\right)/U\left(Y_{n-k,n}\right)}$$

and the core of this part of the proof lies at relation (**??**). Hence, using (**??**) and applying the result in Lemma **??**, related to $\ln U$, we get

$$\frac{1}{k}\sum_{i=1}^{k} \frac{\ln U\left(Y_{n-i+1,n}\right) - \ln U\left(Y_{n-k,n}\right)}{a\left(Y_{n-k,n}\right)/U\left(Y_{n-k,n}\right)} < \frac{1}{k}\sum_{i=1}^{k} \frac{Q_{i,n}^{\gamma_-} - 1}{\gamma_-} + \frac{\epsilon}{k}\sum_{i=1}^{k} Q_{i,n}^{\gamma_- + \epsilon}. \tag{4.19}$$

Let us look at the right hand-side of (**??**): for an intermediate sequence $k = k_n$, the law of large numbers and (**??**) ensures that the first term converges in probability towards $\int_1^\infty y^{-2}\left(y^{\gamma_-} - 1\right)\ dy/\gamma_- = (1-\gamma_-)^{-1}$, as $n \to \infty$, while for any $\epsilon > 0$, the second term is such that

$$\frac{1}{k}\sum_{i=1}^{k} Q_{i,n}^{\gamma_- + \epsilon} \underset{n\to\infty}{\overset{P}{\longrightarrow}} \int_1^\infty y^{-2}\ y^{\gamma_- + \epsilon}\ dy < \infty.$$

A similar reasoning can be applied to the obvious lower bound that comes from Lemma **??**, and we get

$$\frac{1}{k}\sum_{i=1}^{k}\frac{\ln U\left(Y_{n-i+1,n}\right)-\ln U\left(Y_{n-k,n}\right)}{a\left(Y_{n-k,n}\right)/U\left(Y_{n-k,n}\right)} \underset{n\to\infty}{\overset{P}{\longrightarrow}} \frac{1}{1-\gamma_-}.$$

Hence, (**??**) and Slutsky's theorem yield the result.

The moment statistics $L_n^{(j)}(k)$, $j=1,2,3$ are based on the random terms

$$\frac{X_{n-k,n}}{a(n/k)}\left(1-\frac{X_{n-k,n}}{X_{n-i+1,n}}\right) \overset{d}{=} \frac{a\left(Y_{n-k,n}\right)}{a(n/k)}\,\frac{U\left(Y_{n-i+1,n}\right)-U\left(Y_{n-k,n}\right)}{a\left(Y_{n-k,n}\right)}\,\frac{U\left(Y_{n-k,n}\right)}{U\left(Y_{n-i+1,n}\right)}, \quad i=1,2,\cdots,k.$$

Under condition (**??**), Lemma **??** guarantees that, for any $\epsilon>0$, and sufficiently large n,

$$\frac{1}{k}\sum_{i=1}^{k}\frac{U\left(Y_{n-i+1,n}\right)-U\left(Y_{n-k,n}\right)}{a\left(Y_{n-k,n}\right)}\,\frac{U\left(Y_{n-k,n}\right)}{U\left(Y_{n-i+1,n}\right)} \;<\; \frac{1+\epsilon}{k}\sum_{i=1}^{k}\left(\frac{Q_{i.n}^{\gamma}-1}{\gamma}+\epsilon\, Q_{i,n}^{\gamma+\epsilon}\right)Q_{i,n}^{-\gamma_++\epsilon}.$$

The law of large numbers and (**??**) ensures that this upper bound is equal in distribution to

$$\frac{1+\epsilon}{k}\sum_{i=1}^{k}\frac{Y_i^{\gamma}-1}{\gamma}\,Y_i^{-\gamma_++\epsilon}+\frac{\epsilon}{k}\sum_{i=1}^{k}Y_i^{\gamma_-+\epsilon}$$

and converges in probability towards

$$\frac{1+\epsilon}{\gamma}\int_1^{\infty}y^{-2}\left(y^{\gamma-\gamma_++\epsilon}-y^{\gamma_+-\epsilon}\right)\,dy+\epsilon\int_1^{\infty}y^{-2}y^{\gamma_-+\epsilon}\,dy.$$

Since $\epsilon>0$ is arbitrary, Lebesgue' s dominated convergence theorem may be applied to the first integral with dominating function given by $g(y)=((1+\epsilon_0)/\gamma)\left(y^{\gamma_-+\epsilon_0}-y^{-(\gamma_+-\epsilon_0)}\right)$, for all $y\geq 1$ and $\epsilon<\epsilon_0<1+|\gamma|$, integrable against dy/y^2 and such that

$$\lim_{\epsilon\downarrow 0}\frac{1+\epsilon}{\gamma}\int_1^{\infty}\left(y^{\gamma_-+\epsilon_0}-y^{-(\gamma_+-\epsilon_0)}\right)\frac{dy}{y^2}=\frac{1}{\gamma}\int_1^{\infty}\left(y^{\gamma_-}-y^{-\gamma_+}\right)\frac{dy}{y^2}=\frac{1}{(1-\gamma_-)(1+\gamma_+)}=\frac{1}{1+|\gamma|}.$$

The second integral is finite for any $\epsilon<1-\gamma_-$. Consequently, the second sum in the upper bound is stochastically bounded. We can establish a similar asymptotic lower bound, and apply a similar reasoning to that lower bound. From (**??**) we thus get

$$\left(\frac{X_{n-k,n}}{a(n/k)}\right)L_n^{(1)}(k)\underset{n\to\infty}{\overset{P}{\longrightarrow}}\frac{1}{1+|\gamma|}. \tag{4.20}$$

Now, (**??**) jointly with (**??**) imply the convergence

$$\frac{X_{n-k,n}}{a(n/k)} \overset{d}{=} \frac{a\left(Y_{n-k,n}\right)}{a(n/k)}\,\frac{U\left(Y_{n-k,n}\right)}{a\left(Y_{n-k,n}\right)} \underset{n\to\infty}{\overset{P}{\longrightarrow}} \frac{1}{\gamma_+} \tag{4.21}$$

and (**??**) together with (**??**) imply that

$$\left(\frac{X_{n-k,n}}{a(n/k)}\right)^2 L_n^{(1)}(k)\underset{n\to\infty}{\overset{P}{\longrightarrow}}\frac{\gamma_+^{-1}}{1+|\gamma|}.$$

The proofs of the other limiting results are similar and are therefore omitted. ■

Lemma 4.6. *Let $k = k_n$ be an intermediate sequence. If the underlying quantile function U satisfies (??) with $\gamma \leq 0$, then, for any $j \geq m \geq 1$,*

$$\left(\frac{U\left(Y_{n-k,n}\right)}{a(n/k)}\right)^m \frac{1}{k}\sum_{i=1}^{k}\frac{U\left(Y_{n-i+1,n}\right)}{U\left(Y_{n-k,n}\right)}\left(1-\frac{U\left(Y_{n-k,n}\right)}{U\left(Y_{n-i+1,n}\right)}\right)^j = O_p\left(\left(\frac{a(n/k)}{U\left(Y_{n-k,n}\right)}\right)^{j-m}\right).$$

Proof. Since $U\left(Y_{n-k,n}\right) \leq U\left(Y_{n-i+1,n}\right)$, for $i = 1, 2, \cdots, k$, we may write

$$\left(\frac{U\left(Y_{n-k,n}\right)}{a(n/k)}\right)^{j-m} \frac{1}{k}\sum_{i=1}^{k}\left(\frac{U\left(Y_{n-k,n}\right)}{U\left(Y_{n-i+1,n}\right)}\right)^{j-1}\left(\frac{U\left(Y_{n-i+1,n}\right)-U\left(Y_{n-k,n}\right)}{a\left(Y_{n-k,n}\right)}\right)^j$$
$$\leq \left(\frac{U\left(Y_{n-k,n}\right)}{a(n/k)}\right)^{j-m}\left\{\frac{1}{k}\sum_{i=1}^{k}\left(\frac{U\left(Y_{n-i+1,n}\right)-U\left(Y_{n-k,n}\right)}{a\left(Y_{n-k,n}\right)}\right)^j\right\}.$$

It follows from the first inequality of Lemma **??** that the second factor is $O_p(1)$. ■

Remark 4.4. *If we consider $a^*(t)$ in (??), since $a^*(t) \sim a(t)$, as $t \to \infty$, the results in Lema ?? and Lema ?? obviously hold if we replace $a(n/k)$ by $a^*(n/k)$.*

The two following lemmas follow immediately from the central limit theorem (cf. Billingsley, 1979, Theorem 29.5):

Lemma 4.7. *For $\gamma > 0$,*

$$\sqrt{k}\left(\frac{1}{k}\sum_{i=1}^{k}\ln Y_i - 1,\ \frac{1}{k}\sum_{i=1}^{k}\left(1-Y_i^{-\gamma}\right)-\frac{\gamma}{1+\gamma},\ \frac{1}{k}\sum_{i=1}^{k}\frac{\left(1-Y_i^{-\gamma}\right)^2}{2}-\frac{\gamma^2}{(1+\gamma)(1+2\gamma)}\right)$$
$$\xrightarrow[k\to\infty]{d} \left(P_0, P_1, P_2\right),$$

where (P_0, P_1, P_2) has a joint normal distribution with mean zero and covariance matrix given by

$$\begin{cases} E\left(P_0^2\right) = 1, & E\left(P_1^2\right) = \frac{\gamma^2}{(1+\gamma)^2(1+2\gamma)}, & E\left(P_2^2\right) = \frac{\gamma^4(5+11\gamma)}{(1+\gamma)^2(1+2\gamma)^2(1+3\gamma)(1+4\gamma)} \\ E\left(P_0P_1\right) = \frac{\gamma}{(1+\gamma)^2}, & E\left(P_0P_2\right) = \frac{\gamma^2(2+3\gamma)}{(1+\gamma)^2(1+2\gamma)^2}, & E\left(P_1P_2\right) = \frac{2\gamma^3}{(1+\gamma)^2(1+2\gamma)(1+3\gamma)} \end{cases}.$$

Lemma 4.8. *For $\gamma \leq 0$,*

$$\sqrt{k}\left(\frac{1}{k}\sum_{i=1}^{k}\left(\frac{Y_i^\gamma-1}{\gamma}\right)-\frac{1}{1-\gamma},\ \frac{1}{k}\sum_{i=1}^{k}\frac{1}{2}\left(\frac{Y_i^\gamma-1}{\gamma}\right)^2-\frac{1}{(1-\gamma)(1-2\gamma)}\right)\xrightarrow[k\to\infty]{d}\left(P_1^*, P_2^*\right),$$

where (P_1^*, P_2^*) has a joint normal distribution with mean zero and covariance matrix given by

$$E\left((P_1^*)^2\right) = \frac{1}{(1-\gamma)^2(1-2\gamma)},\quad E\left((P_2^*)^2\right) = \frac{5-11\gamma}{(1-\gamma)^2(1-2\gamma)^2(1-3\gamma)(1-4\gamma)},\quad E\left(P_1^*P_2^*\right) = \frac{2}{(1-\gamma)^2(1-2\gamma)(1-3\gamma)}.$$

Again with Y a unit Pareto r.v, let us denote, for any $\alpha,\ \beta \le 0$,

$$d_{\alpha,\beta} := \mathbb{E}\left(Y^{\alpha}\left(\frac{Y^{\beta}-1}{\beta}\right)\right) = \frac{1}{(1-\alpha)(1-\alpha-\beta)}. \tag{4.22}$$

Proposition 4.3. *Assume that the second order condition (**??**) holds with $\gamma > 0$. If $k = k_n$ is an intermediate sequence, then, as $n \to \infty$,*

$$\begin{aligned}
\frac{X_{n-k,n}}{a(n/k)} &= \frac{1}{\gamma} + \frac{B}{\sqrt{k}} - \frac{\overline{A}(n/k)}{\gamma^2}(1+o_p(1)) + o_p\left(\widetilde{A}(n/k)\right) + o_p\left(1/\sqrt{k}\right)\\
M_n^{(1)}(k) &= \gamma + \frac{\gamma\ P_0}{\sqrt{k}} + d_{M_1}\widetilde{A}(n/k) + o_p\left(\widetilde{A}(n/k)\right) + o_p\left(1/\sqrt{k}\right),\\
L_n^{(1)}(k) &= \frac{\gamma}{1+\gamma} + \frac{P_1}{\sqrt{k}} + d_{L_1}\widetilde{A}(n/k) + o_p\left(\widetilde{A}(n/k)\right) + o_p\left(1/\sqrt{k}\right),\\
\left\{L_n^{(1)}(k)\right\}^2 &= \frac{\gamma^2}{(1+\gamma)^2} + \frac{2\gamma}{1+\gamma}\ \frac{P_1}{\sqrt{k}} + \frac{2\gamma\ d_{L_1}\widetilde{A}(n/k)}{1+\gamma} + o_p\left(\widetilde{A}(n/k)\right) + o_p\left(1/\sqrt{k}\right),\\
L_n^{(2)}(k) &= \frac{2\ \gamma^2}{(1+\gamma)(1+2\gamma)} + \frac{2\ P_2}{\sqrt{k}} + d_{L_2}\widetilde{A}(n/k) + o_p\left(\widetilde{A}(n/k)\right) + o_p\left(1/\sqrt{k}\right),
\end{aligned}$$

*where $\widetilde{A}(t)$, $\widetilde{\rho}$ and $d_{\alpha,\beta}$ are given in (**??**), (**??**) and (**??**), respectively, and B is a standard normal r.v. independent of $(P_0,\ P_1,\ P_2)$, the normally distributed vector in Lemma **??**. Moreover, we get*

$$d_{M_1} = d_{0,\widetilde{\rho}} = \frac{1}{1-\widetilde{\rho}}, \qquad d_{L_1} = d_{-\gamma,\widetilde{\rho}} = \frac{1}{(1+\gamma)(1+\gamma-\widetilde{\rho})},$$

$$d_{L_2} = 2\left(d_{-\gamma,\widetilde{\rho}} - d_{-2\gamma,\widetilde{\rho}}\right) = \frac{2\gamma(2+3\gamma-\widetilde{\rho})}{(1+\gamma)(1+2\gamma)(1+\gamma-\widetilde{\rho})(1+2\gamma-\widetilde{\rho})}.$$

Proof. Since (**??**) holds, $(k\ Y_{n-k,n}/n)^{\gamma} = 1 + \gamma\ B/\sqrt{k} + o_p\left(\widetilde{A}(n/k)\right) + o_p\left(1/\sqrt{k}\right)$, with B a standard normal r.v. (see de Haan and Ferreira, 2006, Theorem 2.4.2) and $X_{n-k,n} \stackrel{d}{=} U\left(Y_{n-k,n}\right)$, we may write, as $n \to \infty$,

$$\begin{aligned}
\frac{X_{n-k,n}}{a(n/k)} &= \frac{U(n/k)}{a(n/k)}\left(\left(\frac{k\ Y_{n-k,n}}{n}\right)^{\gamma} - \widetilde{A}\left(\frac{n}{k}\right)\left(\frac{k\ Y_{n-k,n}}{n}\right)^{2\gamma} K_{\gamma,\rho}\left(\frac{k\ Y_{n-k,n}}{n}\right) + o_p\left(\widetilde{A}\left(\frac{n}{k}\right)\right)\right)\\
&= \frac{U(n/k)}{a(n/k)}\left(1 + \frac{\gamma\ B}{\sqrt{k}} + o_p\left(\widetilde{A}(n/k)\right) + o_p\left(1/\sqrt{k}\right)\right).
\end{aligned}$$

Notice now that from (**??**), $a(n/k)/U(n/k) = \gamma + \overline{A}(n/k)$, and consequently,

$$\frac{U(n/k)}{a(n/k)} = \frac{1}{\gamma}\left(1 - \frac{\overline{A}(n/k)}{\gamma}(1+o(1))\right).$$

Consequently,

$$\frac{X_{n-k,n}}{a(n/k)} = \frac{1}{\gamma} + \frac{B}{\sqrt{k}} - \frac{\overline{A}(n/k)}{\gamma^2}(1+o_p(1)) + o_p\left(\widetilde{A}(n/k)\right) + o_p\left(1/\sqrt{k}\right).$$

The use of (**??**) and (**??**), together with the inequalities in Lemma **??**, for $\gamma > 0$, leads us to

$$
\begin{aligned}
M_n^{(1)}(k) &= \frac{\gamma}{k}\sum_{i=1}^{k}\ln Y_i + \widetilde{A}\left(\frac{n}{k}\right)\left(\frac{1}{k}\sum_{i=1}^{k}\widetilde{K}_{\gamma,\rho}(Y_i) + \frac{o_p(1)}{k}\sum_{i=1}^{k}Y_i^{\widetilde{\rho}+\epsilon}\right),\\
L_n^{(1)}(k) &= \frac{1}{k}\sum_{i=1}^{k}\left(1-Y_i^{-\gamma}\right) - \widetilde{A}\left(\frac{n}{k}\right)\left(\frac{1}{k}\sum_{i=1}^{k}K_{\gamma,\rho}(Y_i) + \frac{o_p(1)}{k}\sum_{i=1}^{k}Y_i^{-\gamma+\widetilde{\rho}+\epsilon}\right)\\
L_n^{(2)}(k) &= \frac{1}{k}\sum_{i=1}^{k}\left(1-Y_i^{-\gamma}\right)^2 - \widetilde{A}\left(\frac{n}{k}\right)\left(\frac{2}{k}\sum_{i=1}^{k}\left(1-Y_i^{-\gamma}\right)K_{\gamma,\rho}(Y_i) + \frac{o_p(1)}{k}\sum_{i=1}^{k}Y_i^{-\gamma+\widetilde{\rho}+\epsilon}\right),
\end{aligned}
$$

with $\left(L_n^{(j)}(k), M_n^{(j)}(k)\right)$, $K_{\gamma,\rho}(x)$, $\widetilde{A}(t)$ and $\widetilde{K}_{\gamma,\rho}(x)$ given in (**??**), (**??**), (**??**) and (**??**), respectively. The law of large numbers implies that the partial sums associated with the $o_p(1)$ terms are negligible. Hence, all the results in the proposition follow straightforwardly. ∎

Let us now define, again with $a^*(t)$ given in (**??**),

$$M_1^* := \left(\frac{X_{n-k,n}}{a^*(n/k)}\right) M_n^{(1)}(k) \underset{n\to\infty}{\overset{P}{\longrightarrow}} \frac{1}{1-\gamma_-}, \tag{4.23}$$

$$L_j^* := \left(\frac{X_{n-k,n}}{a^*(n/k)}\right)^j L_n^{(j)}(k), \quad j \geq 1 \tag{4.24}$$

$$R_{k,n}^{(m,j)} := \left(\frac{U\left(Y_{n-k,n}\right)}{a^*(n/k)}\right)^m \frac{1}{k}\sum_{i=1}^{k}\frac{U\left(Y_{n-i+1,n}\right)}{U\left(Y_{n-k,n}\right)}\left(1-\frac{U\left(Y_{n-k,n}\right)}{U\left(Y_{n-i+1,n}\right)}\right)^j, \quad j \geq m \geq 1. \tag{4.25}$$

Proposition 4.4. *Under the conditions in Proposition* **??**, *for* $\gamma \leq 0$, *with* L_j^*, $j = 0, 1$ *given in* (**??**), *and if* $k = k_n$ *is an intermediate sequence,*

$$L_1^* \overset{d}{=} \frac{1}{1-\gamma} + \frac{P_1^*}{\sqrt{k}} + d_{L_1}^* A^*\left(\frac{n}{k}\right) + o_p\left(A^*\left(\frac{n}{k}\right)\right) + o_p\left(\frac{1}{\sqrt{k}}\right),$$

$$(L_1^*)^2 \overset{d}{=} \frac{1}{(1-\gamma)^2} + \frac{2}{1-\gamma}\frac{P_1^*}{\sqrt{k}} + \frac{2\, d_{L_1}^*}{1-\gamma} A^*\left(\frac{n}{k}\right) + o_p\left(A^*\left(\frac{n}{k}\right)\right) + o_p\left(\frac{1}{\sqrt{k}}\right),$$

$$L_2^* \overset{d}{=} \frac{2}{(1-\gamma)(1-2\gamma)} + \frac{2\, P_2^*}{\sqrt{k}} + 2\, d_{L_2}^* A^*\left(\frac{n}{k}\right) + o_p\left(A^*\left(\frac{n}{k}\right)\right) + o_p\left(\frac{1}{\sqrt{k}}\right),$$

as $n \to \infty$, *where* $(P_1^*,\ P_2^*)$ *is normally distributed with mean vector zero and covariance matrix given in Lemma* **??**, *and with* ρ^* *and* $d_{\alpha,\beta}$ *given in* (**??**) *and* (**??**), *respectively,*

$$d_{L_1}^* = \begin{cases} d_{\gamma,0} = \frac{1}{(1-\gamma)^2} & \text{if } \gamma < \rho = 0 \\ d_{0,\gamma+\rho^*} = \frac{1}{1-\gamma-\rho^*} & \text{if } \gamma < \rho < 0 \text{ or } \rho < \gamma < 0 \\ 2 & \text{if } \rho < \gamma = 0 \end{cases}, \tag{4.26}$$

$$d^*_{L_2} = \begin{cases} \frac{1}{\gamma}\left(d_{2\gamma,0} - d_{\gamma,0}\right) = \frac{2-3\gamma}{(1-\gamma)^2(1-2\gamma)^2} & \textit{if} \quad \gamma < \rho = 0 \\ \frac{1}{\gamma}\left(d_{\gamma,\gamma+\rho^*} - d_{0,\gamma+\rho^*}\right) = \frac{2-2\gamma-\rho^*}{(1-\gamma)(1-\gamma-\rho^*)(1-2\gamma-\rho^*)} & \textit{if} \quad \gamma < \rho < 0 \ \textit{or} \ \rho < \gamma < 0 \\ 6 & \textit{if} \quad \rho < \gamma = 0 \end{cases}. \tag{4.27}$$

Proof. The results in Lemma **??** for $\gamma \le 0$, lead us to the following distributional representations,

$$L_1^* \stackrel{d}{=} \frac{1}{k}\sum_{i=1}^{k} \frac{Q_{i,n}^{\gamma} - 1}{\gamma} + A^*\left(Y_{n-k,n}\right)\left(\frac{1}{k}\sum_{i=1}^{k} K^*_{\gamma,\rho}\left(Q_{i,n}\right) + \frac{o_p(1)}{k}\sum_{i=1}^{k}\left(Q_{i,n}\right)^{\gamma+\rho^*+\epsilon}\right)$$

and

$$L_2^* \stackrel{d}{=} \frac{1}{k}\sum_{i=1}^{k}\left(\frac{Q_{i,n}^{\gamma} - 1}{\gamma}\right)^2 + A^*\left(Y_{n-k,n}\right)\left(\frac{1}{k}\sum_{i=1}^{k} 2\left(\frac{Q_{i,n}^{\gamma} - 1}{\gamma}\right) K^*_{\gamma,\rho}\left(Q_{i,n}\right) + \frac{o_p(1)}{k}\sum_{i=1}^{k}\left(Q_{i,n}\right)^{\gamma+\rho^*+\epsilon}\right),$$

with $K^*_{\gamma,\rho}(x)$ and $A^*(t)$ given in (**??**) and (**??**), respectively. Since (**??**) holds for any intermediate sequence $k = k_n$, the use of (**??**) and (**??**) enables us to write:

$$\begin{aligned} L_1^* &\stackrel{d}{=} \frac{1}{k}\sum_{i=1}^{k} \frac{Y_i^{\gamma} - 1}{\gamma} + A^*\left(\frac{n}{k}\right)\left(\frac{1}{k}\sum_{i=1}^{k} K^*_{\gamma,\rho}\left(Y_i\right) + \frac{o_p(1)}{k}\sum_{i=1}^{k} Y_i^{\gamma+\rho^*+\epsilon}\right) \\ L_2^* &\stackrel{d}{=} \frac{1}{k}\sum_{i=1}^{k}\left(\frac{Y_i^{\gamma} - 1}{\gamma}\right)^2 + A^*\left(\frac{n}{k}\right)\left(\frac{1}{k}\sum_{i=1}^{k} 2\left(\frac{Y_i^{\gamma} - 1}{\gamma}\right) K^*_{\gamma,\rho}\left(Y_i\right) + \frac{o_p(1)}{k}\sum_{i=1}^{k} Y_i^{\gamma+\rho^*+\epsilon}\right). \end{aligned}$$

The random term associated to $o_p(1)$ is stochastically bounded. Indeed the integral appearing in the limit of that term is finite for any $\epsilon < 1 - \gamma - \rho^*$. The law of large numbers ensures then that $d^*_{L_1} = \mathbb{E}\left(K^*_{\gamma,\rho}(Y)\right)$ and $d^*_{L_2} = \mathbb{E}\left(\left(\frac{Y^{\gamma}-1}{\gamma}\right) K^*_{\gamma,\rho}(Y)\right)$. Taking P_1^* and P_2^* as the normal limiting r.v.'s of Lemma **??** and noticing that

$$(L_1^*)^2 - \frac{1}{(1-\gamma)^2} = \left(L_1^* - \frac{1}{1-\gamma}\right)\left(L_1^* + \frac{1}{1-\gamma}\right) = \frac{2}{1-\gamma}\left(L_1^* - \frac{1}{1-\gamma}\right)(1 + o_p(1)),$$

we bring the proof to an end. ■

5 Proof of the main results

Proof. [**Theorem ??**]. The proof of Theorem **??** relies essentially on the results in Proposition **??**. Note first that the estimator in (**??**) can be written as

$$\widehat{\varphi}_n(k) = \frac{\left(X_{n-k,n}/a^*(n/k)\right) M_n^{(1)}(k) - \left(X_{n-k,n}/a^*(n/k)\right) L_n^{(1)}(k)}{\left(X_{n-k,n}/a^*(n/k)\right)\left(L_n^{(1)}(k)\right)^2} = \frac{M_1^* - L_1^*}{\left(a^*(n/k)/X_{n-k,n}\right)(L_1^*)^2}.$$

We shall now consider the cases $\gamma > 0$ and $\gamma \leq 0$ separately. If $\gamma > 0$, the use of Proposition **??**, together with Slutsky's argument, leads us to:

$$\widehat{\varphi}_n(k) \underset{n\to\infty}{\overset{P}{\longrightarrow}} \frac{1-(1+\gamma)^{-1}}{\gamma(1+\gamma)^{-2}} = 1+\gamma.$$

If $\gamma \leq 0$,

$$\begin{aligned} M_n^{(1)}(k) - L_n^{(1)}(k) &\overset{d}{=} \frac{1}{k}\sum_{i=1}^{k}\left\{\ln\frac{U(Y_{n-i+1,n})}{U(Y_{n-k,n})} - \left(1-\frac{U(Y_{n-k,n})}{U(Y_{n-i+1,n})}\right)\right\} \\ &= \frac{1}{k}\sum_{i=1}^{k}\left\{-\ln\left(1-\left(1-\frac{U(Y_{n-k,n})}{U(Y_{n-i+1,n})}\right)\right) - \left(1-\frac{U(Y_{n-k,n})}{U(Y_{n-i+1,n})}\right)\right\} \quad (5.1)\end{aligned}$$

Note now that

$$0 < -\ln(1-x) - x - \frac{x^2}{2} < \frac{x^3}{3(1-x)}$$

and use it to deal with (**??**), after assigning $x := 1 - \frac{U(Y_{n-k,n})}{U(Y_{n-i+1,n})}$. Then, with M_1^* and L_j^*, $j=1,2$ given in (**??**) and (**??**), respectively, we get, with $R_{k.n}^{(m,j)}$ defined in (**??**),

$$\frac{L_2^*}{2} < \frac{U(Y_{n-k,n})}{a^*(n/k)}\left(M_1^* - L_1^*\right) < \frac{L_2^*}{2} + \frac{R_{k,n}^{(2,3)}}{3}.$$

Then, since $a^*(n/k)/U(Y_{n-k,n}) \underset{n\to\infty}{\overset{P}{\longrightarrow}} 0$, $R_{k,n}^{(2,3)} = O_p\left(a^*(n/k)/U(Y_{n-k,n})\right) = o_p(1)$. From (**??**) and Proposition **??** we know that $(L_1^*)^2 \underset{n\to\infty}{\overset{P}{\longrightarrow}} 1/(1-\gamma)^2$ and $L_2^*/2 \underset{n\to\infty}{\overset{P}{\longrightarrow}} 1/\left((1-\gamma)(1-2\gamma)\right)$. Therefore, Slutsky's theorem leads us to the desired result, i.e.,

$$\widehat{\varphi}_n(k) = \frac{U(Y_{n-k,n})}{a^*(n/k)}\frac{M_1^* - L_1^*}{(L_1^*)^2} \overset{d}{=} \frac{L_2^*}{2\,(L_1^*)^2} \underset{n\to\infty}{\overset{P}{\longrightarrow}} \frac{1-\gamma}{1-2\gamma}, \quad \gamma \leq 0. \quad (5.2)$$

■

Proof. [**Corollary ??**]. Theorem **??** ensures the consistency of $\widehat{\varphi}_n(k)$ as an estimator of the strictly increasing function $\varphi(\gamma)$, in (**??**). Hence, by simple inversion and a Slutsky's argument, we get the consistency of $\widehat{\gamma}_n^{MM}(k)$, in (**??**), for any real γ. ■

Proof. [**Theorem ??**]. Let us consider first the case $\gamma > 0$. Due to the definition of $\widehat{\varphi}_n(k)$ in (**??**) and given the limits in probability provided in Proposition **??**, we write

$$\begin{aligned}\sqrt{k}\left\{\widehat{\varphi}_n(k) - (1+\gamma)\right\} &= \sqrt{k}\,\frac{M_n^{(1)}(k) - L_n^{(1)}(k) - (1+\gamma)\left(L_n^{(1)}(k)\right)^2}{\left(L_n^{(1)}(k)\right)^2} = \left(\frac{1+\gamma}{\gamma}\right)^2\left[\sqrt{k}\,\left(M_n^{(1)}(k) - \gamma\right)\right. \\ &\left. -\sqrt{k}\left(L_n^{(1)}(k) - \frac{\gamma}{1+\gamma}\right) - (1+\gamma)\sqrt{k}\left(\left(L_n^{(1)}(k)\right)^2 - \left(\frac{\gamma}{1+\gamma}\right)^2\right)\right](1+o_p(1)).\end{aligned}$$

Now, the results in Proposition **??** enable us to write

$$\sqrt{k}\,\{\widehat{\varphi}_n(k)-(1+\gamma)\} = \left(\frac{1+\gamma}{\gamma}\right)^2 \left[(\gamma\ P_0-(1+2\gamma)P_1)+\sqrt{k}\ \widetilde{A}\left(\frac{n}{k}\right)\left(d_{_{M_1}}-(1+2\gamma)d_{_{L_1}}\right)\right] + o_p\left(\sqrt{k}\ \widetilde{A}\left(\frac{n}{k}\right)\right)+o_p(1),$$

and, on the basis of Lemma **??**, we easily get an asymptotic variance equal to $(1+\gamma)^2$.

- If $c=\gamma/(\gamma+\rho)$, $\widetilde{\rho}=\rho$, $\widetilde{A}(n/k)=\gamma\ A(n/k)/(\gamma+\rho)$ and we get

  $$\sqrt{k}\,\{\widehat{\varphi}_n(k)-(1+\gamma)\} = \left(\frac{1+\gamma}{\gamma}\right)^2 \left[(\gamma\ P_0-(1+2\gamma)P_1)+\frac{\sqrt{k}\ A\left(\frac{n}{k}\right)\gamma^2}{(1-\rho)(1+\gamma)(1+\gamma-\rho)}\right] + o_p\left(\sqrt{k}\ A\left(\frac{n}{k}\right)\right)+o_p(1),$$

  and the asymptotic bias provided for the region where $c=\gamma/(\gamma+\rho)$ follows.

- If $c=\pm\infty$, $\widetilde{\rho}=-\gamma$, $\widetilde{A}(t)=\overline{A}(t)$ and $d_{_{M_1}}-d_{_{L_1}}\equiv 0$. Consequently, the asymptotic bias of $\sqrt{k}\,\{\widehat{\varphi}_n(k)-(1+\gamma)\}$ is null provided we consider levels such that $\sqrt{k}\ \overline{A}(n/k)\to\lambda$, finite, as $n\to\infty$.

If $\gamma\le 0$, we use the distributional representation in (**??**). Similarly to the proof of Theorem **??**, if we now use the inequalities, $0<-\ln(1-x)-x-x^2/2-x^3/3<x^4/(4(1-x))$, $0<x<1$, for $x=1-U\left(Y_{n-k,n}\right)/U\left(Y_{n-i+1,n}\right)$, we get

$$\frac{L_2^*}{2}+\frac{a^*(n/k)}{U\left(Y_{n-k,n}\right)}\ \frac{L_3^*}{3}<\frac{U\left(Y_{n-k,n}\right)}{a^*(n/k)}\left(M_1^*-L_1^*\right)<\frac{L_2^*}{2}+\frac{a^*(n/k)}{U\left(Y_{n-k,n}\right)}\ \frac{L_3^*}{3}+\frac{R_{k,n}^{(2,4)}}{4},$$

with $R_{k,n}^{(m,j)}$ in (**??**). Due to the limits in probability provided in Proposition **??**, if we consider levels k such that $\sqrt{k}\ \overline{A}(n/k)\to\lambda$, finite, and with $\overline{A}$ given in (**??**),

$$\sqrt{k}\ \overline{A}(n/k)\ \frac{L_3^*}{3}\ \underset{n\to\infty}{\overset{P}{\longrightarrow}}\ \frac{2\lambda}{(1-\gamma)(1-2\gamma)(1-3\gamma)}.$$

The same condition and Lemma **??** imply $\sqrt{k}\ R_{k,n}^{(2,4)}\ \underset{n\to\infty}{\overset{P}{\longrightarrow}}\ 0$. From Proposition **??** we get the asymptotic normality of L_2^* and $\left(L_1^*\right)^2$ and we easily derive

$$\sqrt{k}\ \left(\widehat{\varphi}_n(k)-\varphi(\gamma)\right)\ \overset{d}{=}\ (1-\gamma)^2\left(\sqrt{k}\left(\frac{L_2^*}{2}-\frac{1}{(1-\gamma)(1-2\gamma)}\right)-\frac{1-\gamma}{1-2\gamma}\sqrt{k}\left(\left(L_1^*\right)^2-\frac{1}{(1-\gamma)^2}\right) +\sqrt{k}\ \overline{A}(n/k)\frac{L_3^*}{3}\right)+o_p(1),$$

which finally leads to

$$\sqrt{k}\ (\widehat{\varphi}_n(k)-\varphi(\gamma))\ \stackrel{d}{=}\ (1-\gamma)^2\left(P_2^*-\frac{2\ P_1^*}{1-2\gamma}\right)+\sqrt{k}\ A^*(n/k)(1-\gamma)^2\ \left(d^*_{L_2}-\frac{2\ d^*_{L_1}}{1-2\gamma}\right)(1+o_p(1))$$
$$+\sqrt{k}\ \overline{A}(n/k)\ \frac{2(1-\gamma)}{(1-2\gamma)(1-3\gamma)}(1+o_p(1))+o_p(1)$$

Lema **??** leads then to the asymptotic variance in the theorem. From (**??**) and (**??**), we get

$$(1-\gamma)^2\left(d^*_{L_2}-\frac{2\ d^*_{L_1}}{1-2\gamma}\right)=\begin{cases}\frac{\rho(1-\gamma)}{(1-2\gamma)(1-\gamma-\rho)(1-2\gamma-\rho)} & \text{if}\ \ \gamma<\rho<0\\ \frac{\gamma(1-\gamma)}{(1-2\gamma)^2(1-3\gamma)} & \text{if}\ \ \rho<\gamma<0\\ \frac{\gamma}{(1-2\gamma)^2} & \text{if}\ \ \gamma<\rho=0\\ 2 & \text{if}\ \ \rho<\gamma=0\end{cases}.$$

From Proposition **??**, and from the fact that $c=0$ in the region $\gamma<\rho\leq 0$ (see (**??**)), we get

$$\lim_{t\to\infty}\frac{\overline{A}(t)}{A^*(t)}=\begin{cases}0 & \text{if}\ \ \gamma<\rho\leq 0\\ -\frac{\gamma}{2} & \text{if}\ \ \rho<\gamma<0\\ -1 & \text{if}\ \ \rho<\gamma=0\end{cases}.$$

This finally enables us to get, for the asymptotic bias, the value λb_φ in the theorem, equal to 0 whenever $\rho<\gamma=0$. ■

Proof. [**Corollary ??**]. Under the conditions of Theorem **??**, $\widehat{\varphi}_n(k)$ is a consistent estimator of $\varphi(\gamma)$. The extreme value index is given by the inverse function φ^{-1}, independent of the sample size n, with positive and continuous derivative everywhere. In fact $\frac{d}{d\gamma}\ \varphi^{-1}(\gamma)=1$ if $\gamma>0$ and $\frac{d}{d\gamma}\ \varphi^{-1}(\gamma)=(1-2\gamma)^{-2}$, otherwise. Hence, Cramér's delta-method yields that the Mixed Moment estimator in (**??**) satisfies

$$\sqrt{k}\left(\widehat{\gamma}_n^{MM}(k)-\gamma\right)\ \underset{n\to\infty}{\stackrel{d}{\longrightarrow}}\begin{cases}P & \text{if}\ \ \gamma>0\\ \frac{P}{(1-2\gamma)^2} & \text{if}\ \ \gamma\leq 0\end{cases},$$

where P stands for the normal limiting r.v. of $\sqrt{k}\,(\widehat{\varphi}_n(k)-\varphi(\gamma))$. ■

Proof. [**Proposition ??**]. In order to get a possibly non-null asymptotic bias for the case $\gamma>0$, $c=\pm\infty$, it is imperative to consider the terms provided in the expansions (**??**) and (**??**). From (**??**), and again with $d_{\alpha,\beta}$ given in (**??**), we get

$$\sqrt{k}\left(M_n^{(1)}(k)-\gamma\right)\ \stackrel{d}{=}\ \gamma\ P_0+\sqrt{k}\ \overline{A}\left(\frac{n}{k}\right)\ d_{0,-\gamma}+\frac{\gamma\ \sqrt{k}\ A(n/k)}{\gamma+\rho}\left(d_{0,\rho}-d_{0,-\gamma}\right)(1+o_p(1))$$
$$+\sqrt{k}\ \overline{A}^2\left(\frac{n}{k}\right)\frac{1}{\gamma}\left(d_{0,-2\gamma}-d_{0,-\gamma}\right)(1+o_p(1)).\quad(5.3)$$

From (**??**), we get

$$\sqrt{k}\left(L_n^{(1)}(k) - \frac{\gamma}{1+\gamma}\right) \stackrel{d}{=} P_1 + \sqrt{k}\,\overline{A}\left(\frac{n}{k}\right)\, d_{-\gamma,-\gamma} + \frac{\gamma\,\sqrt{k}\,A(n/k)}{\gamma+\rho}\,(d_{-\gamma,\rho} - d_{-\gamma,-\gamma})\,(1+o_p(1)) + \sqrt{k}\,\overline{A}^2\left(\frac{n}{k}\right)\frac{2}{\gamma}\,(d_{-\gamma,-2\gamma} - d_{-\gamma,-\gamma})\,(1+o_p(1)), \quad (5.4)$$

and consequently,

$$\sqrt{k}\left(\left(L_n^{(1)}(k)\right)^2 - \frac{\gamma^2}{(1+\gamma)^2}\right) \stackrel{d}{=} \frac{2\gamma\,P_1}{1+\gamma} + \frac{2\gamma\,d_{-\gamma,-\gamma}}{1+\gamma}\,\sqrt{k}\,\overline{A}\left(\frac{n}{k}\right) + \frac{2\gamma^2\,\sqrt{k}\,A(n/k)}{(1+\gamma)(\gamma+\rho)}\,(d_{-\gamma,\rho} - d_{-\gamma,-\gamma})\,(1+o_p(1)) + \sqrt{k}\,\overline{A}^2\left(\frac{n}{k}\right)\left(\frac{4\,(d_{-\gamma,-2\gamma} - d_{-\gamma,-\gamma})}{1+\gamma} + d_{-\gamma,-\gamma}^2\right)(1+o_p(1)), \quad (5.5)$$

Since $|A| \in RV_\rho$ and $\overline{A}^2 \in RV_{2\gamma}$, A dominates $\overline{A}^2$ if $\gamma > -\rho/2$, but $\overline{A}^2$ dominates A if $\gamma \le -\rho/2$. In the region $-\rho/2 < \gamma \le -\rho$, $l \neq 0$, with l in (**??**), we may still report the asymptotic bias of $\sqrt{k}\,\{\widehat{\varphi}_n(k) - (1+\gamma)\}$ to the function A. Whenever $\sqrt{k}\,A(n/k) \to \lambda$, we get the bias λb_0, with

$$\begin{aligned} b_0 &= \left(\frac{1+\gamma}{\gamma}\right)^2\left(\frac{\gamma}{\gamma+\rho}\right)[(d_{0,\rho} - (1+2\gamma)d_{-\gamma,\rho}) - (d_{0,-\gamma} - (1+2\gamma)d_{-\gamma,-\gamma})] \\ &= \left(\frac{1+\gamma}{\gamma}\right)^2\left(\frac{\gamma}{\gamma+\rho}\right)(d_{0,\rho} - (1+2\gamma)d_{-\gamma,\rho}) = \frac{1+\gamma}{(1-\rho)(1+\gamma-\rho)} \equiv b_\varphi, \end{aligned}$$

the same value we got for the region where $c = \gamma/(\gamma+\rho)$.

If we are working with a model such that $0 < \gamma < -\rho/2$, $l \neq 0$, $\overline{A}^2$ dominates A and a non-null bias is related to levels k such that $\sqrt{k}\,\overline{A}^2(n/k) \to \lambda$. The asymptotic bias is then related to the scale factor associated to $\sqrt{k}\,\overline{A}^2\,(n/k)$ in the distributional representation of $\sqrt{k}\,(\widehat{\varphi}_n(k) - \varphi(\gamma))$. Such a factor comes directly from the distributional representations in (**??**), (**??**) and (**??**), and is given by

$$\begin{aligned} &\left(\frac{1+\gamma}{\gamma}\right)^2\left(\frac{d_{0,-2\gamma} - d_{0,-\gamma}}{\gamma} - \frac{2\,(d_{-\gamma,-2\gamma} - d_{-\gamma,-\gamma})}{\gamma} - 4\,(d_{-\gamma,-2\gamma} - d_{-\gamma,-\gamma}) - (1+\gamma)d_{-\gamma,-\gamma}^2\right) \\ &\quad = \frac{(1+\gamma)^2}{\gamma^3}\left(d_{0,-2\gamma} + (1+2\gamma)d_{-\gamma,-\gamma} - 2(1+2\gamma)d_{-\gamma,-2\gamma} - \gamma(1+\gamma)d_{-\gamma,-\gamma}^2\right) \\ &\quad = \frac{2(1+\gamma)}{(1+2\gamma)^2(1+3\gamma)}. \end{aligned}$$

Hence the result. The case $\gamma = -\rho/2$ has been excluded: then everything depends on the relative behaviour of A and $\overline{A}^2$, both regularly varying functions with the same index of regular variation ρ.

If $\rho < \gamma = 0$, $A(t) = o\left(\overline{A}^2(t)\right)$ and we get, from (**??**),

$$\frac{U(t)}{a(t)}\left(1 - \frac{U(t)}{U(tx)}\right) = \ln x - \overline{A}(t)\ \ln^2 x + \overline{A}^2(t)\ \ln^3 x(1 + o(1)).$$

Again in the lines of the proofs of Theorems **??** and **??**, we need now to use the inequalities, $0 < -\ln(1-x) - x - x^2/2 - x^3/3 - x^4/4 < x^5/(5(1-x))$, $0 < x < 1$, for $x = 1 - U(Y_{n-k,n})/U(Y_{n-i+1,n})$, and we get

$$\frac{L_2^*}{2} + \frac{a^*(n/k)}{U(Y_{n-k,n})}\ \frac{L_3^*}{3} + \left(\frac{a^*(n/k)}{U(Y_{n-k,n})}\right)^2 \frac{L_4^*}{4} < \frac{U(Y_{n-k,n})}{a^*(n/k)}\,(M_1^* - L_1^*)$$
$$< \frac{L_2^*}{2} + \frac{a^*(n/k)}{U(Y_{n-k,n})}\ \frac{L_3^*}{3} + \left(\frac{a^*(n/k)}{U(Y_{n-k,n})}\right)^2 \frac{L_4^*}{4} + \frac{R_{k,n}^{(2,5)}}{5},$$

again with $R_{k,n}^{(m,j)}$ in (**??**). Due to the limit in probability provided in Proposition **??** and related to L_4^*, if we consider levels k such that $\sqrt{k}\ \overline{A}^2(n/k) \to \lambda$, finite, and with $\overline{A}$ given in (**??**),

$$\sqrt{k}\ \overline{A}^2(n/k)\ \frac{L_4^*}{4} \underset{n\to\infty}{\overset{P}{\longrightarrow}} \frac{6\lambda}{(1-\gamma)(1-2\gamma)(1-3\gamma)} = 6\ \lambda.$$

The same condition and Lemma **??** imply $\sqrt{k}\ R_{k,n}^{(2,5)} \underset{n\to\infty}{\overset{P}{\longrightarrow}} 0$. The contributions to the asymptotic bias provided by $L_2^*/2$, $(L_1^*)^2$, $L_3^*/3$ and $L^*/4$ are then given by $36\ \overline{A}^2$, $-16\ \overline{A}^2$, $-24\ \overline{A}^2$ and $6\ \overline{A}^2$, respectively. Hence, the limiting value 2λ, whenever $\sqrt{k}\ \overline{A}^2(n/k) \to \lambda$, finite, as $n \to \infty$. ■

Proof. [**Theorem ??**]. The proof of (**??**) follows immediatelly from Proposition **??** and Slutsky's theorem. The asymptotic normality result is based on the distributional representations provided in Propositions **??** and **??**. We shall here consider separately the cases $\gamma > 0$ and $\gamma \le 0$.

- If $\gamma > 0$, we may write

$$\frac{\widehat{a}(n/k)}{a(n/k)} - 1 = \frac{\frac{X_{n-k,n}}{a(n/k)}\ \frac{L_n^{(1)}(k)\ L_n^{(2)}(k)}{2} - \left(L_n^{(2)}(k) - \left(L_n^{(1)}(k)\right)^2\right)}{L_n^{(2)}(k) - \left(L_n^{(1)}(k)\right)^2}.$$

  Proposition **??**, enables us to write,

$$\frac{X_{n-k,n}}{a(n/k)}\left(\frac{L_n^{(1)}(k)\ L_n^{(2)}(k)}{2}\right) = \frac{\gamma^2}{(1+\gamma)^2(1+2\gamma)} + \frac{\gamma\ P_1}{(1+\gamma)(1+2\gamma)\sqrt{k}} + \frac{P_2}{(1+\gamma)\sqrt{k}}$$
$$+ \frac{\gamma^3\ B}{(1+\gamma)^2(1+2\gamma)\sqrt{k}} + o_p\left(\frac{1}{\sqrt{k}}\right) + \frac{\widetilde{A}(n/k)}{1+\gamma}\left(d_{L_2} + \frac{\gamma\ d_{L_1}}{1+2\gamma}\right)(1 + o_p(1))$$
$$- \frac{\gamma\ \overline{A}(n/k)}{(1+\gamma)^2(1+2\gamma)}(1 + o_p(1)),$$

and

$$L_n^{(2)}(k) - \left(L_n^{(1)}(k)\right)^2 = \frac{\gamma^2}{(1+\gamma)^2(1+2\gamma)} - \frac{2\gamma\ P_1}{(1+\gamma)\sqrt{k}} + \frac{2\ P_2}{\sqrt{k}} + o_p\left(\frac{1}{\sqrt{k}}\right) + 2\widetilde{A}(n/k)\left(d_{L_2} - \frac{\gamma\ d_{L_1}}{1+\gamma}\right)(1+o_p(1)).$$

Hence,

$$\sqrt{k}\left(\frac{\widehat{a}(n/k)}{a(n/k)} - 1\right) = \frac{(1+\gamma)(3+4\gamma)P_1}{\gamma} - \frac{(1+\gamma)(1+2\gamma)^2 P_2}{\gamma^2} + \gamma\ B + o_p(1)) + \sqrt{k}\left(\widetilde{A}(n/k)\left(\frac{(1+\gamma)(3+4\gamma)\ d_{L_1}}{\gamma} - \frac{(1+\gamma)(1+2\gamma)^2\ d_{L_2}}{\gamma^2}\right) - \frac{\overline{A}(n/k)}{\gamma}\right)(1+o_p(1)).$$

The asymptotic bias follows then from the results in Proposition **??** and the asymptotic variance can be calculated on the basis of Lemma **??**.

- If $\gamma \leq 0$, and on the basis of the results in Proposition **??**, we may write

$$\widehat{a}(n/k) = a^*(n/k)\ \frac{L_1^*\ L_2^*/2}{L_2^* - (L_1^*)^2} = a(n/k)\ \frac{a^*(n/k)}{a(n/k)}\ \frac{L_1^*\ L_2^*/2}{L_2^* - (L_1^*)^2},$$

and consequently,

$$\sqrt{k}\left(\frac{\widehat{a}(n/k)}{a(n/k)} - 1\right) \stackrel{d}{=} \sqrt{k}\ \left(\frac{\left(\frac{a^*(n/k)}{a(n/k)}\right)\ \frac{L_1^*\ L_2^*}{2} - L_2^* + (L_1^*)^2}{L_2^* - (L_1^*)^2}\right)$$

with $a^*(t)$ given in (**??**), i.e., with

$$\frac{a^*(t)}{a(t)} = \begin{cases} 1 - A^*(t) & \text{if} \quad \gamma < 0,\ \rho \leq 0 \\ 1 & \text{if} \quad \rho < \gamma = 0 \end{cases}$$

Proposition **??** ensures that

$$\frac{L_1^*\ L_2^*}{2} = \frac{1}{(1-\gamma)^2(1-2\gamma)} + \frac{P_1^*}{(1-\gamma)(1-2\gamma)\sqrt{k}} + \frac{P_2^*}{(1-\gamma)\sqrt{k}} + o_p\left(\frac{1}{\sqrt{k}}\right) + A^*(n/k)\left(\frac{d_{L_1}^*}{(1-\gamma)(1-2\gamma)} + \frac{d_{L_2}^*}{1-\gamma}\right)(1+o_p(1)). \quad (5.6)$$

Consequently, in the region $\gamma < 0,\ \rho \leq 0$,

$$\frac{a^*(n/k)}{a(n/k)}\ \frac{L_1^*\ L_2^*}{2} = \frac{1}{(1-\gamma)^2(1-2\gamma)} + \frac{P_1^*}{(1-\gamma)(1-2\gamma)\sqrt{k}} + \frac{P_2^*}{(1-\gamma)\sqrt{k}} + o_p\left(\frac{1}{\sqrt{k}}\right) + A^*(n/k)\left(\frac{d_{L_1}^*}{(1-\gamma)(1-2\gamma)} + \frac{d_{L_2}^*}{1-\gamma} - \frac{1}{(1-\gamma)^2(1-2\gamma)}\right)(1+o_p(1)).$$

Since

$$L_2^* - (L_1^*)^2 = \frac{1}{(1-\gamma)^2(1-2\gamma)} - \frac{2\ P_1^*}{(1-\gamma)\sqrt{k}} + \frac{2\ P_2^*}{\sqrt{k}} + o_p\left(\frac{1}{\sqrt{k}}\right) + 2A^*(n/k)\left(d^*_{L_2} - \frac{d^*_{L_1}}{1-\gamma}\right)(1+o_p(1)), \quad (5.7)$$

$$\frac{a^*(n/k)}{a(n/k)}\ \frac{L_1^*\ L_2^*}{2} - L_2^* + (L_1^*)^2 = \frac{(3-4\gamma)P_1^*}{(1-\gamma)(1-2\gamma)\sqrt{k}} - \frac{(1-2\gamma)P_2^*}{(1-\gamma)\sqrt{k}} + o_p\left(\frac{1}{\sqrt{k}}\right) + A^*(n/k)\left(\frac{(3-4\gamma)\ d^*_{L_1}}{(1-\gamma)(1-2\gamma)} - \frac{(1-2\gamma)\ d^*_{L_2}}{1-\gamma} - \frac{1}{(1-\gamma)^2(1-2\gamma)}\right)(1+o_p(1)).$$

We thus get

$$\sqrt{k}\left(\frac{\widehat{a}(n/k)}{a(n/k)} - 1\right) = (1-\gamma)(3-4\gamma)\ P_1^* - (1-\gamma)(1-2\gamma)^2\ P_2^* + o_p(1) + \sqrt{k}\ A^*(n/k)\left((1-\gamma)(3-4\gamma)d^*_{L_1} - (1-\gamma)(1-2\gamma)^2\ d^*_{L_2} - 1\right)(1+o_p(1)),$$

where the normal r.v.'s (P_1^*, P_2^*) and the constants $d^*_{L_1}$ and d_{L_2} are defined in Lemma **??**, (**??**) and (**??**), respectively. The asymptotic bias and variance are then obtained by straightforward calculations. In the region $\rho < \gamma = 0$, directly from (**??**) and (**??**), we get

$$\frac{L_1^*\ L_2^*}{2} - L_2^* + (L_1^*)^2 = \left(\frac{3\ P_1^*}{\sqrt{k}} - \frac{P_2^*}{\sqrt{k}} + A^*(n/k)\left(3\ d^*_{L_1} - d^*_{L_2}\right)\right)(1+o_p(1))$$

and, $3\ d^*_{L_1} - d^*_{L_2} \equiv 0$. Hence the null asymptotic bias associated to levels k such that $\sqrt{k}\ \overline{A}(n/k) \to \lambda$, finite, as $n \to \infty$.

■

Proof. [**Proposition ??**]. We only need to take into account the following expansion, obtained under the second order condition (**??**):

$$\begin{aligned}\frac{X_{n-k,n} - U(n/k)}{a(n/k)} &\overset{d}{=} \frac{U\left(Y_{n-k,n}\right) - U(n/k)}{a(n/k)} \\ &= \frac{((k/n)\ Y_{n-k,n})^\gamma - 1}{\gamma} + A(n/k)\left(H_{\gamma,\rho}\left(\frac{k}{n}\ Y_{n-k,n}\right) + o_p(1)\right) \\ &= \frac{B}{\sqrt{k}} + o_p\left(\frac{1}{\sqrt{k}}\right) + o_p\left(A(n/k)\right),\end{aligned}$$

as $n \to \infty$, where B is the standard normal r.v. in Proposition **??**. The result comes from Theorems **??** and **??**, after straightforward calculations. ■

Proof. [**Theorem ??**]. Under the conditions of the theorem, both the random term ruling the normal behaviour as well as the bias are the same as the ones appearing for the ML estimator. Note first that, with $\{W_n(s)\}_{s\geq 0}$ denoting a sequence of Brownian motions, Theorem 2.4.2 of de Haan and Ferreira (2006) gives for $\gamma \neq 0$, and as $n \to \infty$,

$$\left(\frac{k}{n}\, Y_{n-k,n}\right)^{\gamma} = 1 + \frac{\gamma}{\sqrt{k}}\left(W_n(1) + o_p(1)\right).$$

Then, Corollary 2.4.6 *ibidem* implies for $0 < s \leq 1$,

$$\left(\frac{k}{n}\, Y_{n-[ks],n}\right)^{\gamma} = s^{-\gamma}\left(1 + \frac{\gamma}{\sqrt{k}}\, s^{-1}W_n(s)\right) + \frac{o_p(1)}{\sqrt{k}} \max\left(1, s^{-\gamma-\frac{1}{2}-\epsilon}\right),$$

where the o_p-term is uniform in s. Defining $Q_n(s) := Y_{n-[ks],n}/Y_{n-k,n}$, it follows that, for any $\epsilon > 0$,

$$Q_n^{\gamma}(s) = \left(\frac{Y_{n-[ks],n}}{Y_{n-k,n}}\right)^{\gamma} = s^{-\gamma}\left(1 + \frac{\gamma}{\sqrt{k}}\left(s^{-1}W_n(s) - W_n(1)\right)\right) + \frac{o_p(1)}{\sqrt{k}} \max\left(1, s^{-\gamma-\frac{1}{2}-\epsilon}\right). \quad (5.8)$$

Similarly, we get

$$\ln Q_n(s) = -\ln s + \frac{1}{\sqrt{k}}\left(s^{-1}W_n(s) - W_n(1)\right) + \frac{o_p(1)}{\sqrt{k}} \max\left(1, s^{-\frac{1}{2}-\epsilon}\right). \quad (5.9)$$

For $\gamma > 0$, we get from Proposition **??**, in connection with Lemma **??**,

$$\sqrt{k}\left(M_n^{(1)}(k) - \gamma\right) = \gamma\sqrt{k}\left(\int_0^1 \ln Q_n(s)ds - 1\right) + bias + o_p(1),$$

$$\sqrt{k}\left(L_n^{(1)}(k) - \frac{\gamma}{1+\gamma}\right) = \gamma\sqrt{k}\left(\int_0^1 \left(1 - Q_n^{-\gamma}(s)\right) ds - \frac{1}{1+\gamma}\right) + bias + o_p(1).$$

Then, by (**??**) and (**??**),

$$\sqrt{k}\left(M_n^{(1)}(k) - \gamma\right) = \gamma \int_0^1 \left(s^{-1}W_n(s) - W_n(1)\right) ds + bias + o_p(1),$$

$$\sqrt{k}\left(L_n^{(1)}(k) - \frac{\gamma}{1+\gamma}\right) = \gamma \int_0^1 s^{\gamma}\left(s^{-1}W_n(s) - W_n(1)\right) ds + bias + o_p(1).$$

Now, using Cramer's delta method as in the previous proof, we obtain under the conditions of the theorem

$$\sqrt{k}\left(\widehat{\gamma}_n^{MM}(k) - \gamma\right) = \frac{(1+\gamma)^2}{\gamma} \int_0^1 \left(s^{-1} - (1+2\gamma)s^{\gamma-1}\right) W_n(s)ds + (1+\gamma)W_n(1) + bias + o_p(1),$$

For $\gamma \leq 0$, we get

$$\sqrt{k}\left(L_1^* - \frac{1}{1-\gamma}\right) = \sqrt{k}\left(\int_0^1 \frac{Q_n^{\gamma}(s) - 1}{\gamma}\, ds - \frac{1}{1-\gamma}\right) + bias + o_p(1)$$

and

$$\sqrt{k}\left(L_2^* - \frac{2}{(1-\gamma)(1-2\gamma)}\right) = \sqrt{k}\left(\int_0^1 \frac{Q_n^\gamma(s)-1}{\gamma}^2 \, ds - \frac{2}{(1-\gamma)(1-2\gamma)}\right) + bias + o_p(1).$$

So, by (**??**) and for $\gamma < 0$,

$$\sqrt{k}\left(L_1^* - \frac{1}{1-\gamma}\right) = -\int_0^1 s^{-\gamma}\left(s^{-1}W_n(s) - W_n(1)\right) + bias + o_p\left(\sqrt{k}A^*(n/k)\right) o_p(1),$$

$$\sqrt{k}\left(L_2^* - \frac{2}{(1-\gamma)(1-2\gamma)}\right) = -\int_0^1 \frac{s^{-\gamma}-1}{\gamma} s^{-\gamma}\left(s^{-1}W_n(s) - W_n(1)\right) + bias + o_p\left(\sqrt{k}A^*(n/k)\right) o_p(1)).$$

Now, as shown in the proof of Theorem **??** and Corollary **??**,

$$\sqrt{k}\left(\widehat{\gamma}_n^{MM}(k) - \frac{\frac{L_2^*}{2} - (L_1^*)^2}{L_2^* - (L_1^*)^2}\right) \xrightarrow[n\to]{P} 0.$$

Therefore,

$$\sqrt{k}\left(\widehat{\gamma}_n^{MM}(k) - \gamma\right) = \frac{(1-2\gamma)\sqrt{k}\left(\frac{L_2^*}{2} - \frac{1}{(1-\gamma)(1-2\gamma)}\right) - (1-\gamma)\sqrt{k}\left((L_1^*)^2 - \frac{1}{(1-\gamma)^2}\right)}{L_2^* - (L_1^*)^2} + o_p(1).$$

Since $L_2^* - (L_1^*)^2 \longrightarrow_{n\to\infty}^{P} \left((1-\gamma)^2(1-2\gamma)\right)^{-1}$, we may write

$$\begin{aligned}
\sqrt{k}\left(\widehat{\gamma}_n^{MM}(k) - \gamma\right) &= (1-\gamma)(1-2\gamma)\Bigg[-(1-2\gamma)\int_0^1 \frac{s^{-\gamma}-1}{\gamma}\; s^{-\gamma}\left(s^{-1}W_n(s) - W_n(1)\right) ds \\
&\quad + 2\int_0^1 s^{-\gamma}\left(s^{-1}W_n(s) - W_n(1)\right) ds + bias + o_p(1)\Bigg] \\
&= (1-\gamma)(1-2\gamma)\left[\int_0^1 \left(2 - (1-2\gamma)\frac{s^{-\gamma}-1}{\gamma}\right) s^{-\gamma-1}W_n(s)ds - \frac{1}{1-\gamma}\; W_n(1)\right] + bias + o_p(1).
\end{aligned}$$

For $\gamma = 0$ this becomes

$$\begin{aligned}
\sqrt{k}\;\widehat{\gamma}_n^{MM}(k) &= \int_0^1 (2+\ln s)\; s^{-1}W_n(s)ds - W_n(1) + bias + o_p(1) \\
&= \int_0^1 (2+\ln s)\left(\; s^{-1}W_n(s)ds - W_n(1)\right) d,
\end{aligned}$$

again the same r.v. as for $\widehat{\gamma}_n^{ML}$. Consequently, and under the conditions in the theorem, both the functional of the Brownian motion and the bias are the same as in Theorem 2.1 of Drees *et al.* (2004), page 1183. Hence the result. ■

Proof. [**Theorem ??**]. We first prove that the estimator $\widehat{\gamma}_n^{MM}(k;p)$ has exactly the same asymptotic behavior as $\widehat{\widehat{\gamma}}_n^{MM}(k;p)$, defined as $\widehat{\gamma}_n^{MM}(k)$ in (**??**), but with $X_{n-i+1,n}$ replaced everywhere by $X_{n-i+1,n} - Q(p)$, $1 \le i \le n$, with Q the inverse of the d.f. F. In fact this statement is true for all the statistics in (**??**). We shall prove it for $M_n^{(1)}(k)$. For $i = 1, 2, \cdots, k$,

$$\ln\left(X_{n-i+1,n} - X_{[np]+1:n}\right) = \ln\left(X_{n-i+1,n} - Q(p)\right) + \ln\left(1 - \frac{X_{[np]+1,n} - Q(p)}{X_{n-i+1,n} - Q(p)}\right).$$

The last term is at most

$$\ln\left(1 - \frac{X_{[np]+1,n} - Q(p)}{X_{n,n} - Q(p)}\right).$$

Since $X_{[np]+1,n} - Q(p) = O_p\left(n^{-1/2}\right)$ and $X_{n,n} - Q(p) \underset{n\to\infty}{\overset{P}{\longrightarrow}} Q(1) - Q(p) \in (0, \infty]$, we have

$$\sqrt{k}\ \ln\left(1 - \frac{X_{[np]+1,n} - Q(p)}{X_{n,n} - Q(p)}\right) = (1 + o_p(1))\sqrt{\frac{k}{n}}\ \frac{\sqrt{n}\left(X_{[np]+1,n} - Q(p)\right)}{Q(1) - Q(p)} = \sqrt{\frac{k}{n}}\ O_p(1) \underset{n\to\infty}{\overset{P}{\longrightarrow}} 0.$$

Similarly for $L_n^{(1)}(k)$, $L_n^{(2)}(k)$ and thus for $\widehat{\gamma}_n^{MM}(k)$. Now, $\widehat{\widehat{\gamma}}_n^{MM}(k;p)$ differs from $\widehat{\gamma}_n^{MM}(k)$ only by a shift: we replace $U(t)$ by $\widetilde{U}(t) := U(t) - U\left((1-p)^{-1}\right)$. Hence condition (**??**) is valid for $\widetilde{U}$ and Theorems **??** and **??** are valid for the adjusted estimators albeit that, for $\rho < \gamma < 0$, the bias term has to be multiplied by $x^F / \left(x^F - U\left(1/(1-p)\right)\right)$. ■

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