Resampling-Based Methodologies in Statistics of Extremes: Environmental and Financial Applications*

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Abstract

Resampling computer intensive methodologies, like the *jackknife* and the *bootstrap* are important tools for a reliable semi-parametric estimation of parameters of extreme or even rare events. Among these parameters we mention the *extreme value index*, γ, the primary parameter in *statistics of extremes*. Most of the semi-parametric estimators of this parameter show the same type of behaviour: nice asymptotic properties, but a high variance for small k, the number of upper order statistics used in the estimation, a high bias for large k, and the need for an adequate choice of k. After a brief reference to some estimators of the aforementioned parameter and their asymptotic properties we present an algorithm that deals with an adaptive reliable estimation of γ.

1 A brief introduction

Let us assume that we have access to a sample, $(X_1, \ldots, X_n)$ of independent, identically distributed (i.i.d.), or even stationary and weakly

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dependent, random variables (r.v.'s) from an underlying model F, and let us denote by $(X_{1:n} \leq \cdots \leq X_{n:n})$ the sample of associated ascending order statistics (o.s.'s). Let us further assume that it is possible to normalize the sequence of maximum values, $\{X_{n:n}\}_{n\geq 1}$ so that we get a non-degenerate limit. Then (Gnedenko, 1943), that limiting r.v. has a distribution function (d.f.) of the type of the *general extreme value* (GEV) d.f., given by

$$G_\gamma(x) = \begin{cases} \exp\left(-(1+\gamma x)^{-1/\gamma}\right), \ 1+\gamma x \geq 0, & \text{if} \quad \gamma \neq 0, \\ \exp(-\exp(-x)), \ x \in \mathbb{R}, & \text{if} \quad \gamma = 0, \end{cases} \tag{1.1}$$

and γ is the so-called *extreme value index* (EVI), the primary parameter in *statistics of univariate extremes* (SUE). We then say that F is in the max-domain of attraction of G_γ, in (1.1), and use the notation $F \in \mathcal{D}_{\mathcal{M}}(G_\gamma)$.

The *extreme value index* γ measures essentially the weight of the right tail-function $\overline{F} := 1 - F$, as illustrated in Figure 1.

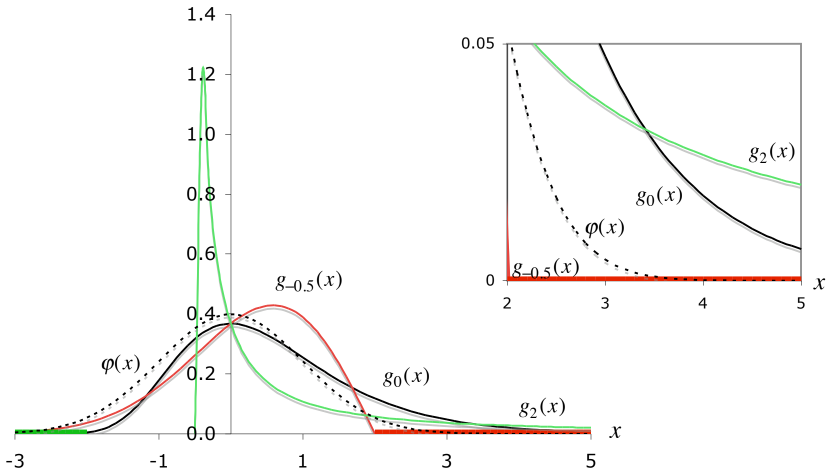

Figure 1: Probability density function (p.d.f.) $g_\gamma(x) = dG_\gamma(x)/dx$, for $\gamma = -0.5$, $\gamma = 0$ and $\gamma = 2$, together with the normal p.d.f., $\phi(x) = \exp(-x^2/2)/\sqrt{2\pi}, \ x \in \mathbb{R}$

- If $\gamma < 0$, the right tail is light, and F has a finite *right endpoint*, i.e. $x^F := \sup\{x : F(x) < 1\} < +\infty$;
- If $\gamma > 0$, the right tail is heavy, of a negative polynomial type, and F has an infinite *right endpoint*;
- If $\gamma = 0$, the right tail is of an exponential type. The *right endpoint* can then be either finite or infinite.

Slightly more restrictively than the full max-domain of attraction of the GEV d.f., we now consider a positive EVI, i.e. we work with heavy-tailed models F in $\mathcal{D}_{\mathcal{M}}\left(G_\gamma\right)_{\gamma>0} =: \mathcal{D}_{\mathcal{M}}^{+}$. As usual, we shall further use the notations

$$F^{\leftarrow}(y) := \inf\left\{x : F(x) \geq y\right\}$$

for the *generalized inverse* function of F, and $\mathcal{R}_{\mathrm{a}}$ for the class of *regularly varying* functions at infinity with an index of regular variation a, i.e. positive Borel measurable functions $g(\cdot)$ such that $g(tx)/g(t) \to x^{\mathrm{a}}$, as $t \to \infty$, for all $x > 0$. Let us further use the notation

$$U(t) := F^{\leftarrow}\big(1 - 1/t\big),$$

for the (reciprocal) or tail quantile function.

Equivalently to say that $F \in \mathcal{D}_{\mathcal{M}}^{+}$, we can say (Gnedenko, 1943) that the tail function

$$\overline{F} := 1 - F$$

belongs to $\mathcal{R}_{-1/\gamma}$ or that $U \in \mathcal{R}_\gamma$ (de Haan, 1984), i.e. for heavy-tailed models we have the validity of the so-called *first-order conditions*,

$$F \in \mathcal{D}_{\mathcal{M}}^{+} \quad \Longleftrightarrow \quad \overline{F} \in \mathcal{R}_{-1/\gamma} \quad \Longleftrightarrow \quad U \in \mathcal{R}_\gamma. \tag{1.2}$$

For these heavy-tailed models, and given a sample $\underline{\mathbf{X}}_n = (X_1, \ldots, X_n)$, the classical EVI-estimators are Hill estimators (Hill, 1975), with the functional expression

$$H_{k,n} \equiv H(k; \underline{\mathbf{X}}_n) := \frac{1}{k}\sum_{i=1}^{k} V_{ik},$$
$$V_{ik} := \ln X_{n-i+1:n} - \ln X_{n-k:n}, \quad 1 \leq i \leq k < n. \tag{1.3}$$

The Hill EVI-estimators are thus the average of the k log-excesses above a random level $X_{n-k:n}$, that compulsory needs to be an *intermediate* o.s., i.e.

$$k = k_n \to \infty \quad \text{and} \quad k/n \to 0, \text{ as } n \to \infty, \tag{1.4}$$

so that we have consistent EVI-estimation in the whole $\mathcal{D}_{\mathcal{M}}^{+}$.

Under adequate second-order conditions that rule the rate of convergence in any of the first-order conditions in (1.2), Hill estimators, $H_{k,n}$, in (1.3), have usually a high asymptotic bias, i.e., $\sqrt{k}\,(H_{k,n} - \gamma)$ is asymptotically normal with variance γ^2 and a non-null mean value for the moderate k-values that lead to minimal mean square error (MSE), as sketched in Section 2.2. This non-null asymptotic bias and a rate of convergence of the order of $1/\sqrt{k}$ lead to sample paths with a high variance for small k, a high bias for large k, and a very peaked MSE pattern. Recently, several authors have considered different ways of reducing bias in the area of SUE (see the overviews in Gomes *et al.*, 2007b, Chapter 6 of Reiss and Thomas, 2007; Gomes *et al.*, 2008a; Beirlant *et al.*, 2012). A simple class of *minimum-variance reduced-bias* (MVRB) EVI-estimators is the class studied in Caeiro *et al.* (2005), to be introduced in Section 2.1. These MVRB EVI-estimators depend on the estimation of second-order parameters, and their asymptotic behaviour is presented in Section 2.2. Both the Hill and the MVRB EVI-estimators are invariant to changes in scale, but they are not invariant to changes in location. And particularly the Hill EVI-estimators can suffer drastic changes when we induce an arbitrary shift in the data. This was one of the reasons that led Araújo Santos *et al.* (2006) to introduce the so-called *peaks over random threshold* (PORT) methodology, to be sketched in Section 2.3.

Resampling methodologies, introduced in Section 3, have recently revealed to be quite fruitful in the field of SUE. We mention the importance of the *generalized jackknife* (GJ), detailed in Gray and Schucany (1972), in the reduction of bias, revisited recently in the field of extremes by Gomes *et al.* (2013b). We further refer the relevance of the *bootstrap* (Efron, 1979) in the estimation of a crucial tuning parameter in the area, the number k of top order statistics involved in the estimation of the tails. Together, these two resampling procedures enable the obtention of reliable semi-parametric estimates of any parameter of extreme or even rare events, like a *high quantile*, the *expected shortfall*, the *return period* of a high level or the two primary parameters of extreme events, the *extreme value index* (EVI) and the *extremal index*, related to the degree of local dependence in the extremes of a stationary sequence. In order to illustrate such topics, we essentially consider the GJ EVI-estimators in Gomes *et al.* (2013b), associated with the simplest class of MVRB estimators of a positive EVI introduced and studied in Caeiro *et al.* (2005).

In Section 4, an application of these methodologies to the analysis of an environmental data set, related to the number of hectares, exceeding 100 ha, burnt during wildfires recorded in Portugal during 14 years (1990-2003), is undertaken. To enhance the relevance of the PORT methodology, we further consider an application to financial data.

2 Second-order reduced-bias, MVRB and PORT EVI-estimators

As mentioned above, for consistent semi-parametric EVI-estimation, in the whole $\mathcal{D}^+_{\mathcal{M}}$, we merely need to work with adequate functionals, dependent on an *intermediate tuning* parameter k, the number of top o.s.'s involved in the estimation, i.e. (1.4) should hold. To obtain full information on the non-degenerate asymptotic behaviour of semi-parametric EVI-estimators, we need further assuming a *second-order condition*, ruling the rate of convergence in the *first-order condition*, or even a *third* or *fourth-order condition*. Whenever dealing with reduced-bias estimators of parameters of extreme events, like the EVI, and essentially due to technical reasons, we slightly restrict the domain of attraction, $\mathcal{D}^+_{\mathcal{M}}$, and consider a Pareto-type class of models, assuming that, with $C, \gamma > 0$, $\rho < 0$, and $\beta \neq 0$,

$$U(t) = Ct^{\gamma}\Big(1 + A(t)/\rho + o(t^{\rho})\Big), \quad A(t) := \gamma\beta t^{\rho}, \tag{2.1}$$

as $t \to \infty$, i.e. we assume that the slowly varying function $L_{_U}(t) = t^{-\gamma}U(t)$ tends to a finite non-null constant. To obtain information on the bias of MVRB EVI-estimators it is even common to slightly restrict our class of models, further assuming the following third-order condition,

$$U(t) = Ct^{\gamma}\Big(1 + A(t)/\rho + \beta' t^{2\rho} + o(t^{2\rho})\Big), \tag{2.2}$$

as $t \to \infty$, with $\beta' \neq 0$. And if we deal with GJ-MVRB EVI-estimators, to be detailed in Section 3.2, and also want to obtain full information on their asymptotic bias, we can further assume, in the lines of Taylor series, that

$$U(t) = Ct^{\gamma}\Big(1 + A(t)/\rho + \beta' t^{2\rho} + \beta'' t^{3\rho} + o(t^{3\rho})\Big), \tag{2.3}$$

as $t \to \infty$, with $\beta'' \neq 0$.

More generally than (2.1), it is often merely assumed that there exists a function $A(\cdot)$, such that

$$\lim_{t\to\infty} \frac{\ln U(tx) - \ln U(t)}{A(t)} = \psi_\rho(x) := \begin{cases} (x^\rho - 1)/\rho, & \text{if} \quad \rho \neq 0, \\ \ln x, & \text{if} \quad \rho = 0. \end{cases} \tag{2.4}$$

Then, we compulsory have $|A| \in \mathcal{R}_\rho$. Moreover, if the limit in the right hand-side of (2.4) exists, it is compulsory equal to the above defined $\psi_\rho(\cdot)$ function (Geluk and de Haan, 1987). Further note that the validity of (2.4) with $\rho < 0$ is equivalent to (2.1). Additional details on second and higher-order conditions can be found in de Haan and Ferreira (2006).

As mentioned above, and provided that (1.4) and (2.4) hold, Hill EVI-estimators, $H_{k,n}$, have usually a high asymptotic bias. The adequate accommodation of this bias has recently been extensively addressed. We mention the pioneering papers by Peng (1998), Beirlant *et al.* (1999), Feuerverger and Hall (1999) and Gomes *et al.* (2000), among others. In these papers, authors are led to *second-order reduced-bias* EVI-estimators, with asymptotic variances larger than or equal to $(\gamma\ (1-\rho)/\rho)^2$, where $\rho(<0)$ is the aforementioned 'shape' second-order parameter, ruling the rate of convergence of the distribution of the normalized sequence of maximum values towards the limiting law G_γ, in (1.1).

2.1 MVRB EVI-estimation

Recently, Caeiro *et al.* (2005) and Gomes *et al.* (2007a; 2008c) have been able to *reduce the bias without increasing the asymptotic variance*, kept at γ^2, just as happens with the Hill EVI-estimators. Those estimators, called MVRB EVI-estimators, are all based on an adequate 'external' and a bit more than consistent estimation of the pair of second-order parameters, $(\beta,\rho) \in (\mathbb{R},\mathbb{R}^-)$, in (2.1), done through adequate estimators denoted by $(\hat\beta,\hat\rho)$, and outperform the classical estimators for all k. Different algorithms for the estimation of (β,ρ) can be found in Gomes and Pestana (2007), among others.

Among the most common MVRB EVI-estimators, we now consider the class in Caeiro *et al.* (2005), used for Value-at-Risk (VaR) estimation in the aforementioned seminal paper by Gomes and Pestana (2007). Such a class, denoted by $\overline{H} \equiv \overline{H}_{k,n}$, has the functional form

$$\overline{H}_{k,n} \equiv \overline{H}_{\hat\beta,\hat\rho}(k;\underline{\mathbf{X}}_n) := H_{k,n}\Big(1-\hat\beta(n/k)^{\hat\rho}/(1-\hat\rho)\Big), \qquad (2.5)$$

where $(\hat\beta,\hat\rho)$ is an adequate consistent estimator of (β,ρ), with $\hat\beta$ and $\hat\rho$ based on a number of top o.s.'s k_1 usually of a higher order than the number of top o.s.'s k used in the EVI-estimation. Further details on such estimation are given in Section 3.3.

2.2 A brief asymptotic comparison of classical and MVRB EVI-estimators

The Hill estimators reveal usually a high asymptotic bias. Indeed, from the results of de Haan and Peng (1998), and with $\mathcal{N}_{\mu,\sigma^2}$ denoting a normal r.v. with mean value μ and variance σ^2,

$$\sqrt{k}\,(H_{k,n}-\gamma) \;\stackrel{d}{=}\; \mathcal{N}_{0,\gamma^2} + b_{_H}\sqrt{k}A(n/k) + o_p\big(\sqrt{k}A(n/k)\big), \qquad (2.6)$$

where the bias $b_{_H}\sqrt{k}A(n/k)$ can be very large, moderate or small (i.e. go to ∞, constant or 0) as $n \to \infty$. Under the same conditions as before, $\sqrt{k}\,\big(\overline{H}_{k,n}-\gamma\big)$ is asymptotically normal with variance also equal to γ^2 but with a null mean value. Indeed, under the validity of the aforementioned third-order condition in (2.2), related to Pareto-type class of models, we can then adequately estimate the vector of second-order parameters, (β,ρ) so that $\overline{H}_{k,n}$ outperforms $H_{k,n}$ for all k. Indeed, we can write (Caeiro *et al.*, 2009)

$$\sqrt{k}\,\big(\overline{H}_{k,n}-\gamma\big) \stackrel{d}{=} \mathcal{N}_{0,\gamma^2} + b_{_{\overline{H}}}\sqrt{k}A^2(n/k) + o_p\big(\sqrt{k}A^2(n/k)\big). \qquad (2.7)$$

And when we try answering the question whether it is still possible to improve the performance of these MVRB EVI-estimators through the use of resampling methods, we are led to a positive answer, as provided in Section 3.2.

2.3 PORT and quasi-PORT EVI-estimation

The estimators in (1.3) and (2.5) are scale invariant but not location invariant. In order to achieve location invariance, Araújo Santos *et al.* (2006) introduced the so-called PORT EVI-estimators, functionals of a sample of excesses over a random level $X_{n_q:n}$, $n_q := \lfloor nq \rfloor + 1$, with $\lfloor x \rfloor$ denoting the integer part of x, i.e. functionals of the sample,

$$\underline{\mathbf{X}}_n^{(q)} := \big(X_{n:n}-X_{n_q:n}, \ldots, X_{n_q+1:n}-X_{n_q:n}\big)\,. \qquad (2.8)$$

Generally, we can have $0<q<1$, for any $F \in \mathcal{D}_{\mathcal{M}}^{+}$ (*the random level is an empirical quantile*). If the underlying model F has a finite *left endpoint*, $x_{_F} := \inf\{x : F(x) \geq 0\}$, we can also use $q=0$ (*the random level can then be the minimum*).

If we think, for instance, on Hill EVI-estimators, in (1.3), the new classes of PORT-Hill EVI-estimators, theoretically studied in Araújo

Santos *et al.* (2006), and for finite samples in Gomes *et al.* (2008b), are given by

$$\begin{aligned} H_{k,n}^{(q)} &:= H(k;\underline{\mathbf{X}}_n^{(q)}) \\ &= \frac{1}{k}\sum_{i=1}^{k}\Big\{\ln \frac{X_{n-i+1:n}-X_{n_q:n}}{X_{n-k:n}-X_{n_q:n}}\Big\}, \quad 0 \le q < 1. \end{aligned} \tag{2.9}$$

Similarly, if we think on the MVRB EVI-estimators, in (2.5), the new classes of PORT-MVRB EVI-estimators, studied for finite samples in Gomes *et al.* (2011a; 2013a), are given by

$$\begin{aligned} \overline{H}_{k,n}^{(q)} &:= \overline{H}_{\hat\beta_q,\hat\rho_q}(k;\underline{\mathbf{X}}_n^{(q)}) \\ &= H_{k,n}^{(q)}\Big(1-\hat\beta(n_q/k)^{\hat\rho}/(1-\hat\rho)\Big), \quad 0 \le q < 1, \end{aligned} \tag{2.10}$$

with $H_{k,n}^{(q)}$ in (2.9), and $\hat\beta \equiv \hat\beta_q := \hat\beta(\underline{\mathbf{X}}_n^{(q)})$, $\hat\rho \equiv \hat\rho_q := \hat\rho(\underline{\mathbf{X}}_n^{(q)})$ any adequate estimator of (β,ρ) based on the sample $\underline{\mathbf{X}}_n^{(q)}$, in (2.8).

These PORT EVI-estimators are thus dependent on a *tuning parameter* q, $0 \le q < 1$, that makes them highly flexible. Moreover, they are invariant to changes in both location and scale. We shall further use the notation $X_{n+1:n} \equiv 0$, and work with $0 \le q \le 1$, so that with $\overline{H}$ and $\overline{H}^{(q)}$, given in (2.5) and (2.10), respectively, we can consider that $\overline{H} = \overline{H}^{(q)}$ for $q = 1$ ($n_1 = n+1, \hat\beta_1 = \hat\beta,\ \hat\rho_1 = \hat\rho$).

We get to know that the second-order MVRB EVI-estimators in (2.5) are not location invariant, but they are 'approximately' location invariant. Almost equivalent to the PORT-MVRB EVI-estimators in (2.10), we can consider, in the lines of Figueiredo *et al.* (2012), quasi-PORT-MVRB EVI-estimators, with a functional expression similar to the one in (2.10) but where for all $0 \le q < 1$, $(\hat\beta,\hat\rho) = (\hat\beta_1,\hat\rho_1)$ are the (β,ρ)-estimators based on the original sample, i.e.

$$\overline{\overline{H}}_{k,n}^{(q)} := H_{k,n}^{(q)}\overline{H}_{k,n}/H_{k,n}, \tag{2.11}$$

with $H_{k,n}$, $\overline{H}_{k,n}$ and $H_{k,n}^{(q)}$ given in (1.3), (2.5) and (2.9), respectively.

3 Resampling methodologies in SUE

The use of resampling methodologies (Efron, 1979) has revealed to be promising in the estimation of the nuisance parameter k, or equivalently, in the estimation of the optimal sample fraction (OSF), k/n, as well as in the reduction of bias of any estimator of a parameter of extreme events. If we ask how to choose the tuning parameter k in the EVI-estimation, either through $H_{k,n}$ or $\overline{H}^{(q)}_{k,n}$ or $\overline{\overline{H}}^{(q)}_{k,n}$, $0 \le q \le 1$, generally denoted $E_{k,n}$, we usually consider the estimation of

$$k_{0|E}(n) := \arg\min_k \mathrm{MSE}(E_{k,n}). \tag{3.1}$$

3.1 Bootstrap methodology and OSF-estimation

To obtain estimates of $k_{0|E}(n)$, in (3.1), one can use a *double-bootstrap* method applied to an adequate *auxiliary statistic* like

$$T_{k,n} \equiv T_{k,n|E} := E_{\lfloor k/2 \rfloor,n} - E_{k,n}, \quad k = 2, \ldots, n-1, \tag{3.2}$$

which tends to the well-known value **zero** and has an asymptotic behaviour similar to the one of $E_{k,n}$ (see Gomes and Oliveira, 2001, among others, for the estimation through $H_{k,n}$ and Gomes *et al.*, 2012, for the the estimation through MVRB EVI-estimators). See also Section 3.3 of this article. At such optimal levels, we have a non-null asymptotic bias, and if we still want to remove such a bias, we can then make use of the GJ methodology.

3.2 The GJ methodology and bias reduction

The main objectives of the *jackknife methodology* are:

1. Bias and variance estimation of a certain statistic, only through manipulation of observed data $\underline{x}$.
2. The building of estimators with bias and MSE smaller than those of an initial set of estimators.

The jackknife or the GJ are resampling methodologies, which usually give a positive answer to the question: '*May the combination of information improve the quality of estimators of a certain parameter or functional*?' The pioneering EVI reduced-bias estimators are, in a certain sense, *generalized jackknife* estimators, i.e., affine combinations of well-known estimators of γ.

The *generalized jackknife* statistic was introduced by Gray and Shucany (1972), and the main objective of the method is related to bias reduction. Let $E_n^{(1)}$ and $E_n^{(2)}$ be two biased estimators of γ, with similar bias properties, i.e.,

$$\mathrm{Bias}(E_n^{(i)}) = \gamma + \phi(\gamma) d_i(n), \quad i = 1, 2.$$

Then, and trivially, if

$$p = p_n = d_1(n)/d_2(n) \neq 1,$$

the affine combination

$$E_n^{GJ} := \left(E_n^{(1)} - pE_n^{(2)}\right)/(1-p)$$

is an unbiased estimator of γ.

3.2.1 GJ-MVRB EVI-ESTIMATION

Given $\overline{H}$, in (2.5), the most natural GJ r.v. is the one associated with the random pair $\left(\overline{H}_{k,n}, \overline{H}_{\lfloor\theta k\rfloor,n}\right)$, $0<\theta<1$, i.e.

$$\overline{H}_{k,n}^{GJ(q,\theta)} := \frac{\overline{H}_{k,n} - p\ \overline{H}_{\lfloor\theta k\rfloor,n}}{1-p}, \quad 0<\theta<1,$$

with

$$p = p_n = \frac{Bias_\infty[\overline{H}_{k,n}]}{Bias_\infty[\overline{H}_{\lfloor\theta k\rfloor,n})]} = \frac{A^2(n/k)}{A^2(n/\lfloor\theta k\rfloor)} \underset{n/k\to\infty}{\longrightarrow} \theta^{2\rho}.$$

It is thus sensible to consider $p=\theta^{2\rho}$, $\theta = 1/2$ (see Gomes *et al.*, 2002, for further details on the choice of θ), and, with $\hat\rho$ a consistent estimator of ρ, the GJ-MVRB EVI-estimators,

$$\overline{\overline{H}}_{k,n} \equiv \overline{H}_{k,n}^{GJ} := \frac{2^{2\hat\rho}\ \overline{H}_{k,n} - \overline{H}_{\lfloor k/2\rfloor,n}}{2^{2\hat\rho}-1}. \tag{3.3}$$

Then, and provided that $\widehat{\rho}-\rho = o_p(1)$,

$$\sqrt{k}\left(\overline{\overline{H}}_{k,n} - \gamma\right) \overset{d}{=} \mathcal{N}_{0,\sigma^2_{GJ}} + o_p\left(\sqrt{k}A^2(n/k)\right),$$

with

$$\sigma^2_{GJ} = \gamma^2\left(1+1/(2^{-2\rho}-1)^2\right),$$

just as proved in Gomes *et al.* (2013b). More precisely, and under the fourth-order framework in (2.3), we can write

$$\sqrt{k}\left(\overline{\overline{H}}_{k,n} - \gamma\right) \overset{d}{=} \mathcal{N}_{0,\sigma^2_{GJ}} + b_{GJ}\sqrt{k}A^3(n/k) + o_p\left(\sqrt{k}A^3(n/k)\right). \tag{3.4}$$

We have thus again a trade-off between variance and bias. The bias decreases, but the variance increases. Anyway, we are able to reach a better performance at optimal levels, as desired.

Consequently, even if

$$\sqrt{k}\ A(n/k) \to \infty, \quad \text{with} \quad \sqrt{k}\ A^2(n/k) \to \lambda_{_A}, \text{ finite},$$

the type of levels k where the MSE of $\overline{H}_{k,n}$ is minimized,

$$\sqrt{k}\ \left(\overline{H}_{k,n} - \gamma\right) \underset{n\to\infty}{\overset{d}{\longrightarrow}} \mathcal{N}_{\lambda_{_A} b_{\overline{H}}, \sigma^2_{\overline{H}}}$$

$$\text{and} \quad \sqrt{k}\ \left(\overline{\overline{H}}_{k,n} - \gamma\right) \underset{n\to\infty}{\overset{d}{\longrightarrow}} \mathcal{N}_{0,\sigma^2_{GJ}}.$$

If $\sqrt{k}\ A^3(n/k) \longrightarrow \lambda_{_A}$, finite

$$\sqrt{k}\ \left(\overline{\overline{H}}_{k,n} - \gamma\right) \underset{n\to\infty}{\overset{d}{\longrightarrow}} \mathcal{N}_{\lambda_{_A} b_{GJ}, \sigma^2_{GJ}},$$

now the type of levels where the MSE of $\overline{\overline{H}}_{k,n} \equiv \overline{H}^{GJ}_{k,n}$ is minimized.

Let E denote either H or $\overline{H}$ or $\overline{\overline{H}} \equiv \overline{H}^{GJ}$. We then get, on the basis of (2.6), (2.7) and (3.4),

$$\begin{aligned} k_{A|E}(n) &:= \arg\min_k \text{AMSE}\big(E_{k,n}\big) \\ &= \arg\min_k \begin{cases} \sigma^2_E/k + b^2_{_E}\ A^2(n/k), & \text{if} \quad E = H, \\ \sigma^2_E/k + b^2_{_E}\ A^4(n/k), & \text{if} \quad E = \overline{H}, \\ \sigma^2_E/k + b^2_{_E}\ A^6(n/k), & \text{if} \quad E = \overline{\overline{H}} \end{cases} \\ &= k_{0|E}(n)(1 + o(1)), \end{aligned}$$

with $k_{0|E}(n)$ defined in (3.1). See Theorem 1 of Draisma *et al.*, 1999, for a proof of this result, in the case of H. The proof is similar for the cases of $\overline{H}$ and $\overline{\overline{H}}$. Things work more intricately for the PORT-MVRB and quasi-PORT-MVRB EVI-estimators, and we shall consider an algorithm similar to the one devised for the Hill EVI-estimators in case we are working with either $\overline{H}^{(q)}$ or $\overline{\overline{H}}^{(q)}$, $0 \le q < 1$, since only for specific values of q will these estimators be second-order reduced-bias. The bootstrap methodology enables us to estimate the OSF, $k_{0|E}(n)/n$, on the basis of a consistent estimator of $k_{0|E}(n)$, in (3.1), in a way similar to the one used for the classical EVI-estimators, now through the use of an auxiliary statistic like the one in (3.2), a method detailed in Gomes *et al.* (2011b; 2012) for the MVRB EVI-estimation. Indeed, under the above-mentioned fourth-order framework in (2.3),

$$T^E_{k,n} \overset{d}{=} \frac{\gamma\ P^E_k}{\sqrt{k}} + \begin{cases} b_{_E}(2^\rho - 1)\ A(n/k)(1 + o_p(1)), & \text{if} \quad E = H, \\ b_{_E}(2^{2\rho} - 1)\ A^2(n/k)(1 + o_p(1)), & \text{if} \quad E = \overline{H}, \\ b_{_E}(2^{3\rho} - 1)\ A^3(n/k)(1 + o_p(1)), & \text{if} \quad E = \overline{\overline{H}}, \end{cases}$$

with P_k^E asymptotically standard normal.

Consequently, denoting $k_{0|T}(n) := \arg\min_k \mathrm{MSE}(T_{k,n})$, we have

$$k_{0|E}(n) = k_{0|T}(n) \times \begin{cases} (1-2^{\rho})^{\frac{2}{1-2\rho}}\,(1+o(1)), & \text{if} \quad E = H, \\ (1-2^{2\rho})^{\frac{2}{1-4\rho}}\,(1+o(1)), & \text{if} \quad E = \overline{H}, \\ (1-2^{3\rho})^{\frac{2}{1-6\rho}}\,(1+o(1)), & \text{if} \quad E = \overline{\overline{H}}. \end{cases}$$

3.3 Adaptive EVI-estimation

In the following Algorithm, and with the notation $X_{n+1:n} = 0$, we consider that $\overline{H} \equiv \overline{H}^{(q)}_{k,n} \equiv \overline{\overline{H}}^{(q)}_{k,n}$, for $q = 1$, i.e. we include the MVRB EVI-estimators in the overall selection. Moreover, whenever dealing with $0 \le q < 1$ replace n by $n - n_q$, $n_q = \lfloor nq \rfloor + 1$.

3.3.1 Algorithm—Adaptive bootstrap estimation of γ

1. Given the sample $(x_1, \ldots, x_n)$, compute for the tuning parameters $\tau = 0$ and $\tau = 1$, the observed values of $\hat{\rho}_\tau(k)$, the most simple class of estimators in Fraga Alves *et al.* (2003). Such estimators have the functional form

$$\hat{\rho}_\tau(k) := -\left|3(W^{(\tau)}_{k,n} - 1)/(W^{(\tau)}_{k,n} - 3)\right|,$$

dependent on the statistics

$$W^{(\tau)}_{k,n} := \begin{cases} \dfrac{\left(M^{(1)}_{k,n}\right)^{\tau} - \left(M^{(2)}_{k,n}/2\right)^{\tau/2}}{\left(M^{(2)}_{k,n}/2\right)^{\tau/2} - \left(M^{(3)}_{k,n}/6\right)^{\tau/3}}, & \text{if} \quad \tau \neq 0, \\[2ex] \dfrac{\ln M^{(1)}_{k,n} - \ln\left(M^{(2)}_{k,n}/2\right)/2}{\ln\left(M^{(2)}_{k,n}/2\right)/2 - \ln\left(M^{(3)}_{k,n}/6\right)/3}, & \text{if} \quad \tau = 0, \end{cases}$$

where

$$M^{(j)}_{k,n} := \frac{1}{k}\sum_{i=1}^{k}\Big(\ln X_{n-i+1:n} - \ln X_{n-k:n}\Big)^{j},\ j = 1, 2, 3.$$

2. Consider $\mathcal{K} = \left(\lfloor n^{0.995} \rfloor,\ \lfloor n^{0.999} \rfloor\right)$. Compute the median of $\{\hat{\rho}_\tau(k)\}_{k\in\mathcal{K}}$, denoted χ_τ, and compute $I_\tau := \sum_{k\in\mathcal{K}} \left(\hat{\rho}_\tau(k) - \chi_\tau\right)^2$, $\tau = 0, 1$. Next choose the *tuning parameter* $\tau^* = 0$ if $I_0 \le I_1$; otherwise, choose $\tau^* = 1$.

3. Work with $\hat{\rho} \equiv \hat{\rho}_{\tau^*} = \hat{\rho}_{\tau^*}(k_1)$ and $\hat{\beta} \equiv \hat{\beta}_{\tau^*} := \hat{\beta}_{\hat{\rho}_{\tau^*}}(k_1)$, with $k_1 = \lfloor n^{0.999} \rfloor$, being $\hat{\beta}_{\hat{\rho}}(k)$ the estimator in Gomes and Martins (2002), given by

$$\hat{\beta}_{\hat{\rho}}(k) := \left(\frac{k}{n}\right)^{\hat{\rho}} \frac{d_k(\hat{\rho})\ D_k(0) - D_k(\hat{\rho})}{d_k(\hat{\rho})\ D_k(\hat{\rho}) - D_k(2\hat{\rho})},$$

dependent on the estimator $\hat{\rho} = \hat{\rho}_{\tau^*}(k_1)$, and where, for any $\alpha \le 0$,

$$d_k(\alpha) := \frac{1}{k}\sum_{i=1}^{k} (i/k)^{-\alpha} \quad \text{and} \quad D_k(\alpha) := \frac{1}{k}\sum_{i=1}^{k} (i/k)^{-\alpha}\, U_i,$$

with $U_i = i\,\left(\ln X_{n-i+1:n} - \ln X_{n-i:n}\right)$, $1 \le i \le k < n$, the *scaled log-spacings.*

4. For $k = 1, 2, \ldots$, compute the observed values of $H_{k,n}$, $\overline{H}_{k,n}$ and $\overline{\overline{H}}_{k,n} \equiv \overline{H}^{GJ}_{k,n}$, in (1.3), (2.5) and (3.3), respectively.

5. For $q = 0(0.1)0.9$, execute steps **1.**, **2.** and **3.** for the observed value of the sample of excesses in (2.8), and compute the observed values of $\overline{H}^{(q)}_{k,n}$, in (2.10), $\big($or alternatively $\overline{\overline{H}}^{(q)}_{k,n}$, in (2.11) $\big)$, for all admissible k.

6. Consider sub-sample sizes $m_1 = o(n)$ and $m_2 = [m_1^2/n]+1$, having n the same meaning as $n - \lfloor nq \rfloor - 1$ if $0 \le q < 1$.

7. For l from 1 until $B = 250$, independently generate from the empirical d.f. associated with the underlying sample $(x_1, x_2, \ldots, x_n)$, B bootstrap samples

$$(x_1^*, \ldots, x_{m_2}^*) \quad \text{and} \quad (x_1^*, \ldots, x_{m_2}^*, x_{m_2+1}^*, \ldots, x_{m_1}^*),$$

with sizes m_2 and m_1, respectively.

8. Again generally denoting $E_{k,n}$ any of the aforementioned EVI-estimators, let us denote by $T^*_{k,n|E}$ the bootstrap counterpart of the auxiliary statistic in (3.2), and obtain $t^*_{k,m_1,l|E}$, $1 < k < m_1$, $t^*_{k,m_2,l|E}$, $1 < k < m_2$, $1 \le l \le B$, the observed values of the statistics $T^*_{k,m_i|E}$, $i = 1, 2$, and compute

$$\mathrm{MSE}^*_E(m_i, k) = \frac{1}{B}\sum_{l=1}^{B} \left(t^*_{k,m_i,l|E}\right)^2, \quad k = 1, 2, \ldots, m_i - 1, \quad i = 1, 2.$$

9. Obtain $\hat{k}^*_{0|E}(m_i) := \arg\min_{1\le k\le m_i-1} \text{MSE}^*_E(m_i,k)$, $i=1,2$.

10. Compute

$$\hat{k}_{0|E} := \min\left(n-1, \left[\frac{c_{\hat{\rho}}\ \big(\hat{k}^*_{0|E}(m_1)\big)^2}{\hat{k}^*_{0|E}(m_2)}\right]+1\right),$$

with

$$c_\rho = \begin{cases} (1-2^\rho)^{\frac{2}{1-2\rho}}, & \text{if} \quad E=H \text{ or } \overline{H}^{(q)},\ 0\le q<1, \\ (1-2^{2\rho})^{\frac{2}{1-4\rho}}, & \text{if} \quad E=\overline{H}, \\ (1-2^{3\rho})^{\frac{2}{1-6\rho}}, & \text{if} \quad E=\overline{\overline{H}}, \end{cases}$$

and the OSF's estimates, $\hat{k}_{0|E}/n$.

11. Obtain $H^* = H_{\hat{k}_{0|H},n}$, $\overline{H}^* = \overline{H}_{\hat{k}_{0|\overline{H}},n}$, $\overline{\overline{H}}^* = \overline{\overline{H}}_{\hat{k}_{0|\overline{\overline{H}}},n}$ and $\overline{H}^{*(q)}_{n,m_1} := \overline{H}^{(q)}_{\hat{k}_0^{(q)},n}$, with $\hat{k}_0^{(q)} := \hat{k}_{0|\overline{H}^{(q)}}$.

12. With $B^*_q(m_i,k) = \frac{1}{B}\sum_{l=1}^{B} t^*_{k,m_i,l|\overline{H}^{(q)}}$, $\ k=1,2,\ldots,m_i-1$, $\ i=1,2$, consider

$$\widehat{\text{AMSE}}(k;q) := \frac{\big(\overline{H}^{*(q)}_{n,m_1}\big)^2}{k} + \left(\frac{\big(B^*_q(m_1,k)\big)^2}{\big(2^{\hat{\rho}}-1\big)B^*_q(m_2,k)}\right)^2,$$

with the previously obtained values $\hat{\rho}=\hat{\rho}_q$, and $\overline{H}^{*(q)}_{n,m_1}$.

13. Compute $\hat{q} := \arg\min_q \widehat{\text{AMSE}}(\hat{k}_0^{(q)};q)$.

14. Obtain the final adaptive EVI-estimate,

$$\overline{H}^{**} \equiv \overline{H}^{**}|\hat{q} \equiv \overline{H}^{*(\hat{q})}_{n,m_1} := \overline{H}^{(\hat{q})}_{\hat{k}_0^{(\hat{q})},n}.$$

Remark 3.1. *An analogue procedure can be used for any other parameter of extreme events.*

Remark 3.2. *A few practical questions may be again raised under the set-up developed: How does the asymptotic method work for moderate sample sizes? What is the dependence of the method on the choice of* m_1*? What is the sensitivity of the method with respect to the choice of*

ρ-*estimators? Although aware of the need of* $m_1 = o(n)$*, what happens if we choose* $m_1 = n$*? Answers to these questions were given in* Gomes and Oliveira (2001) *for the* Hill EVI-*estimators, can be addressed here, but are beyond the scope of this article.*

Remark 3.3. *Note that bootstrap confidence intervals associated with the adaptive* EVI-*estimates are easily computed on the basis of the replication of the Algorithm R times, for an adequate R.*

4 Applications to real data

4.1 An environmental application

The first set of data, already considered in Gomes *et al.* (2012), are related to the number of hectares, exceeding 100 ha, burnt during wildfires recorded in Portugal during 14 years (1990-2003). Most of the wildfires are extinguished within a short period of time, with almost negligible effects. However, some wildfires go out of control, burning hectares of land and causing significant and negative environmental and economical impacts. The data (a sample of size $n = 2627$) do not seem to have a significant temporal structure, and we have used it as a whole. A box-and-whiskers plot of the data provides evidence on the heaviness of the right tail, as can be seen in Figure 2.

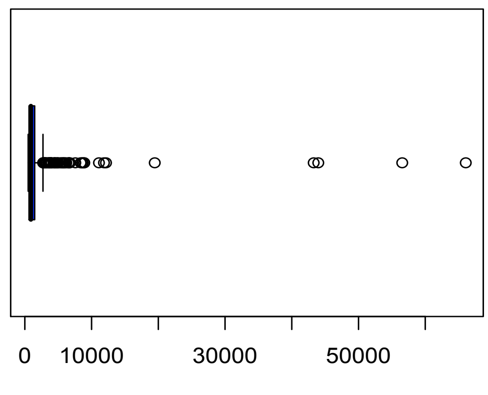

Figure 2: Box-and-whiskers plot associated with the burnt areas in Portugal, above 100 ha, in the period 1990-2003

Let us have a look at the behaviour of the adaptive EVI-estimators under consideration for this data set. We have been led to the ρ-estimate $\hat\rho \equiv \hat\rho_0 = -0.388$, obtained at the level $k_1 = \lfloor n_0^{0.999} \rfloor = 2606$.

The associated β-estimate is $\hat{\beta} \equiv \hat{\beta}_0 = 0.470$. Note that the sample paths of the ρ-estimates associated with $\tau = 0$ and $\tau = 1$ lead us indeed to choose, on the basis of any stability criterion for large k, the estimate associated with $\tau = 0$. The aforementioned double-bootstrap algorithm (until Step 11., and with $q = 1$, so that we are working with $\overline{H}$ only, among the $\overline{H}^{(q)}$ EVI-estimators) depends very weakly on the choice of a subsample size $m_1 = o(n)$ (see Gomes *et al.*, 2012). For $m_1 = \lfloor n^{0.955} \rfloor = 1843$, and B=250 bootstrap replications, we have got

- $\hat{k}_0^H = 157$ $(\hat{k}_0^H/n = 0.060)$ and the Hill EVI-estimate, $H^* = 0.73$,
- $\hat{k}_0^{\overline{H}} = 1319$ $(\hat{k}_0^{\overline{H}}/n = 0.502)$ and the MVRB EVI-estimate, $\overline{H}^* = 0.66$,
- $\hat{k}_0^{\overline{H}^{GJ}} = 2296$ $(\hat{k}_0^{\overline{H}^{GJ}}/n = 0.874)$ and the GJ-MVRB EVI-estimate, $\overline{\overline{H}}^* = 0.65$,

the values presented in Figure 3, together with sample paths of the EVI-estimates under consideration.

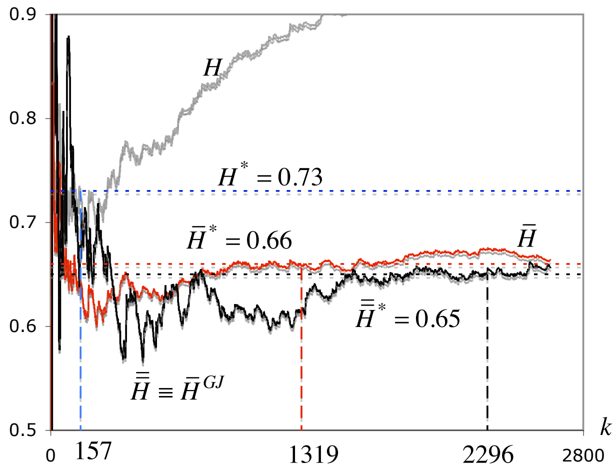

Figure 3: EVI-estimates for the burned areas

For the PORT-MVRB EVI-estimation, illustrated in Figure 4, and with the exclusion of the value $q = 1$, we have been led to the choice $q = 0$, $\hat{k}_0^{(0)} = 242$ $(\hat{k}_0^{(0)}/n_0 = 0.092$, $n_0 = n - 1)$ and $\overline{H}^{**}|0 = 0.670$. With the inclusion of $q = 1$ in the algorithm, we have been led to $\overline{H}$, as expected, due to the data characteristics (positive values only). For this type of data we have thus no particular gain in terms of efficiency when we use the PORT methodology.

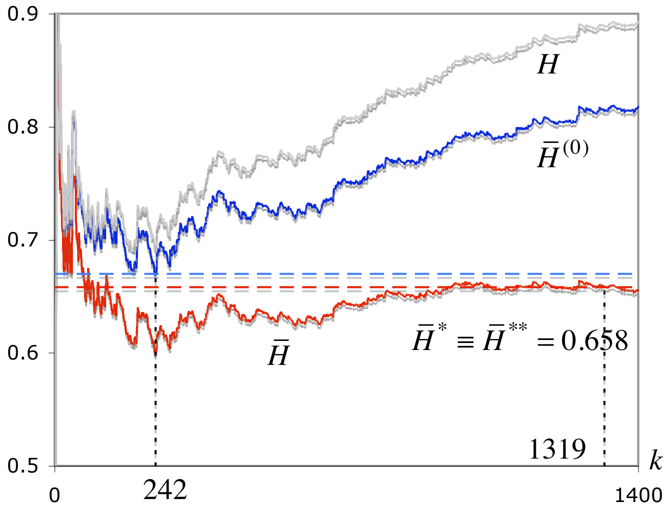

Figure 4: PORT-MVRB EVI-estimation for burned areas

4.2 An application in the area of finances

To enhance the importance of the PORT-MVRB EVI-estimation, we shall further consider an application to the analysis of the log-returns associated with one of the four sets of finance data considered in Gomes and Pestana (2007). Such data, collected over the period from January 4, 1999, until November 17, 2005, and with a size $n = 1762$, were the daily closing values of the Microsoft Corp. (MSFT). Note that these MSFT data have also been analysed in Gomes *et al.* (2011b; 2013a), through the use of different algorithms. Although there is some increasing trend in the volatility of all these log-returns, stationarity and weak dependence is often assumed, under the same considerations as in Drees (2003).

The underlying model has heavy left and right tails. We have thus eliminated the estimators associated with $q = 0$, due to their inconsistency (see Gomes *et al.*, 2008b, for details). The number of positive elements in the available sample of MSFT log-returns is $n_0 = 882$. We have been led to the ρ-estimate $\hat{\rho} \equiv \hat{\rho}_0 = -0.72$, obtained at the level $k_1 = \lfloor n_0^{0.999} \rfloor = 876$. The associated β-estimate is $\hat{\beta} \equiv \hat{\beta}_0 = 1.02$. Just as above, the sample paths of the ρ-estimates associated with $\tau = 0$ and $\tau = 1$ lead us indeed to choose, on the basis of any stability criterion for large k, the estimate associated with $\tau = 0$.

In Figure 5, we present the adaptive and non-adaptive estimates of γ, provided by H, $\overline{H}^{(q)}$, $q = 0.1$, 0.3 and 1 ($\overline{H}^{(1)} = \overline{H}$), with H and $\overline{H}^{(q)}$ given in (1.3) and (2.10), respectively. Note that the Hill estimators $H_{k,n}$, in (1.3), are unbiased for the EVI estimation only when the underlying model is a strict Pareto model. Otherwise, i.e. when we have only Pareto-like tails, as surely happens here and can be seen from Figure 5 (as well as from Figures 3 and 4), it exhibits a quite relevant bias. The PORT-MVRB estimators, $\overline{H}^{(q)}$, in (2.10), which are expected to be 'asymptotically unbiased' for adequate values of q, have a smaller bias, exhibit more stable sample paths as functions of k, and enable us to take a decision upon the estimate of γ and other parameters of extreme events to be used, even with the help of any heuristic stability criterion, like the '*largest run*' method suggested in Gomes *et al.* (2005), and the ones provided in Gomes *et al.* (2013a), among others.

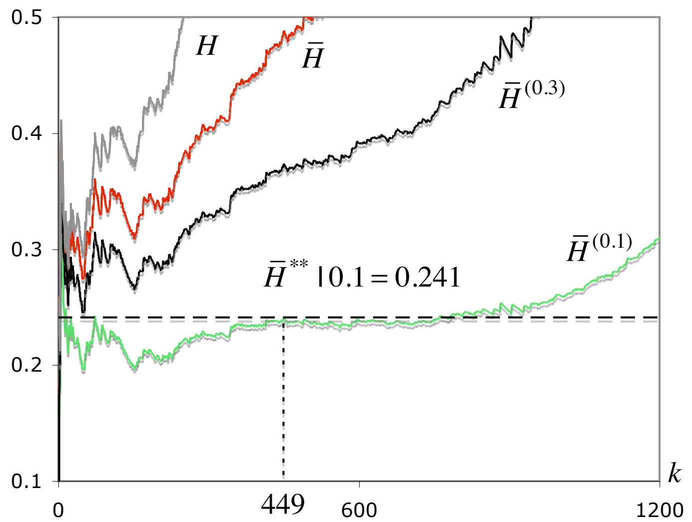

Figure 5: Adaptive and non-adaptive EVI-estimates for the MSFT data set

The *Algorithm* in Section 3.3, for $0 < q \leq 1$, led us to the choice $\hat{q} = 0.1$, $\hat{k}_0^{(0.1)} = 449$, and $\overline{H}^{**} \equiv \overline{H}^{**}|0.1 = 0.241$, as shown in Figure 5. Indeed, the MVRB EVI-estimators, despite of 'asymptotically unbiased' reveal a relevant bias for models like the Student-t, one of the most common candidates in a parametric estimation of log-returns. The PORT-MVRB EVI-estimates are then serious candidates to a reliable EVI-estimation.

5 Some overall conclusions

- The double-bootstrap algorithm, despite of computationally intensive, is quite reliable for the estimation of OSFs.
- The most attractive features of the GJ EVI -estimators are their stable sample paths (for a wide region of k or k/n values).
- The GJ-MVRB EVI-estimate is quite close to the MVRB EVI-estimate, but with a higher OSF-estimate.
- Due to stability reasons we advise for positive data sets the use of the GJ-MVRB or the MVRB EVI-estimators rather than the PORT-MVRB EVI-estimators.
- For the MSFT data set, or for any data set with negative values, we advice the use of the PORT-MVRB EVI-estimators, due to their stable sample paths as functions of k or k/n for an adequate q, enabling a more reliable EVI-estimation.

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