A new partially reduced-bias mean-of-order p class of extreme value index estimators*

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Abstract

A class of partially reduced-bias estimators of a positive *extreme value index* (EVI), related to a mean-of-order-p class of EVI-estimators, is introduced and studied both asymptotically as well as for finite samples through a Monte-Carlo simulation study. We are further interested in the comparison of this class and a representative class of *minimum-variance reduced-bias* (MVRB) EVI-estimators, related to a direct removal of the dominant component of the bias of a classical estimator of a positive EVI, the Hill estimator, performed in such a way that the minimal asymptotic variance is also attained by this MVRB class. Heuristic choices of the *tuning* parameters p and k, the number of top order statistics used in the estimation, are put forward, and applied to simulated and real data.

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1 Introduction and preliminaries

Let $X_1, \ldots, X_n$ be independent, identically distributed (i.i.d.), or possibly weakly dependent and stationary random variables (r.v.'s) from an underlying cumulative distribution function (c.d.f.) F. Let us denote the associated ascending order statistics (o.s.) by $X_{1:n} \leq \cdots \leq X_{n:n}$ and let us assume that there exist sequences of real constants $\{a_n > 0\}$ and $\{b_n \in \mathbb{R}\}$ such that the maximum, linearly normalized, i.e. $(X_{n:n} - b_n)/a_n$, has a non-degenerate limit. Then the limit distribution is necessarily an *extreme value* (EV) distribution, denoted $\mathrm{EV}_\xi(\cdot)$, with the functional form

$$\mathrm{EV}_\xi(x) = \begin{cases} \exp(-(1+\xi x)^{-1/\xi}),\ 1+\xi x > 0, & \text{if } \xi \neq 0, \\ \exp(-\exp(-x)),\ x \in \mathbb{R}, & \text{if } \xi = 0. \end{cases} \tag{1.1}$$

The c.d.f. F is then said to belong to the max-domain of attraction of EV_ξ, and we write $F \in \mathcal{D}_{\mathrm{M}}(\mathrm{EV}_\xi)$. The parameter ξ is the *extreme value index* (EVI), the primary parameter of extreme events, with a low frequency, but with a usually high impact. The EVI measures the heaviness of the *right tail function* (RTF), $\overline{F} := 1 - F$, and the heavier the tail, the larger the EVI is. In this paper we shall work with Pareto-type distributions, with a strictly positive EVI.

1.1 First and second-order conditions for heavy tails

Power laws, such as the Pareto income distribution (Pareto, 1965) and the Zipf's law for city-size distribution (Zipf, 1941), have been observed a long time ago in many important phenomena in economics and biology and have recently seriously attracted scientists. In statistics of extremes, F is often said to be heavy-tailed whenever the RTF, $\overline{F}$, is a *regularly varying* (RV) function with a negative index of regular variation equal to $-1/\xi$, $\xi > 0$, or equivalently, with $F^{\leftarrow}(x) := \inf\{y : F(y) \geq x\}$ denoting the generalized inverse function of F, the *reciprocal tail quantile function* (RTQF), $U(t) := F^{\leftarrow}(1 - 1/t),\ t \geq 1$, is of regular variation with index ξ (for details on regular variation, see Bingham *et al.*, 1987). With the notation RV_a for the class of RV functions with an index of regular variation a, i.e., positive measurable functions $g(\cdot)$ such that $\lim_{t\to\infty} g(tx)/g(t) = x^a$, for all $x > 0$,

$$F \in \mathcal{D}_{\mathrm{M}}(\mathrm{EV}_{\xi>0}) \iff \overline{F} \in \mathrm{RV}_{-1/\xi} \text{ (Gnedenko, 1943)}$$
$$\iff U \in \mathrm{RV}_\xi \text{ (de Haan, 1984)}. \tag{1.2}$$

The second-order parameter, ρ (≤ 0), rules the rate of convergence in any of the first-order conditions in (1.2), and it is the non-positive parameter appearing in the limiting relation

$$\lim_{t\to\infty} \frac{\ln U(tx) - \ln U(t) - \xi \ln x}{A(t)} = \psi_\rho(x) := \begin{cases} \frac{x^\rho - 1}{\rho}, & \text{if } \rho < 0, \\ \ln x, & \text{if } \rho = 0, \end{cases} \tag{1.3}$$

which we assume to hold for every $x > 0$, and where $|A|$ must then be of regular variation with index ρ (Geluk and de Haan, 1987). We shall further assume everywhere in the paper that $\rho < 0$. We shall also assume that we are working in the more strict Hall-Welsh class of Pareto-type models (Hall and Welsh, 1985), with an RTF,

$$\overline{F}(x) = Cx^{-1/\xi}\big(1 + D_1 x^{\rho/\xi} + o\big(x^{\rho/\xi}\big)\big), \text{ as } x \to \infty, \ \xi > 0, \tag{1.4}$$

for $C > 0$, $D_1 \neq 0$, $\rho < 0$. Regarding the RTQF, we can then say that there exist $c > 0$ and $d_1 \neq 0$ such that $U(t) = ct^{\xi}\big(1 + d_1 t^{\rho} + o\big(t^{\rho}\big)\big)$, as $t \to \infty$, Therefore, condition (1.3) holds and we may choose there $A(t) = \alpha t^{\rho}$, for an adequate α, which we reparameterize as,

$$A(t) = \xi\beta t^{\rho}, \ \rho < 0. \tag{1.5}$$

1.2 The class of EVI-estimators under play

For Pareto-type models, the most commonly used EVI-estimators are the Hill estimators (Hill, 1975), which are the averages of the log-excesses, V_{ik}, i.e.

$$\mathrm{H}(k) := \frac{1}{k}\sum_{i=1}^{k} V_{ik}, \quad V_{ik} := \ln\frac{X_{n-i+1:n}}{X_{n-k:n}}, \quad 1 \le i \le k < n. \tag{1.6}$$

But since we can write

$$\mathrm{H}(k) = \sum_{i=1}^{k} \ln\left(\frac{X_{n-i+1:n}}{X_{n-k:n}}\right)^{1/k} = \ln\left(\prod_{i=1}^{k}\frac{X_{n-i+1:n}}{X_{n-k:n}}\right)^{1/k}, \quad 1 \le i \le k < n,$$

the Hill estimator can be thought as the logarithm of the geometric mean (or mean-of-order-0) of $\underline{\mathbb{U}} := \{U_{ik} := X_{n-i+1:n}/X_{n-k:n}, \ 1 \le i \le k < n\}$. More generally, Brilhante *et al.* (2013a) considered as basic statistics the mean-of-order-p (MOP) of $\underline{\mathbb{U}}$, with $p \ge 0$, i.e., the class of statistics

$$A_p(k) = \begin{cases} \left(\frac{1}{k}\sum\limits_{i=1}^{k} U_{ik}^{p}\right)^{1/p}, & \text{if } \ p > 0, \\[2ex] \left(\prod\limits_{i=1}^{k} U_{ik}\right)^{1/k}, & \text{if } \ p = 0, \end{cases}$$

and the class of MOP EVI-estimators,

$$\mathrm{H}_p(k) \equiv \mathrm{MOP}_p(k) := \begin{cases} \Big(1 - A_p^{-p}(k)\Big)/p, & \text{if } \ 0 < p < 1/\xi, \\[2ex] \ln A_0(k) = \mathrm{H}(k), & \text{if } \ p = 0, \end{cases} \tag{1.7}$$

with $\mathrm{H}_0(k) \equiv \mathrm{H}(k)$, given in (1.6). We now state the following result, proved for $p = 0$ in de Haan and Peng (1998) and for $0 < p < 1/(2\xi)$ in Brilhante *et al.* (2013a).

Theorem 1.1 (de Haan and Peng, 1998; Brilhante *et al.*, 2013a)**.** *Under the first-order condition in* (1.2) *and for intermediate* k*, i.e. a sequence of integers* $k = k_n$*,* $1 \le k < n$*, such that*

$$k = k_n \to \infty \quad and \quad k_n = o(n), \;\; as\; n \to \infty, \tag{1.8}$$

the EVI-*estimators* $\mathrm{H}_p(k)$*, in* (1.7) *are consistent for the estimation of* ξ*.*
If we further assume the validity of the second-order condition in (1.3)*, we can write the following asymptotic distributional representation,*

$$\mathrm{H}_p(k) \;\stackrel{d}{=}\; \xi + \frac{\sigma_{_{\mathrm{H}_p}} Z_k^{(p)}}{\sqrt{k}} + b_{_{\mathrm{H}_p}} A(n/k)(1 + o_p(1)), \tag{1.9}$$

where $Z_k^{(p)}$ *is a standard normal r.v.,*

$$b_{_{\mathrm{H}_p}} = b_{_{\mathrm{H}_p}}(\xi, \rho) = \frac{1 - p\xi}{1 - \rho - p\xi} \quad and \quad \sigma^2_{_{\mathrm{H}_p}} = \sigma^2_{_{\mathrm{H}_p}}(\xi) = \frac{\xi^2(1 - p\xi)^2}{1 - 2p\xi}. \tag{1.10}$$

Remark 1.1. *We thus have an asymptotic standard normal behavior for* $\mathrm{H}_p(k)$*, in* (1.7)*, whenever working with values* k *such that* $\sqrt{k}A(n/k) \to \lambda$*, finite, but with a high asymptotic bias when* $\lambda \neq 0$*, i.e. when we slightly increase* k *up to values where the mean square error* (MSE) *is minimized. This high bias at optimal levels has led several authors to deal with bias reduction in the field of extremes. Recent overviews can be found in* Gomes *et al.* (2007c) (*Chapter 6 of* Reiss and Thomas, 2007)*,* Gomes *et al.* (2008a)*,* Beirlant *et al.* (2012) *and* Gomes and Guillou (2014)*.*

Working just for technical simplicity in the particular class of models in (1.4), or equivalently in (1.3) but with $A(\cdot)$ parameterized as in (1.5), the asymptotic distributional representation in (1.9), for $p = 0$, with $b_{_{\mathrm{H}_0}} = 1/(1 - \rho)$ given in (1.10), led Caeiro *et al.* (2005) to directly remove the dominant component of the bias of the Hill EVI-estimator, given by $\xi\beta(n/k)^\rho/(1 - \rho)$, considering the *corrected-Hill* (CH) EVI-estimator,

$$\mathrm{CH}(k) \equiv \mathrm{CH}_{\hat\beta,\hat\rho}(k) := \mathrm{H}(k)\Big(1 - \frac{\hat\beta}{1 - \hat\rho}\left(\frac{n}{k}\right)^{\hat\rho}\Big), \tag{1.11}$$

which can be a *minimum-variance reduced-bias* (MVRB) class of EVI-estimators for adequate second-order parameters' estimators, $(\hat\beta, \hat\rho)$.

Similarly, and with values of p such that the asymptotic normality of the estimators in (1.7) is known to hold, i.e. $0 \le p < 1/(2\xi)$, Brilhante *et al.* (2013b) noticed that there is an optimal value

$$p \equiv p_{_M} = \varphi_\rho/\xi, \quad \text{with} \quad \varphi_\rho = 1 - \rho/2 - \sqrt{\rho^2 - 4\rho + 2}\Big/2, \tag{1.12}$$

which maximises the asymptotic efficiency of the class of estimators in (1.7). These authors considered an *optimal* MOP (OMOP) r.v., defined by

$$\mathrm{OMOP}(k) := \mathrm{H}_{p_{_M}}(k), \tag{1.13}$$

with $\mathrm{H}_p(k)$ given in (1.7), and for which they derived its asymptotic behaviour. Such a behaviour has led Gomes *et al.* (2013a) to introduce an associated *optimal reduced-bias* MOP (ORBMOP) r.v.,

$$\mathrm{RB}(k;\beta,\rho) \equiv \mathrm{ORBMOP}(k) := \mathrm{OMOP}(k)\left(1 - \frac{\beta(1-\varphi_\rho)}{1-\rho-\varphi_\rho}\left(\frac{n}{k}\right)^\rho\right), \tag{1.14}$$

with φ_ρ and $\mathrm{OMOP}(k)$ given in (1.12) and (1.13), respectively. The class of r.v.'s in (1.14) is similar in spirit to the MVRB CH EVI-estimators in (1.11), and the main reasons for such a consideration are also similar to the ones presented before, and related to the asymptotic distributional representation of the MOP class of EVI-estimators, provided in (1.9) and (1.10).

The dependence of $p_{_M}$, in (1.12), on (ξ,ρ), requires adequate estimates for these two parameters in order to have OMOP and ORBMOP EVI-estimators, based in (1.13) and (1.14), respectively. Therefore, it is reasonable to consider the *partially* RBMOP (PRBMOP) class of EVI-estimators based on $\mathrm{H}_p(k)$, in (1.7), i.e.

$$\mathrm{RB}_p(k;\hat\beta,\hat\rho) \equiv \mathrm{PRBMOP}_p(k) := \mathrm{H}_p(k)\left(1 - \frac{\hat\beta(1-\varphi_{\hat\rho})}{1-\hat\rho-\varphi_{\hat\rho}}\left(\frac{n}{k}\right)^{\hat\rho}\right), \tag{1.15}$$

dependent again on a flexible tuning parameter p, and on adequate second-order parameters's estimators, $(\hat\beta,\hat\rho)$.

1.3 Scope of the paper

In this paper, we obtain asymptotic and finite sample distributional properties of the class of PRBMOP EVI-estimators, in (1.15), comparatively to the classes of Hill and MVRB EVI-estimators, in (1.6) and (1.11), respectively. More specifically, in Section 2, we present the asymptotic degenerate and non-degenerate behaviour of the aforementioned PRBMOP class of EVI-estimators. In Section 3, and through the use of Monte-Carlo simulation techniques, we exhibit the performance of these RB-estimators, comparatively to the MVRB and the classical Hill estimators. In Section 4 we provide algorithms for an adaptive PRBMOP EVI-estimation, that also includes an algorithm for the adequate estimation of the two second-order parameters β and ρ, in the lines of the ones presented before in articles related to MVRB estimation of parameters of extreme events. In Section 5 we provide an illustration of the behaviour of the EVI-estimators under study for simulated and real samples, drawing some overall conclusions in Section 6.

2 Asymptotic behaviour of PRBMOP EVI-estimators

2.1 Asymptotic behaviour at a level k

We state and prove the two following theorems.

Theorem 2.1. *Under the validity of the first-order condition, in* (1.2)*, for levels k such that* (1.8) *holds, and with* $\mathrm{RB}_p(k;\hat\beta,\hat\rho)$ *defined in* (1.15)*, the r.v.'s* $\mathrm{RB}_p(k;\beta,\rho)$ *are consistent for the estimation of ξ, provided that $0 \le p < 1/\xi$. If we further assume the second-order condition, in* (1.3)*, and with $Z_k^{\mathrm{RB}_p}$ asymptotically standard normal r.v.'s, we can write*

$$\mathrm{RB}_p(k;\beta,\rho) \overset{d}{=} \xi + \frac{\sigma_{\mathrm{RB}_p} Z_k^{\mathrm{RB}_p}}{\sqrt{k}} + b_{\mathrm{RB}_p} A(n/k)(1+o_p(1)), \tag{2.1}$$

for $0 \le p < 1/(2\xi)$, with $\sigma_{\mathrm{RB}_p} = \sigma_{\mathrm{H}_p}$, defined in (1.10)*, and*

$$b_{\mathrm{RB}_p} = \frac{\rho(p\xi - \varphi_\rho)}{(1-p\xi-\rho)(1-\rho-\varphi_\rho)}. \tag{2.2}$$

Consequently, if $\sqrt{k}\, A(n/k) \to \lambda$, finite,

$$\sqrt{k}\,(\mathrm{RB}(k;\beta,\rho) - \xi) \underset{n\to\infty}{\overset{d}{\longrightarrow}} \mathcal{N}\Big(\lambda b_{\mathrm{RB}_p}, \sigma^2_{\mathrm{RB}_p}\Big), \tag{2.3}$$

with a null bias if and only if $\lambda = 0$ or $\lambda \ne 0$ and $p = p_{_M} = \varphi_\rho/\xi$, with φ_ρ given in (1.12).

Proof. Under the second-order framework, in (1.3), we know from Brilhante *et al.* (2013a) that (1.9) and (1.10) hold, i.e. we have the asymptotic distributional representation

$$\mathrm{H}_p(k) \overset{d}{=} \xi + \frac{\xi(1-p\xi)V_k^{(p)}}{\sqrt{k}\sqrt{1-2p\xi}} + \frac{(1-p\xi)A(n/k)}{1-p\xi-\rho} + o_p\left(A(n/k)\right),$$

with $V_k^{(p)}$ asymptotically standard normal.

Noticing now that

$$\mathrm{RB}_p(k;\beta,\rho) := \mathrm{H}_p(k)\Big(1 - \frac{\beta(1-\varphi_\rho)}{1-\rho-\varphi_\rho}\left(\frac{n}{k}\right)^\rho\Big),$$

we easily derive that the dominant component of the bias is given by

$$\frac{(1-p\xi)A(n/k)}{1-p\xi-\rho} - \frac{(1-\varphi_\rho)A(n/k)}{1-\rho-\varphi_\rho} = \frac{\rho(p\xi-\varphi_\rho)A(n/k)}{(1-p\xi-\rho)(1-\rho-\varphi_\rho)},$$

i.e. it is null only for $p = \varphi_\rho/\xi$. Consequently, with $Z_k^{\mathrm{RB_p}} := V_k^{(p)}$, (2.1), (2.2) and (2.3) follow. □

Theorem 2.2. *Under the same conditions of* Theorem 2.1*, let us consider the* RB*-class of* EVI*-estimators,* $\mathrm{RB}_p(k;\hat\beta,\hat\rho)$*, in* (1.15)*, for any consistent estimators* $(\hat\beta,\hat\rho)$ *such that*

$$\hat\rho - \rho = o_p(1/\ln n), \quad \textit{as} \quad n \to \infty. \tag{2.4}$$

Then, (2.1) *and* (2.3) *hold with* $\mathrm{RB}_p(k;\beta,\rho)$ *replaced by* $\mathrm{RB}_p(k;\hat\beta,\hat\rho)$*, the* PRBMOP EVI-*estimator, in* (1.15).

Proof. If we estimate consistently β and ρ through the estimators $\hat{\beta}$ and $\hat{\rho}$, we can use Cramer's delta-method, and obtain for any of the estimators $\mathrm{RB}_p(k;\hat{\beta},\hat{\rho})$, in (1.15), with $a_{\mathrm{RB}_p} = -1/(1-\rho-p\xi)$,

$$\begin{aligned}
&\mathrm{RB}_p\big(k;\hat{\beta},\hat{\rho}\big) - \mathrm{RB}_p\big(k;\beta,\rho\big) \\
&\qquad \overset{p}{\sim}\ a_{\mathrm{RB}_p}(1-p\xi)A(n/k)\Big\{\Big(\frac{\hat{\beta}-\beta}{\beta}\Big) + (\hat{\rho}-\rho)\Big[\ln(n/k)\ - a_{\mathrm{RB}_p}\Big]\Big\}, \qquad (2.5)
\end{aligned}$$

where $a_n \sim b_n$ means that a_n/b_n converge in probability to one, as $n \to \infty$. Indeed, we can write,

$$\begin{aligned}
&\frac{\partial \mathrm{RB}_p(k;\beta,\rho)}{\partial\beta} \overset{p}{\sim} -\frac{A(n/k)(1-p\xi)}{\beta(1-\rho-p\xi)}, \\
&\qquad\qquad \frac{\partial \mathrm{RB}_p(k;\beta,\rho)}{\partial\rho} \overset{p}{\sim} -A(n/k)\Big(\frac{1-p\xi}{1-\rho-p\xi}\ln(n/k) - \frac{1-p\xi}{(1-\rho-p\xi)^2}\Big).
\end{aligned}$$

The result in the theorem follows thus straightforwardly from (2.4) and (2.5). □

2.2 Asymptotic comparison of PRBMOP and Hill EVI-estimators at optimal levels

With $\sigma_p = \sigma_{\mathrm{RB}_p}(\xi)$ and $b_p = b_{\mathrm{RB}_p}(\xi,\rho)$ given in (1.10) and (2.2), respectively, the so-called *asymptotic mean square error* (AMSE) is given by

$$\mathrm{AMSE}\left(\mathrm{RB}_p(k)\right) := \sigma_p^2/k + b_p^2\, A^2(n/k).$$

Regular variation theory, used in the lines of de Haan and Peng (1998), Gomes and Martins (2001), Gomes *et al.* (2005, 2007a,b, 2013c), Gomes and Henriques-Rodrigues (2010) and Caeiro and Gomes (2011), enable us to assert that, whenever $b_p \neq 0$, there exists a function $\eta(n) = \eta(n,\xi,\rho)$, such that

$$\lim_{n\to\infty} \eta(n)\ \mathrm{AMSE}\left(\mathrm{RB}_{p0}\right) = \big(\sigma_p^2\big)^{-\frac{2\rho}{1-2\rho}}\big(b_p^2\big)^{\frac{1}{1-2\rho}} =: \mathrm{LMSE}\left(\mathrm{RB}_{p0}\right),$$

where $\mathrm{RB}_{p0} := \mathrm{RB}_p(k_{0|p}(n))$ and $k_{0|p}(n) := \arg\min_k \mathrm{MSE}\left(\mathrm{RB}_p(k)\right)$. Moreover, if we slightly restrict the second-order condition in (1.3), assuming that (1.5) holds, i.e. $A(t) = \xi\beta t^\rho$, $\rho < 0$, we can write for $p\xi \neq \varphi_\rho$,

$$k_{p0} \equiv k_{p0}(n) = \arg\min_k \mathrm{MSE}\left(\mathrm{RB}_p(k)\right) = \big(\sigma_p^2\ n^{-2\rho}/\big(b_p^2\xi^2\beta^2(-2\rho)\big)\big)^{1/(1-2\rho)}(1+o(1)).$$

We again consider the usual *asymptotic relative efficiency* (AREFF),

$$\mathrm{AREFF}_{p|0} \equiv \mathrm{AREFF}_{\mathrm{RB}_{p0}|\mathrm{H}_{00}} := \sqrt{\mathrm{LMSE}\big(\mathrm{H}_{00}\big)/\mathrm{LMSE}\big(\mathrm{RB}_{p0}\big)}. \qquad (2.6)$$

Note that we can further write,

$$\text{AREFF}_{p|0} = \left(\left(\frac{\sqrt{1-2p\xi}}{1-p\xi}\right)^{-2\rho}\left|\frac{(1-\rho-p\xi)(1-\rho-\varphi_\rho)}{\rho(1-\rho)(p\xi-\varphi_\rho))}\right|\right)^{\frac{1}{1-2\rho}}, \tag{2.7}$$

i.e. $\text{AREFF}_{p|0}$ depends on (p,ξ) through $p\xi$. If $p\xi = \varphi_\rho$, with φ_ρ given in (1.12), we get an infinite AREFF-indicator. The AREFF-indicators, $\text{AREFF}_{p|0}$, are presented in Figure 1 for a few values of $(p\xi, \rho)$, $0 \le p\xi < 0.5$.

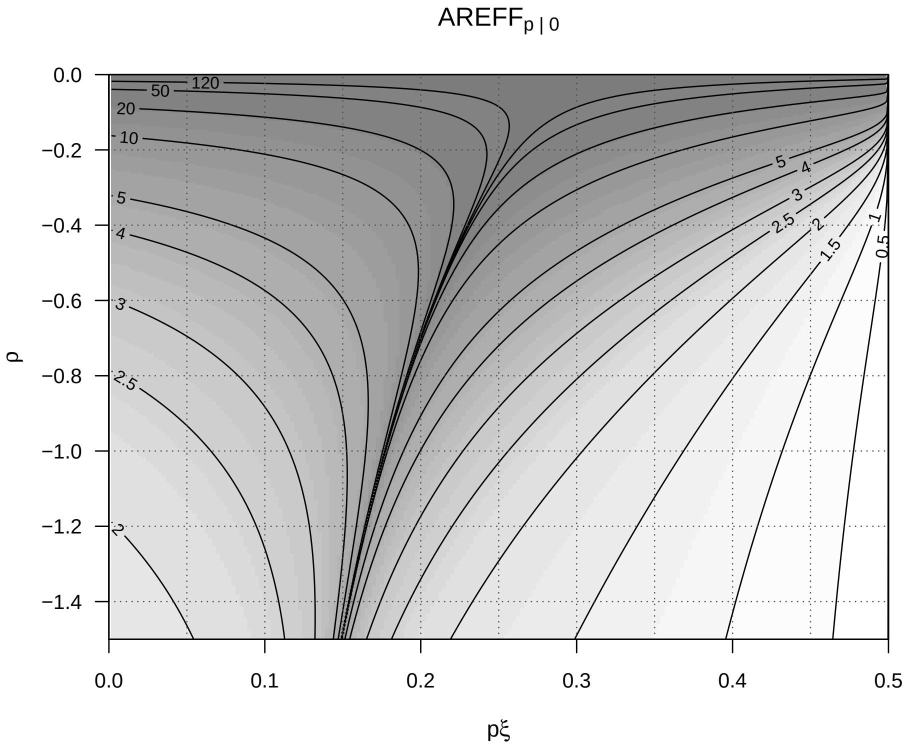

Figure 1: Values of $AREFF_{p|0}$, in (2.7), for different values of $(p\xi, \rho)$

For the CH EVI-estimator in (1.11), we now know that we can adequately estimate the vector of second-order parameters (β, ρ) so that the asymptotic distributional representation

$$\text{CH}(k) \overset{d}{=} \xi + \frac{\sigma_{\text{CH}} Z_k^{\text{CH}}}{\sqrt{k}} + o_p(A(n/k))$$

holds under the second-order framework in (1.3) and with $\sigma_{\text{CH}} = \xi$ (see Caeiro *et al.*, 2005, for a proof). Consequently, if we consider an AREFF-indicator of the type of the one in (2.6), but for the comparison of CH_0 and H_{00}, we get an infinite AREFF-indicator. The analogue AREFF-indicator, $\text{AREFF}_{\text{RB}_{p0}|\text{CH}_0}$, is zero unless $p\xi = \varphi_\rho$, and in this case, i.e. when we consider an ORBMOP EVI-estimator, a study of $\text{AREFF}_{\text{RB}_{p0}|\text{CH}_0}$ involves deeply a third-order framework which is beyond the scope of this article.

3 Monte-Carlo simulations

We have run Monte-Carlo simulations for the following heavy-tailed models:

1. the Fréchet$_\xi$ model, with c.d.f. $F(x) = \exp(-x^{-1/\xi})$, $x \geq 0$, $\xi > 0$, for which $\rho = -1$,

2. the EV_ξ model, with c.d.f. $F(x) = \text{EV}_\xi(x)$, given in (1.1) ($\rho = -\xi$), and

3. the Student's t_ν-model with ν degrees of freedom, with a probability density function

$$f_{t_\nu}(t) = \Gamma((\nu+1)/2)\left[1+t^2/\nu\right]^{-(\nu+1)/2} / (\sqrt{\pi\nu}\;\Gamma(\nu/2)),\ t \in \mathbb{R} \quad (\nu > 0),$$

for which $\xi = 1/\nu$ and $\rho = -2/\nu$.

We have further considered, out of Hall-Welsh's class, and also out of the scope of Theorems 2.1 and 2.2,

4. a model for which the second-order condition in (1.3) does not hold, the sin-Burr$_{\xi,\rho}$ model, with an RTQF, $U(t) = (t^{-\rho} - \sin t^{-\rho})^{-\xi/\rho}$, $t \geq 1$.

In all Monte-Carlo simulation experiments we have considered multi-sample simulations of size 5000×20 and sample sizes $n = 100,\ 200,\ 500,\ 1000,\ 2000$ and 5000. For details on multi-sample simulation, we refer Gomes and Oliveira (2001).

3.1 Mean values and MSE patterns of the EVI-estimators, as functions of k

For each value of n and for each of the aforementioned models, we have first simulated the mean values (E) and root MSEs (RMSEs) of the EVI-estimators in (1.11) and (1.15), for values of $p = a/(10\xi)$, $a = 0(1)9$, as functions of k, the number of top order statistics k involved in the estimation, and on the basis of the first run of size 5000. As an illustration, we present Figures 2, 3 and 4, respectively associated with $\text{EV}_{0.1}$, Student t_4 and Sin-Burr$_{1,-0.25}$ parents. Note that for $a = 5(1)9$, we have consistency of the estimators either in (1.7) or in (1.15), but no guarantee of asymptotic normality, and even of bias reduction in (1.15) comparatively to the Hill EVI-estimators. However, as can be seen in all figures, some of the EVI-estimators in this region can beat the EVI-estimators in the region $0 \leq p < 1/(2\xi)$, both regarding bias and RMSE.

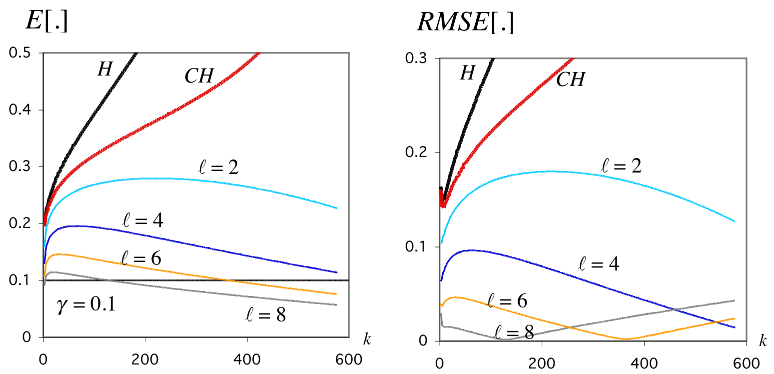

Figure 2: Mean values (left) and RMSEs (right) of H(k), CH(k) and RB$_p(k)$, $p = a/10\xi$, $a = 2, 4, 6, 8$, for an EV$_{0.1}$ ($\rho = -\xi = -0.1$) underlying parent

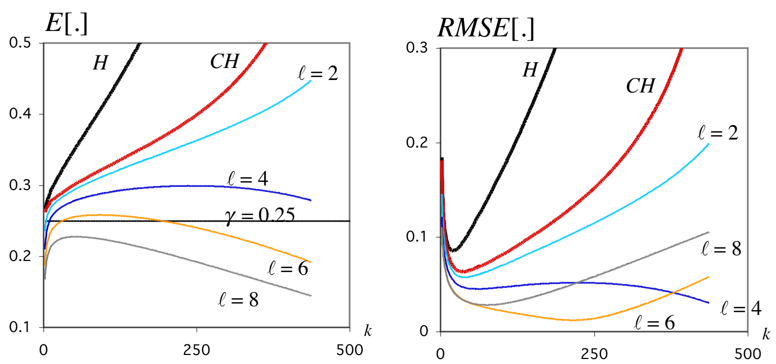

Figure 3: Mean values (left) and RMSEs (right) of H(k), CH(k) and RB$_p(k)$, $p = a/10\xi$, $a = 2, 4, 6, 8$, for a Student t_ν, $\nu = 4$ ($\xi = 1/\nu = 0.25, \rho = -2/\nu = -0.5$) underlying parent

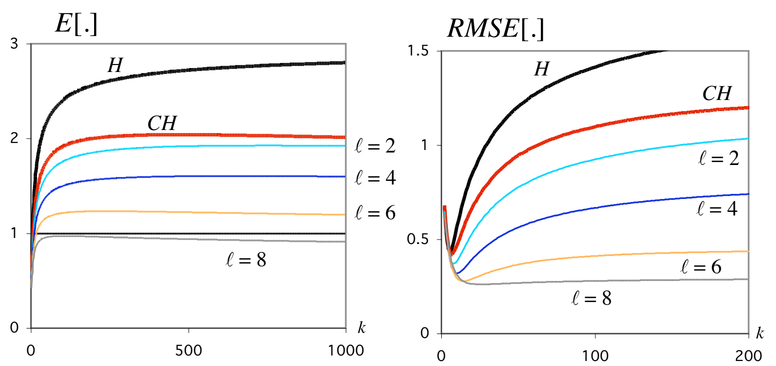

Figure 4: Mean values (left) and RMSEs (right) of H(k), CH(k) and RB$_p(k)$, $p = a/10\xi$, $a = 2, 4, 6, 8$, for a Sin-Burr$_{1,-0.25}$ underlying parent

3.1.1 Mean values of the EVI-estimators at optimal levels

We have further computed the Hill estimator at the simulated value of $k_{0|\mathrm{H}} := \arg\min_k \mathrm{RMSE}\big(\mathrm{H}(k)\big)$, the simulated optimal k in the sense of minimum RMSE, not relevant in practice, but providing an indication of the best possible performance of Hill's estimator. Such an estimator is denoted by H_{00}. We have also computed RB_{p0}, i.e. the PRBMOP EVI-estimator $\mathrm{RB}_p(k)$ computed at the simulated value of $k_{0|\mathrm{RB}_p} := \arg\min_k \mathrm{RMSE}\big(\mathrm{RB}_p(k)\big)$. As an illustration of the bias reduction achieved with the PRBMOP EVI-estimators in (1.15) at optimal levels (levels where RMSE are minimal as functions of k), i.e. the bias of H_{00}, CH_0 and RB_{p0}, we present in the following tables, for $n = 100,\ 200,\ 500,\ 1000,\ 2000$ and 5000, the simulated mean values at optimal levels of $\mathrm{H}(k)$, $\mathrm{CH}(k)$ and $\mathrm{RB}_p(k)$, in (1.6), (1.11) and (1.15), respectively, for $p = a/(10\xi)$, and the two regions, $a = 1(1)4$, where we can guarantee consistency and asymptotic normality under the second-order framework in (1.3), and $a = 5(1)9$, where only consistency is assured by Theorem 2.2, and for models in Hall-Wesh class. Information on 95% confidence intervals, computed on the basis of the 20 replicates with 5000 runs each, is also provided. Among the estimators considered, the one providing the smallest squared bias in each of the three regions is written in **bold** whenever it overpasses the best estimator in the previous region. If an estimator is not the best one in a region but it outperforms the best EVI-estimator in the previous region, we write it in *italic*.

Remark 3.1. *Note that, as proved in* Gomes *et al.* (2013a), *for* Fréchet *models, T/ξ does not depend on ξ, again with T denoting any of the aforementioned* MOP *or* RBMOP *EVI-estimators. Also, for* sin-Burr *models, and just like happens with* Burr *models* (see Gomes *et al.*, 2013a)*, any of the statistics T under consideration is also such that T/ξ is independent of ξ.*

Table 1: Simulated mean values, at optimal levels, of $\mathrm{H}(k)/\xi$, $\mathrm{CH}(k)/\xi$ and $\mathrm{RB}_p(k)/\xi$, $p = a/(10\xi)$, $a = 1(1)9$, for Fréchet underlying parents, together with 95% confidence intervals

n	100	200	500	1000	2000	5000
Fréchet parents						
H	1.109 ± 0.0027	1.085 ± 0.0290	1.063 ± 0.0013	1.049 ± 0.0014	1.039 ± 0.0009	1.029 ± 0.0008
CH	**0.982** ± 0.0030	**0.986** ± 0.0395	**0.995** ± 0.0016	**0.999** ± 0.0008	**1.000** ± 0.0005	**1.000** ± 0.0002
$a = 1$	0.969 ± 0.0018	0.975 ± 0.0008	0.979 ± 0.0007	0.980 ± 0.0005	0.980 ± 0.0005	0.980 ± 0.0002
$a = 2$	**0.995** ± 0.0030	*1.004* ± 0.0303	*1.001* ± 0.0010	1.001 ± 0.0008	1.001 ± 0.0006	1.001 ± 0.0004
$a = 3$	*0.994* ± 0.0030	*1.000* ± 0.0340	*1.000* ± 0.0012	*1.001* ± 0.0004	*1.000* ± 0.0003	1.000 ± 0.0004
$a = 4$	*0.989* ± 0.0013	**1.000** ± 0.0164	**1.000** ± 0.0011	**1.000** ± 0.0005	**1.000** ± 0.0003	**1.000** ± 0.0003
$a = 5$	0.943 ± 0.0012	0.976 ± 0.0181	1.000 ± 0.0020	0.998 ± 0.0005	0.999 ± 0.0008	0.999 ± 0.0003
$a = 6$	0.866 ± 0.0011	0.893 ± 0.0156	0.912 ± 0.0004	0.922 ± 0.0004	0.934 ± 0.0003	0.945 ± 0.0002
$a = 7$	0.855 ± 0.0016	0.889 ± 0.0206	0.919 ± 0.0008	0.935 ± 0.0008	0.948 ± 0.0006	0.960 ± 0.0004
$a = 8$	0.822 ± 0.0012	0.857 ± 0.0152	0.890 ± 0.0005	0.909 ± 0.0006	0.924 ± 0.0006	0.939 ± 0.0004
$a = 9$	0.785 ± 0.0012	0.821 ± 0.0150	0.854 ± 0.0005	0.874 ± 0.0005	0.890 ± 0.0006	0.907 ± 0.0004

Table 2: Simulated mean values, at optimal levels, of H(k), CH(k) and RB$_p$(k), $p = a/(10\xi)$, $a = 1(1)9$, for EV$_\xi$ underlying parents, $\xi = 0.1, 0.25$ and 1, together with 95% confidence intervals

n	100	200	500	1000	2000	5000
EV$_\xi$ parent, $\xi = 0.1$ ($\rho = -0.1$)						
H	0.334 $\pm$ 0.0009	0.284 $\pm$ 0.0007	0.243 $\pm$ 0.0005	0.223 $\pm$ 0.0016	0.209 $\pm$ 0.0014	0.195 $\pm$ 0.0011
CH	**0.276** $\pm$ 0.0016	**0.258** $\pm$ 0.0014	**0.234** $\pm$ 0.0012	**0.221** $\pm$ 0.0013	**0.208** $\pm$ 0.0015	**0.194** $\pm$ 0.0012
$a = 1$	*0.251* $\pm$ 0.0014	*0.238* $\pm$ 0.0007	*0.217* $\pm$ 0.0005	*0.203* $\pm$ 0.0008	*0.192* $\pm$ 0.0009	*0.182* $\pm$ 0.0014
$a = 2$	*0.222* $\pm$ 0.0009	*0.227* $\pm$ 0.0009	*0.199* $\pm$ 0.0004	*0.188* $\pm$ 0.0005	*0.178* $\pm$ 0.0004	*0.166* $\pm$ 0.0005
$a = 3$	*0.156* $\pm$ 0.0009	*0.156* $\pm$ 0.0008	*0.153* $\pm$ 0.0003	*0.152* $\pm$ 0.0005	*0.152* $\pm$ 0.0003	*0.152* $\pm$ 0.0001
$a = 4$	**0.118** $\pm$ 0.0008	**0.117** $\pm$ 0.0006	**0.115** $\pm$ 0.0002	**0.114** $\pm$ 0.0004	**0.114** $\pm$ 0.0002	**0.114** $\pm$ 0.0001
$a = 5$	*0.098* $\pm$ 0.0003	*0.099* $\pm$ 0.0001	*0.100* $\pm$ 0.0001	*0.100* $\pm$ 0.0001	*0.100* $\pm$ 0.0001	*0.100* $\pm$ 0.0001
$a = 6$	*0.098* $\pm$ 0.0003	*0.100* $\pm$ 0.0001	*0.100* $\pm$ 0.0001	*0.100* $\pm$ 0.0001	*0.100* $\pm$ 0.0001	*0.100* $\pm$ 0.0001
$a = 7$	**0.099** $\pm$ 0.0003	**0.100** $\pm$ 0.0001	**0.100** $\pm$ 0.0001	**0.100** $\pm$ 0.0001	**0.100** $\pm$ 0.0001	**0.100** $\pm$ 0.0001
$a = 8$	*0.096* $\pm$ 0.0004	*0.100* $\pm$ 0.0001	*0.100* $\pm$ 0.0001	*0.100* $\pm$ 0.0001	*0.100* $\pm$ 0.0001	*0.100* $\pm$ 0.0001
$a = 9$	*0.088* $\pm$ 0.0005	*0.095* $\pm$ 0.0002	*0.099* $\pm$ 0.0001	*0.100* $\pm$ 0.0001	*0.100* $\pm$ 0.0001	*0.100* $\pm$ 0.0001
EV$_\xi$ parent, $\xi = 0.25$ ($\rho = -0.25$)						
H	0.427 $\pm$ 0.0012	0.391 $\pm$ 0.0026	0.365 $\pm$ 0.0019	0.348 $\pm$ 0.0012	0.335 $\pm$ 0.0013	0.321 $\pm$ 0.0010
CH	**0.382** $\pm$ 0.0027	**0.372** $\pm$ 0.0021	**0.353** $\pm$ 0.0014	**0.342** $\pm$ 0.0017	**0.330** $\pm$ 0.0008	**0.317** $\pm$ 0.0008
$a = 1$	*0.368* $\pm$ 0.0025	*0.360* $\pm$ 0.0018	*0.346* $\pm$ 0.0014	*0.335* $\pm$ 0.0014	*0.326* $\pm$ 0.0007	*0.314* $\pm$ 0.0008
$a = 2$	*0.353* $\pm$ 0.0024	*0.345* $\pm$ 0.0017	*0.335* $\pm$ 0.0013	*0.327* $\pm$ 0.0014	*0.319* $\pm$ 0.0007	*0.310* $\pm$ 0.0009
$a = 3$	*0.340* $\pm$ 0.0024	*0.353* $\pm$ 0.0051	*0.323* $\pm$ 0.0012	*0.316* $\pm$ 0.0012	*0.311* $\pm$ 0.0007	*0.304* $\pm$ 0.0009
$a = 4$	**0.283** $\pm$ 0.0011	**0.291** $\pm$ 0.0009	**0.291** $\pm$ 0.0004	**0.291** $\pm$ 0.0007	**0.290** $\pm$ 0.0005	**0.290** $\pm$ 0.0002
$a = 5$	**0.242** $\pm$ 0.0007	**0.248** $\pm$ 0.0003	**0.250** $\pm$ 0.0001	**0.250** $\pm$ 0.0001	**0.250** $\pm$ 0.0001	**0.250** $\pm$ 0.0001
$a = 6$	*0.236* $\pm$ 0.0008	*0.246* $\pm$ 0.0003	*0.249* $\pm$ 0.0001	*0.250* $\pm$ 0.0001	*0.250* $\pm$ 0.0001	*0.250* $\pm$ 0.0001
$a = 7$	*0.228* $\pm$ 0.0012	*0.242* $\pm$ 0.0004	*0.248* $\pm$ 0.0001	*0.249* $\pm$ 0.0001	*0.250* $\pm$ 0.0001	*0.250* $\pm$ 0.0001
$a = 8$	*0.213* $\pm$ 0.0012	*0.230* $\pm$ 0.0006	*0.242* $\pm$ 0.0003	*0.246* $\pm$ 0.0001	*0.248* $\pm$ 0.0001	*0.249* $\pm$ 0.0001
$a = 9$	0.197 $\pm$ 0.0012	*0.214* $\pm$ 0.0006	*0.226* $\pm$ 0.0003	*0.232* $\pm$ 0.0001	*0.237* $\pm$ 0.0001	*0.242* $\pm$ 0.0001
EV$_\xi$ parent, $\xi = 1$ ($\rho = -1$)						
H	1.159 $\pm$ 0.0049	1.124 $\pm$ 0.0032	1.091 $\pm$ 0.0030	1.072 $\pm$ 0.0020	1.058 $\pm$ 0.0014	1.042 $\pm$ 0.0009
CH	**0.894** $\pm$ 0.0099	**0.975** $\pm$ 0.0046	**1.003** $\pm$ 0.0024	**1.004** $\pm$ 0.0013	**1.003** $\pm$ 0.0007	**1.001** $\pm$ 0.0004
$a = 1$	**0.917** $\pm$ 0.0104	**0.995** $\pm$ 0.0038	1.013 $\pm$ 0.0017	1.011 $\pm$ 0.0011	1.007 $\pm$ 0.0008	1.003 $\pm$ 0.0004
$a = 2$	*0.907* $\pm$ 0.0122	*0.983* $\pm$ 0.0036	1.009 $\pm$ 0.0016	1.008 $\pm$ 0.0009	1.005 $\pm$ 0.0005	1.002 $\pm$ 0.0003
$a = 3$	0.894 $\pm$ 0.0124	0.976 $\pm$ 0.0045	**1.003** $\pm$ 0.0018	1.005 $\pm$ 0.0008	1.003 $\pm$ 0.0008	1.001 $\pm$ 0.0005
$a = 4$	0.853 $\pm$ 0.0091	0.940 $\pm$ 0.0052	0.991 $\pm$ 0.0018	**0.999** $\pm$ 0.0006	**1.000** $\pm$ 0.0005	**1.000** $\pm$ 0.0004
$a = 5$	0.806 $\pm$ 0.0064	0.876 $\pm$ 0.0032	0.912 $\pm$ 0.0015	0.924 $\pm$ 0.0016	0.934 $\pm$ 0.0017	0.946 $\pm$ 0.0011
$a = 6$	0.763 $\pm$ 0.0055	0.840 $\pm$ 0.0048	0.894 $\pm$ 0.0010	0.913 $\pm$ 0.0010	0.927 $\pm$ 0.0008	0.941 $\pm$ 0.0007
$a = 7$	0.736 $\pm$ 0.0052	0.819 $\pm$ 0.0023	0.873 $\pm$ 0.0009	0.894 $\pm$ 0.0008	0.910 $\pm$ 0.0008	0.927 $\pm$ 0.0005
$a = 8$	0.703 $\pm$ 0.0054	0.786 $\pm$ 0.0025	0.842 $\pm$ 0.0007	0.866 $\pm$ 0.0008	0.884 $\pm$ 0.0006	0.904 $\pm$ 0.0005
$a = 9$	0.668 $\pm$ 0.0050	0.750 $\pm$ 0.0022	0.806 $\pm$ 0.0006	0.831 $\pm$ 0.0007	0.850 $\pm$ 0.0006	0.871 $\pm$ 0.0005

Remark 3.2. *The analysis of Tables* 1–4 *allow us to conclude that:*

- *Regarding bias, and with the exception of the* Student-t_2 *model for values of* $n < 200$ (*see* Table 3), the PRBMOP EVI–estimators have outperformed at optimal levels the MVRB EVI-estimators, for all simulated models (even including the Sin-Burr models, as illustrated in Table 4).
- *If we compare these results with the ones in* Brilhante *et al.* (2013*a*) *for* MOP EVI-*estimation, we see that these* PRBMOP EVI-*estimators clearly outperform the* MOP

Table 3: Simulated mean values, at optimal levels, of H(k), CH(k) and RB$_p$(k), $p = a/(10\xi)$, $a = 1(1)9$, for Student t_ν underlying parents, $\nu = 4$ and 2, together with 95% confidence intervals

n	100	200	500	1000	2000	5000
Student$_4$ parent ($\xi = 1/4 = 0.25,\ \rho = -0.5$)						
H	0.361 ± 0.0009	0.339 ± 0.0026	0.317 ± 0.0016	0.305 ± 0.0013	0.296 ± 0.0009	0.286 ± 0.0007
CH	**0.311** ± 0.0023	**0.310** ± 0.0009	**0.300** ± 0.0013	**0.294** ± 0.0008	**0.288** ± 0.0006	**0.281** ± 0.0004
$a = 1$	*0.293* ± 0.0016	*0.296* ± 0.0008	*0.291* ± 0.0011	*0.287* ± 0.0007	*0.283* ± 0.0006	*0.277* ± 0.0004
$a = 2$	*0.300* ± 0.0022	*0.300* ± 0.0010	*0.294* ± 0.0008	*0.289* ± 0.0006	*0.284* ± 0.0005	*0.278* ± 0.0004
$a = 3$	*0.285* ± 0.0036	*0.297* ± 0.0017	*0.290* ± 0.0005	*0.285* ± 0.0007	**0.281** ± 0.0005	**0.276** ± 0.0005
$a = 4$	**0.255** ± 0.0018	**0.271** ± 0.0009	**0.277** ± 0.0004	**0.281** ± 0.0005	*0.282* ± 0.0005	0.283 ± 0.0010
$a = 5$	*0.228* ± 0.0011	*0.242* ± 0.0004	**0.248** ± 0.0002	**0.249** ± 0.0001	**0.250** ± 0.0001	**0.250** ± 0.0001
$a = 6$	*0.213* ± 0.0014	*0.232* ± 0.0007	*0.243* ± 0.0003	*0.247* ± 0.0001	*0.249* ± 0.0001	*0.249* ± 0.0001
$a = 7$	0.200 ± 0.0010	0.221 ± 0.0007	*0.234* ± 0.0003	*0.240* ± 0.0002	*0.244* ± 0.0002	*0.247* ± 0.0001
$a = 8$	0.187 ± 0.0010	0.207 ± 0.0006	0.220 ± 0.0002	0.227 ± 0.0002	0.233 ± 0.0002	0.238 ± 0.0001
$a = 9$	0.174 ± 0.0009	0.193 ± 0.0006	0.207 ± 0.0002	0.214 ± 0.0001	*0.219* ± 0.0002	*0.225* ± 0.0001
Student$_2$ parent ($\xi = 1/2 = 0.5,\ \rho = -1$)						
H	0.601 ± 0.0039	0.577 ± 0.0027	0.556 ± 0.0011	0.544 ± 0.0008	0.535 ± 0.0010	0.526 ± 0.0005
CH	**0.464** ± 0.0123	**0.506** ± 0.0020	**0.512** ± 0.0011	**0.507** ± 0.0006	**0.504** ± 0.0006	**0.502** ± 0.0003
$a = 1$	0.456 ± 0.0126	*0.496* ± 0.0016	*0.501* ± 0.0007	*0.498* ± 0.0005	0.495 ± 0.0003	0.493 ± 0.0002
$a = 2$	0.462 ± 0.0132	0.508 ± 0.0019	*0.512* ± 0.0010	*0.507* ± 0.0005	*0.504* ± 0.0003	*0.502* ± 0.0002
$a = 3$	0.447 ± 0.0128	**0.499** ± 0.0018	*0.510* ± 0.0007	*0.506* ± 0.0006	*0.503* ± 0.0003	*0.501* ± 0.0002
$a = 4$	0.425 ± 0.0119	0.477 ± 0.0018	**0.500** ± 0.0008	**0.502** ± 0.0003	**0.502** ± 0.0001	**0.501** ± 0.0002
$a = 5$	0.395 ± 0.0107	0.435 ± 0.0011	0.451 ± 0.0005	0.455 ± 0.0004	0.459 ± 0.0004	0.463 ± 0.0004
$a = 6$	0.370 ± 0.0089	0.407 ± 0.0011	0.432 ± 0.0006	0.443 ± 0.0005	0.451 ± 0.0004	0.459 ± 0.0004
$a = 7$	0.352 ± 0.0081	0.395 ± 0.0009	0.420 ± 0.0005	0.432 ± 0.0006	0.441 ± 0.0003	0.450 ± 0.0003
$a = 8$	0.335 ± 0.0074	0.378 ± 0.0009	0.405 ± 0.0004	0.418 ± 0.0004	0.428 ± 0.0003	0.437 ± 0.0002
$a = 9$	0.318 ± 0.0067	0.361 ± 0.0008	0.388 ± 0.0005	0.401 ± 0.0004	0.411 ± 0.0003	0.421 ± 0.0002

Table 4: Simulated mean values, at optimal levels, of H(k)$/\xi$, CH(k)$/\xi$ and RB$_p$(k)$/\xi$, $p = a/(10\xi)$, $a = 1(1)9$, for Sin-Burr underlying parents with $\rho = -0.25$, together with 95% confidence intervals

Sin-Burr parents with $(\xi, \rho) = (1, -0.25)$						
n	100	200	500	1000	2000	5000
H	2.195 ± 0.0064	1.921 ± 0.0032	1.439 ± 0.0016	1.122 ± 0.0139	1.040 ± 0.0106	1.016 ± 0.0066
CH	**2.057** ± 0.0609	**1.664** ± 0.0305	**1.208** ± 0.0152	**1.007** ± 0.0169	**0.983** ± 0.0055	**0.992** ± 0.0037
$a = 1$	1.875 ± 0.0607	1.567 ± 0.0304	1.154 ± 0.0152	1.002 ± 0.0156	0.987 ± 0.0054	1.000 ± 0.0041
$a = 2$	1.706 ± 0.0519	1.460 ± 0.0260	1.094 ± 0.0130	**1.002** ± 0.0072	**0.989** ± 0.0032	**1.000** ± 0.0031
$a = 3$	1.526 ± 0.0081	1.354 ± 0.0041	1.035 ± 0.0020	0.998 ± 0.0048	0.987 ± 0.0060	0.998 ± 0.0024
$a = 4$	**1.389** ± 0.0070	**1.250** ± 0.0035	**1.006** ± 0.0018	0.987 ± 0.0041	0.986 ± 0.0043	1.003 ± 0.0021
$a = 5$	1.260 ± 0.0060	1.151 ± 0.0030	1.016 ± 0.0015	0.978 ± 0.0083	0.977 ± 0.0025	1.002 ± 0.0019
$a = 6$	1.143 ± 0.0052	1.059 ± 0.0026	0.980 ± 0.0013	0.963 ± 0.0078	0.969 ± 0.0027	0.997 ± 0.0018
$a = 7$	**1.039** ± 0.0045	**1.015** ± 0.0022	0.953 ± 0.0011	0.943 ± 0.0109	0.958 ± 0.0016	0.991 ± 0.0014
$a = 8$	0.965 ± 0.0100	0.973 ± 0.0050	0.930 ± 0.0025	0.916 ± 0.0133	0.935 ± 0.0018	0.976 ± 0.0009
$a = 9$	0.889 ± 0.0113	0.899 ± 0.0056	0.873 ± 0.0028	0.863 ± 0.0125	0.884 ± 0.0015	0.930 ± 0.0012

EVI-*estimators for* $a \le 7$. *Again, note that for* $a \ge 5$ *the reduction in bias has not been correctly done, due to the lack of information on the dominant component of such* *a bias. Despite of that, and comparatively to the* H *and even* CH EVI-*estimators, we do notice a reduction of bias for* $a = 5$ *and* $a = 7$*, whenever* $|\rho| < 1$ *and* n *is large.*

3.1.2 Mean square errors and relative efficiency indicators at optimal levels

We next present the simulated values of the indicators,

$$\mathrm{REFF}_{\mathrm{RB}_p|\mathrm{H}} := \frac{\mathrm{RMSE}\,(\mathrm{H}_{00})}{\mathrm{RMSE}\,(\mathrm{RB}_{p0})}.$$

A similar REFF-indicator has also been computed for the CH versus H EVI-estimators.

Remark 3.3. *An indicator higher than one means a better performance than the* Hill *estimator. Consequently, the higher these indicators are, the better the associated* EVI-*estimators perform, comparatively to* H_{00}.

Again as an illustration of the results obtained for $\mathrm{RB}_p(k)$, in (1.15), we present Tables 5–8. In the first row, we provide RMSE_{00}, the RMSE of H_{00}, so that we can easily recover the RMSE of all other estimators. The following rows provide the REFF-indicators of CH and RB_p, for the same values of p in the previous section, i.e. $p = a/(10\xi)$, $a = 1(1)9$. Similar marks (*italic* or **bold**) are used with the same meaning as before. Confidence intervals are not provided for REFF-indicators larger than 10, but are available from the authors upon request.

Table 5: Simulated RMSE of H/ξ (first row) and REFF-indicators of CH and RB_p, $p = a/(10\xi)$, $a = 1(1)9$, (independent on ξ), for Fréchet parents, together with 95% confidence intervals

	Fréchet parents					
n	100	200	500	1000	2000	5000
RMSE_{00}	0.212 ± 0.1547	0.164 ± 0.9989	0.117 ± 0.1432	0.091 ± 0.1345	0.071 ± 0.1255	0.052 ± 0.1136
CH	**1.257** ± 0.0072	**1.237** ± 0.1591	**1.337** ± 0.0080	**1.459** ± 0.0123	**1.574** ± 0.0123	**1.795** ± 0.0111
$a = 1$	**1.323** ± 0.0076	**1.306** ± 0.1659	**1.413** ± 0.0088	*1.539* ± 0.0078	*1.655* ± 0.0112	*1.883* ± 0.0099
$a = 2$	*1.308* ± 0.0086	*1.290* ± 0.2260	*1.405* ± 0.0090	**1.541** ± 0.0084	**1.666** ± 0.0122	**1.899** ± 0.0094
$a = 3$	1.294 ± 0.0093	1.273 ± 0.2541	1.390 ± 0.0101	1.533 ± 0.0117	1.662 ± 0.0130	1.897 ± 0.0103
$a = 4$	*1.306* ± 0.0104	*1.273* ± 0.2822	*1.386* ± 0.0118	*1.526* ± 0.0154	*1.655* ± 0.0137	*1.884* ± 0.0106
$a = 5$	1.305 ± 0.0092	1.296 ± 0.2978	1.395 ± 0.0132	1.528 ± 0.0150	1.644 ± 0.0150	1.859 ± 0.0126
$a = 6$	1.100 ± 0.0072	1.052 ± 0.1779	1.003 ± 0.0054	0.956 ± 0.0052	0.918 ± 0.0063	0.849 ± 0.0056
$a = 7$	0.939 ± 0.0063	0.887 ± 0.1251	0.823 ± 0.0038	0.782 ± 0.0036	0.740 ± 0.0050	0.693 ± 0.0048
$a = 8$	0.891 ± 0.0063	0.834 ± 0.1160	0.760 ± 0.0034	0.709 ± 0.0035	0.657 ± 0.0044	0.595 ± 0.0038
$a = 9$	0.826 ± 0.0058	0.760 ± 0.1077	0.672 ± 0.0031	0.611 ± 0.0032	0.551 ± 0.0037	0.480 ± 0.0030

Remark 3.4. *For the* REFF-*indicators obtained:*

- *If we restrict ourselves to the region of* p-*values where we can guarantee asymptotic normality, the best results were obtained for* $p = 4/(10\xi)$ *for all simulated models with* $|\rho| < 1$*, including the* sin-Burr *models.*

Table 6: Simulated RMSE of H_{00} (first row) and REFF-indicators of CH and RB_p, $p = a/(10\xi)$, $a = 1(1)9$, for EV_ξ underlying parents, $\xi = 0.1, 0.25, 1$, together with 95% confidence intervals

n	100	200	500	1000	2000	5000
EV_ξ parent, $\xi = 0.1$ ($\rho = -0.1$)						
RMSE_{00}	0.268 ± 0.3511	0.216 ± 0.3211	0.174 ± 0.2890	0.151 ± 0.2698	0.133 ± 0.2531	0.114 ± 0.2340
CH	**1.245** ± 0.0025	**1.140** ± 0.0027	**1.070** ± 0.0019	**1.045** ± 0.0011	**1.029** ± 0.0011	**1.019** ± 0.0008
$a = 1$	*1.477* ± 0.0058	*1.322* ± 0.0032	*1.217* ± 0.0012	*1.173* ± 0.0016	*1.140* ± 0.0018	*1.107* ± 0.0016
$a = 2$	*2.092* ± 0.0130	*1.682* ± 0.0126	*1.451* ± 0.0020	*1.378* ± 0.0025	*1.322* ± 0.0035	*1.266* ± 0.0040
$a = 3$	*4.312* ± 0.0791	*3.785* ± 0.0537	*3.246* ± 0.0234	*2.872* ± 0.0305	*2.559* ± 0.0239	*2.190* ± 0.0109
$a = 4$	**9.836** ± 0.3893	**10.909**	**11.099**	**10.323**	**9.478** ± 0.1815	**8.161** ± 0.0673
$a = 5$	*15.632*	*25.159*	*42.630*	*62.455*	*87.575*	*129.216*
$a = 6$	16.399	26.972	46.391	68.072	95.280	139.886
$a = 7$	**17.586**	**29.985**	*52.521*	*77.303*	*108.198*	*158.585*
$a = 8$	**18.433**	**34.833**	63.818	94.852	133.195	195.479
$a = 9$	*15.440*	*27.741*	**66.819**	**123.644**	**189.829**	**290.002**
EV_ξ parent, $\xi = 0.25$ ($\rho = -0.25$)						
RMSE_{00}	0.246 ± 0.3353	0.200 ± 0.3077	0.157 ± 0.2743	0.133 ± 0.2526	0.113 ± 0.2332	0.092 ± 0.2108
CH	**1.328** ± 0.0108	**1.237** ± 0.0056	**1.171** ± 0.0042	**1.130** ± 0.0021	**1.101** ± 0.0021	**1.072** ± 0.0020
$a = 1$	*1.420* ± 0.0083	*1.319* ± 0.0051	*1.236* ± 0.0034	*1.185* ± 0.0027	*1.148* ± 0.0190	*1.112* ± 0.0013
$a = 2$	*1.624* ± 0.0091	*1.478* ± 0.0057	*1.351* ± 0.0034	*1.277* ± 0.0033	*1.222* ± 0.0027	*1.171* ± 0.0011
$a = 3$	*2.150* ± 0.0117	*1.747* ± 0.0164	*1.521* ± 0.0040	*1.412* ± 0.0045	*1.328* ± 0.0040	*1.252* ± 0.0018
$a = 4$	**3.707** ± 0.1013	**3.981** ± 0.0705	**3.637** ± 0.0362	**3.183** ± 0.0568	**2.782** ± 0.0349	**2.293** ± 0.0148
$a = 5$	**5.035** ± 0.2271	**8.099** ± 0.3140	**13.777**	**20.164**	**28.274**	**42.280**
$a = 6$	5.036 ± 0.2181	7.970 ± 0.2922	13.269	19.062	26.194	38.046
$a = 7$	*4.823* ± 0.1968	*7.501* ± 0.2558	*12.315*	*17.314*	*23.111*	*31.805*
$a = 8$	4.329 ± 0.1496	6.209 ± 0.1731	9.543 ± 0.2106	13.130 ± 0.1620	17.199	22.789
$a = 9$	*3.736* ± 0.1015	*4.629* ± 0.0852	*5.629* ± 0.0789	*6.444* ± 0.0588	*7.331* ± 0.0480	*8.792* ± 0.0817
EV_ξ parent, $\xi = 1$ ($\rho = -1$)						
RMSE_{00}	0.314 ± 0.4875	0.239 ± 0.3105	0.170 ± 0.2741	0.132 ± 0.2455	0.104 ± 0.2192	0.076 ± 0.1883
CH	0.814 ± 0.1168	**1.182** ± 0.0230	**1.410** ± 0.0212	**1.678** ± 0.0192	**2.005** ± 0.0192	**2.500** ± 0.0218
$a = 1$	0.842 ± 0.1246	**1.237** ± 0.0226	1.449 ± 0.0196	1.696 ± 0.0181	1.999 ± 0.0181	2.472 ± 0.0213
$a = 2$	0.823 ± 0.1244	*1.221* ± 0.0247	*1.468* ± 0.0219	*1.760* ± 0.0223	*2.109* ± 0.0199	*2.654* ± 0.0230
$a = 3$	0.807 ± 0.1231	*1.202* ± 0.0279	*1.497* ± 0.0261	*1.848* ± 0.0281	*2.257* ± 0.0234	*2.896* ± 0.0270
$a = 4$	0.795 ± 0.1181	1.176 ± 0.0307	**1.537** ± 0.0324	**1.963** ± 0.0355	**2.459** ± 0.0282	**3.256** ± 0.0366
$a = 5$	0.779 ± 0.1086	1.081 ± 0.0252	1.254 ± 0.0240	1.341 ± 0.0198	1.358 ± 0.0289	1.293 ± 0.0244
$a = 6$	0.758 ± 0.0965	0.951 ± 0.0138	0.960 ± 0.0064	0.936 ± 0.0052	0.893 ± 0.0050	0.821 ± 0.0058
$a = 7$	0.748 ± 0.0879	0.910 ± 0.0124	0.901 ± 0.0057	0.859 ± 0.0046	0.803 ± 0.0042	0.720 ± 0.0048
$a = 8$	0.735 ± 0.0800	0.862 ± 0.0110	0.833 ± 0.0049	0.778 ± 0.0044	0.712 ± 0.0039	0.624 ± 0.0042
$a = 9$	0.715 ± 0.0718	0.806 ± 0.0093	0.753 ± 0.0040	0.685 ± 0.0040	0.613 ± 0.0034	0.521 ± 0.0034

- *Regarding* RMSE, *the consistent and asymptotically normal* PRBMOP EVI-*estimators* $\big(0 < p < 1/(2\xi)\big)$, *at optimal levels, can always beat the* MVRB EVI *estimators, also computed at optimal levels. In* Hall-Welsh*'s class, they can however be beaten by the only consistent* PRBMOP EVI-*estimators* $\big(1/(2\xi) \le p < 1/\xi\big)$, *at optimal levels, for all simulated parents again with* $|\rho| < 1$.
- *For large values of* n, *the* MVRB *and the* MOP *methodologies do not work for the* sin-Burr *models with* ρ *either quite close to zero or to one. For these same models, if* $-1 < \rho < -0.1$ *the new* PRBMOP *provide the best results.*

Table 7: Simulated RMSE of H_{00} (first row) and REFF-indicators of CH and RB_p, $p = a/(10\xi)$, $a = 1(1)9$, for Student t_ν underlying parents, $\nu = 4, 2$, together with 95% confidence intervals

n	100	200	500	1000	2000	5000
Student-t_4 parent ($\xi = 1/4 = 0.25, \rho = -0.5$)						
$RMSE_{00}$	0.183 ± 0.1537	0.143 ± 0.1511	0.106 ± 0.1420	0.085 ± 0.1340	0.070 ± 0.1260	0.054 ± 0.1157
CH	**1.435** ± 0.0468	**1.398** ± 0.0084	**1.361** ± 0.0053	**1.322** ± 0.0057	**1.283** ± 0.0057	**1.236** ± 0.0048
$a = 1$	*1.497* ± 0.0521	*1.460* ± 0.0076	*1.408* ± 0.0048	*1.360* ± 0.0051	*1.314* ± 0.0051	*1.261* ± 0.0044
$a = 2$	*1.685* ± 0.0723	*1.628* ± 0.0096	*1.537* ± 0.0063	*1.466* ± 0.0053	*1.399* ± 0.0052	*1.325* ± 0.0045
$a = 3$	*2.019* ± 0.1079	*1.895* ± 0.0133	*1.727* ± 0.0090	*1.613* ± 0.0070	*1.511* ± 0.0064	*1.402* ± 0.0044
$a = 4$	**2.537** ± 0.1657	**3.455** ± 0.0597	**3.319** ± 0.0347	**2.676** ± 0.0445	**2.146** ± 0.0346	**1.607** ± 0.0191
$a = 5$	**2.731** ± 0.1703	**4.452** ± 0.1441	**7.411** ± 0.1724	**10.621**	**13.843**	**18.819**
$a = 6$	2.602 ± 0.1278	3.807 ± 0.1003	5.588 ± 0.1074	7.438 ± 0.0988	9.310 ± 0.1649	12.203
$a = 7$	2.421 ± 0.0903	3.207 ± 0.0683	*4.094* ± 0.0566	*4.877* ± 0.0509	*5.591* ± 0.0823	*6.588* ± 0.0892
$a = 8$	2.225 ± 0.0621	2.675 ± 0.0446	2.963 ± 0.0264	3.124 ± 0.0227	3.215 ± 0.0336	3.321 ± 0.0343
$a = 9$	2.037 ± 0.0430	2.252 ± 0.0300	2.247 ± 0.0146	2.183 ± 0.0118	2.085 ± 0.0165	*1.943* ± 0.0139
Student-t_2 parent ($\xi = 1/2 = 0.5, \rho = -1$)						
$RMSE_{00}$	0.203 ± 0.5207	0.153 ± 0.1433	0.108 ± 0.1376	0.083 ± 0.1295	0.065 ± 0.1209	0.047 ± 0.1093
CH	0.980 ± 0.1394	**1.418** ± 0.0172	**1.706** ± 0.0152	**1.944** ± 0.0179	**2.227** ± 0.0179	**2.641** ± 0.0218
$a = 1$	*1.014* ± 0.1464	*1.459* ± 0.0161	*1.707* ± 0.0147	1.920 ± 0.0146	2.177 ± 0.0183	2.553 ± 0.0209
$a = 2$	**1.014** ± 0.1486	*1.516* ± 0.0218	*1.849* ± 0.0182	*2.107* ± 0.0179	*2.419* ± 0.0215	*2.870* ± 0.0237
$a = 3$	*1.012* ± 0.1499	*1.582* ± 0.0313	*2.083* ± 0.0246	*2.420* ± 0.0229	*2.809* ± 0.0285	*3.380* ± 0.0305
$a = 4$	*1.006* ± 0.1490	**1.617** ± 0.0422	**2.487** ± 0.0525	**3.165** ± 0.0497	**3.834** ± 0.0463	**4.699** ± 0.0390
$a = 5$	0.984 ± 0.1415	1.445 ± 0.0314	1.726 ± 0.0265	1.676 ± 0.0202	1.514 ± 0.0170	1.249 ± 0.0160
$a = 6$	0.949 ± 0.1287	1.213 ± 0.0159	1.209 ± 0.0098	1.133 ± 0.0068	1.052 ± 0.0062	0.931 ± 0.0054
$a = 7$	0.922 ± 0.1187	1.112 ± 0.0120	1.074 ± 0.0077	0.993 ± 0.0063	0.911 ± 0.0053	0.795 ± 0.0044
$a = 8$	0.896 ± 0.1102	1.033 ± 0.0096	0.964 ± 0.0062	0.873 ± 0.0057	0.787 ± 0.0043	0.670 ± 0.0035
$a = 9$	0.866 ± 0.1024	0.957 ± 0.0080	0.862 ± 0.0051	0.765 ± 0.0050	0.675 ± 0.0035	0.560 ± 0.0028

Table 8: Simulated RMSE of H/ξ (first row) and REFF-indicators of CH and RB_p, $p = a/(10\xi)$, $a = 1(1)9$, (independent on ξ), for Sin-Burr underlying parents with $(\xi, \rho) = (1, -0.25)$, together with 95% confidence intervals

Sin-Burr parents with $(\xi, \rho) = (1, -0.25)$						
n	100	200	500	1000	2000	5000
$RMSE_0(H)$	1.454 ± 1.8487	1.158 ± 0.6162	0.697 ± 0.1761	0.444 ± 0.1931	0.308 ± 0.2025	0.193 ± 0.2000
CH	0.894 ± 0.0887	**1.119** ± 0.0296	**1.158** ± 0.0085	**1.014** ± 0.0052	0.942 ± 0.0052	0.918 ± 0.0074
$a = 1$	0.992 ± 0.1207	1.287 ± 0.0702	1.299 ± 0.0615	1.082 ± 0.0554	0.986 ± 0.0059	0.954 ± 0.0082
$a = 2$	1.146 ± 0.1550	1.509 ± 0.0717	1.463 ± 0.0548	1.156 ± 0.0611	1.031 ± 0.0064	0.988 ± 0.0097
$a = 3$	1.371 ± 0.1936	1.803 ± 0.0745	1.645 ± 0.0684	1.242 ± 0.0675	1.080 ± 0.0074	1.020 ± 0.0103
$a = 4$	**1.671** ± 0.2486	**2.189** ± 0.0829	**1.832** ± 0.0737	**1.337** ± 0.0751	**1.137** ± 0.0087	**1.059** ± 0.0119
$a = 5$	2.059 ± 0.3272	2.673 ± 0.1091	2.046 ± 0.0812	1.441 ± 0.0837	1.200 ± 0.0096	1.111 ± 0.0137
$a = 6$	2.530 ± 0.4313	3.217 ± 0.1438	2.262 ± 0.1011	1.543 ± 0.0921	1.265 ± 0.0116	1.177 ± 0.0156
$a = 7$	3.033 ± 0.5449	3.724 ± 0.1816	2.449 ± 0.1019	**1.622** ± 0.0981	**1.315** ± 0.0131	**1.247** ± 0.0181
$a = 8$	3.469 ± 0.6359	4.158 ± 0.2120	2.555 ± 0.1006	1.650 ± 0.0993	1.328 ± 0.0138	1.285 ± 0.0198
$a = 9$	**3.728** ± 0.6622	**4.278** ± 0.2207	**2.509** ± 0.0931	1.595 ± 0.0917	1.268 ± 0.0125	1.203 ± 0.0172

4 Adaptive PRBMOP EVI–estimation

We now proceed with the description of an algorithm for an heuristic adaptive estimation of ξ. In **Steps 2**, **3** and **4**, we present an algorithm similar to the one provided in Gomes and Pestana (2007b), among others, for the estimation of the second-order parameters β and ρ. Such a computation enables us to attain the validity of condition (2.4) for a large variety of underlying parents. This heuristic algorithm, essentially based on sample path stability, is similar to the ones in Gomes *et al.* (2013c) and Neves *et al.* (2014).

Algorithm 1.

Step 1 *Given an observed sample* $(x_1, \ldots, x_n)$*, compute the observed values of* $\mathrm{H}_0(k) \equiv \mathrm{H}(k)$*, in* (1.6)*,* $1 \le k < n$*.*

Step 2 *Compute for the tuning parameters* $\tau = 0$ *and* $\tau = 1$*, the observed values of the estimators*

$$\hat\rho_\tau(k) \equiv \hat\rho(k;\tau) := -\left|3\big(\mathrm{V}_n(k;\tau)-1\big)/\big(\mathrm{V}_n(k;\tau)-3\big)\right|, \qquad (4.1)$$

the most simple class of estimators in Fraga Alves *et al.* (2003)*, using also the same notation* $\hat\rho_\tau(k)$ *for those estimates, where, with* $M_n^{(j)}(k) := \frac{1}{k}\sum_{i=1}^k \{\ln X_{n-i+1:n} - \ln X_{n-k:n}\}^j$*,* $j = 1, 2, 3$*, and the notation* $a^{b\tau} = b \ln a$ *whenever* $\tau = 0$*,*

$$V_n(k;\tau) := \frac{\Big(M_n^{(1)}(k)\Big)^{\tau} - \Big(M_n^{(2)}(k)/2\Big)^{\tau/2}}{\Big(M_n^{(2)}(k)/2\Big)^{\tau/2} - \Big(M_n^{(3)}(k)/6\Big)^{\tau/3}}, \quad \tau \in \mathbb{R}.$$

Step 3 *Consider* $\{\hat\rho_\tau(k)\}_{k\in\mathcal{K}}$*, for large* k*, say* $k \in \mathcal{K} = (\lfloor n^{0.995}\rfloor,\ \lfloor n^{0.999}\rfloor)$*, with* $\lfloor x\rfloor$ *denoting the integer part of* x*, and compute their median, denoted* χ_τ*. Next choose the tuning parameter* $\tau^* := \arg\min_\tau \sum_{k\in\mathcal{K}} \left(\hat\rho_\tau(k) - \chi_\tau\right)^2$*.*

Step 4 *Work then with* $(\hat\rho, \hat\beta) \equiv (\hat\rho_{\tau^*}, \hat\beta_{\tau^*}) := (\hat\rho_{\tau^*}(k_1), \hat\beta_{\hat\rho_{\tau^*}}(k_1))$*, where,*

$$k_1 = \lfloor n^{0.995}\rfloor, \qquad (4.2)$$

and

$$\hat\beta_\rho(k) \equiv \hat\beta(k;\rho) := \frac{\left(\frac{k}{n}\right)^{\rho}\left\{\left(\frac{1}{k}\sum_{i=1}^{k}\left(\frac{i}{k}\right)^{-\rho}\right)\left(\frac{1}{k}\sum_{i=1}^{k}U_i\right) - \left(\frac{1}{k}\sum_{i=1}^{k}\left(\frac{i}{k}\right)^{-\rho} U_i\right)\right\}}{\left(\frac{1}{k}\sum_{i=1}^{k}\left(\frac{i}{k}\right)^{-\rho}\right)\left(\frac{1}{k}\sum_{i=1}^{k}\left(\frac{i}{k}\right)^{-\rho} U_i\right) - \left(\frac{1}{k}\sum_{i=1}^{k}\left(\frac{i}{k}\right)^{-2\rho} U_i\right)}. \qquad (4.3)$$

being $U_i := i\,\{\ln X_{n-i+1:n} - \ln X_{n-i:n}\}$ *and* $\hat\rho_\tau(k)$ *given in* (4.1)*.*

Step 5 *Compute*

$$\hat{k}_{0|\mathrm{H}} := \min\Big(n-1, \Big\lfloor \Big(\frac{(1-\hat{\rho})^2 n^{-2\hat{\rho}}}{-2\hat{\rho}\hat{\beta}^2}\Big)^{1/(1-2\hat{\rho})} \Big\rfloor + 1\Big), \tag{4.4}$$

the estimate of $k_{0|\mathrm{H}} := \arg\min \mathrm{MSE}(\mathrm{H}(k))$ *in* Hall (1982)*, and the adaptive* EVI-*estimate* $\mathrm{H}_{00} = \mathrm{H}_0(\hat{k}_{0|\mathrm{H}})$.

Step 6 *For* $a = 0, 1, 2, \ldots, 9$ *and* $p = a/(10\ \mathrm{H}_{00})$, *with* H_{00} *the estimate obtained in* **Step 5***, compute the observed values of* $T_a(k) := \mathrm{RB}_p(k)$, $1 \le k < n-1$.

Step 7 *For any* $a = 0(1)9$, *obtain* $j_{0,a}$*, the minimum positive integer value of* j*, such that the rounded values, to* j *decimal places, of the estimates* $\mathrm{T}_a(k)$, $1 \le k < n-1$, *are distinct. Define* $r_k^{(\mathrm{T}_a)}(j_{0,a}) = round(\mathrm{T}_a(k), j_{0,a})$, $k = 1, 2, \ldots, n-1$, *the rounded values of* $\mathrm{T}_a(k)$ *to* $j_{0,a}$ *decimal places.*

Step 8 *Consider the sets of* k *values associated with equal consecutive values of* $r_k^{(\mathrm{T}_a)}(j_{0,a})$, *obtained in Step* **7***. Set* $k_{min}^{(\mathrm{T}_a)}$ *and* $k_{max}^{(\mathrm{T}_a)}$ *the minimum and maximum values, respectively, of the set with the largest range. The largest run size is then* $s_{\mathrm{T}_a} := k_{max}^{(\mathrm{T}_a)} - k_{min}^{(\mathrm{T}_a)} + 1$. *If there are ties take the minimum a-value, among those ties.*

Step 9 *Consider all estimates,* $\mathrm{T}_a(k)$, $k_{min} \equiv k_{min}^{(\mathrm{T}_a)} \le k \le k_{max}^{(\mathrm{T}_a)} \equiv k_{max}$, *now with two extra decimal places, i.e. compute* $\mathrm{T}_a(k) = r_k^{(\mathrm{T}_a)}(j_{0,a}+2)$. *Obtain the mode of* $\mathrm{T}_a(k)$ *and denote* $\mathcal{K}_{\mathrm{T}_a}$ *the set of* k*-values associated with this mode.*

Step 10 *Take* $\hat{k}_{\mathrm{T}_a}$ *as the maximum value of* $\mathcal{K}_{\mathrm{T}_a}$, *and consider the adaptive estimate* $\mathrm{T}_a(\hat{k}_{\mathrm{T}_a})$.

Step 11 *The final estimate, denoted* $\hat{\xi}_1$, *is the value of* T_a *that corresponds to the maximum run size* s_{T_a} *computed in Step* **8**.

Remark 4.1. *For asymptotic and finite sample details on the estimators of* ρ *in* (4.1)*, see* Fraga Alves *et al.* (2003)*. The class of* ρ*-estimators in* (4.1) *has been first parameterised in a tuning parameter* $\tau > 0$*, but more generally* τ *can be considered as a real number* (Caeiro and Gomes, 2006)*. Interesting alternative* ρ*-estimators can be found in* Goegebeur *et al.* (2008; 2010), Ciuperca and Mercadier (2010), Deme *et al.* (2013) *and* Caeiro and Gomes (2014a)*. The estimator of* β *in* (4.3) *has been introduced in* Gomes and Martins (2002)*, where conditions that enable its asymptotic normality have been set, whenever* ρ *is estimated at a level* k_1 *of a larger order than the level* k *used for the estimation of* β*. Details on the asymptotic distribution of* $\hat{\beta}_{\hat{\rho}(k;\tau)}(k)$*, in* (4.3)*, can be found in* Gomes *et al.* (2008*b*) *and* Caeiro *et al.* (2009)*. We can find alternative* β*-estimators in* Caeiro and Gomes (2006)*, and more recently in* Gomes *et al.* (2010) *and* Caeiro and Gomes (2012)*.*

Remark 4.2. Step 2 *and* **Step 3** *of* **Algorithm 1** *lead in almost all situations to the tuning parameter* $\tau^* = 0$ *whenever* $|\rho| \le 1$ *and* $\tau^* = 1$*, otherwise. Such an educated guess* *can usually provide better results than a possibly 'noisy' estimation of τ, and it is highly recommended in practice. For details on this and similar algorithms for the ρ-estimation, see* Gomes and Pestana (2007*a*).

Remark 4.3 (Adequate choice of k_1 for the ρ-estimation)**.** *As stated in* Caeiro and Gomes (2008) *the ideal situation would perhaps be the choice of an 'optimal' level $k_1^{\rm opt}$ for the estimation of ρ, in the sense of a $k_1^{\rm opt}$ that enables us to guarantee the asymptotic normality of the ρ-estimators with a non-null asymptotic bias. That level $k_1^{\rm opt}$ would automatically lead to condition* (2.4)*, a condition needed for an adequate* EVI-*estimation. We stress that in practice, such a $k_1^{\rm opt}$ has only a 'limited' interest, at the current state-of-the-art. It is however of a high theoretical interest. If we consider a level k_1 of the order of $n^{1-\epsilon}$, for some small $\epsilon > 0$, we can also guarantee* (2.4) *for a large class of models* (*see* Caeiro *et al.*, 2009*, among others*)*. This is the reason why, such as done in* Caeiro *et al.* (2005)*,* Gomes and Pestana (2007*a*; 2007*b*) *and* Gomes *et al.* (2007*a*; 2008*b*)*, the pioneering papers in* MVRB-*estimation, we advise in practice, as a compromise between theoretical and practical considerations, the use of any intermediate level like $k_1 = \lfloor n^{1-\epsilon}\rfloor$ for some $\epsilon > 0$, small. The choice of ϵ is not crucial, and it is sensible to consider the level k_1 in* (4.2)*. Further considerations on the choice of k_1 can be found in* Caeiro *et al.* (2009).

Remark 4.4. *Alternatively to the choice $\hat{k}_{0|{\rm H}}$, in* (4.4)*, we can also use the graphical tool developed in de* Sousa and Michailidis (2004) *and more generally in* Beirlant *et al.* (2011)*, for a sensible estimation of k through the* Hill *estimator in* (1.6) *or any other* EVI-*estimator.*

Alternatively, in order to simplify the computational procedure of identification of 'runs' and to provide a better visualization tool, we have further implemented an algorithm similar to the one already used in Gomes *et al.* (2013b) for a PORT MVRB EVI-estimation, replacing steps from **7** up to **11** by:

Algorithm 2.

Step 7' *In order to detect the sign of the trend in the* EVI-*estimates, obtain the sign of* $s_a := {\rm T}_a(\lfloor n^{0.95}\rfloor) - {\rm T}_a(\lfloor n^{0.05}\rfloor)$.

Step 8' *For $k > \hat{k}_{0|{\rm H}}/2$, with $\hat{k}_{0|{\rm H}}$ given in* (4.4)*, modify the patterns of the estimates, in the following simple way: If $s_a > 0$, consider*

$$\widetilde{\rm T}_a(k) := \begin{cases} {\rm T}_a(k) & \textit{if} \quad k \le \hat{k}_{0|{\rm H}}/2 \\ \max\big(\widetilde{\rm T}_a(k-1), {\rm T}_a(k)\big) & \textit{if} \quad k > \hat{k}_{0|{\rm H}}/2. \end{cases} \tag{4.5}$$

If $s_a \le 0$, replace in (4.5) *the* max-*operator by the* min-*operator.*

Step 9' *For each $a = 0(1)9$, compute the estimate that provides the "largest run" of estimates* (*equal consecutive estimates*)*, say $\widetilde{\rm T}_a(k)$, $k_{a,1} \le k \le k_{a,2}$, with a size $m_a = k_{a,2} - k_{a,1} + 1$* $\big($*this means that* $\widetilde{\rm T}_a(k_{a,1}) = \widetilde{\rm T}_a(k_{a,1}+1) = \cdots = \widetilde{\rm T}_a(k_{a,2})\big)$.

Step 10' *Choose the maximum value of a associated with $\arg\max_a m_a$, denoted by a^{**}.*

Step 11' *Consider $k^{**} = k_{a^{**},2}$, and the adaptive* EVI*-estimate, $\hat{\xi}_2 := \widetilde{\mathrm{T}}_{a^{**}}(k^{**})$.*

Remark 4.5. *If there are negative elements in the sample, the sample size should be replaced everywhere in the algorithms by n^+, the number of positive elements in the sample.*

5 Simulated samples and case-studies

We have considered two randomly simulated samples of size $n = 500$ from a $\mathrm{Burr}_{0.25,-0.75}$ and a Student t_4 parent ($\gamma = 1/4 = 0.25, \rho = -2/4 = -0.5$). We have further considered an application of the estimators under study to SECURA data, related to the 371 automobile claim amounts exceeding 1,200,000 Euro over the period 1988–2001, gathered from several European insurance companies co-operating with the re-insurer Secura Belgian Re (Beirlant *et al.* 2004). In Figures 5, 6 and 7, respetively associated with the three above mentioned samples, we consider on the left the sample paths of $\mathrm{RB}_p(k)$, $p = a/H_{00}$, $a = 0(1)9$, and on the right the sample path associated with the selected value of a and the final adaptive EVI-estimate, $\hat{\xi}$, obtained in **Step 11** of **Algorithm 1**. Those estimates were respectively given by $\hat{\xi}_1 = 0.219$ ($\xi = 0.25$, $\mathrm{H}_{00} = 0.244$), $\hat{\xi}_1 = 0.232$ ($\xi = 0.25$, $\mathrm{H}_{00} = 0.282$) and $\hat{\xi}_1 = 0.219$ ($\mathrm{H}_{00} = 0.305$). In Tables 9, 10 and 11, we present for the aforementioned samples and different values of a, the values of k_{min}, k_{max} and s_{T_a}, in **Step 8** of **Algorithm 1**. The number of decimal places, $j_{0,a}$, in **Step 7** of **Algorithm 1**, was always equal to one.

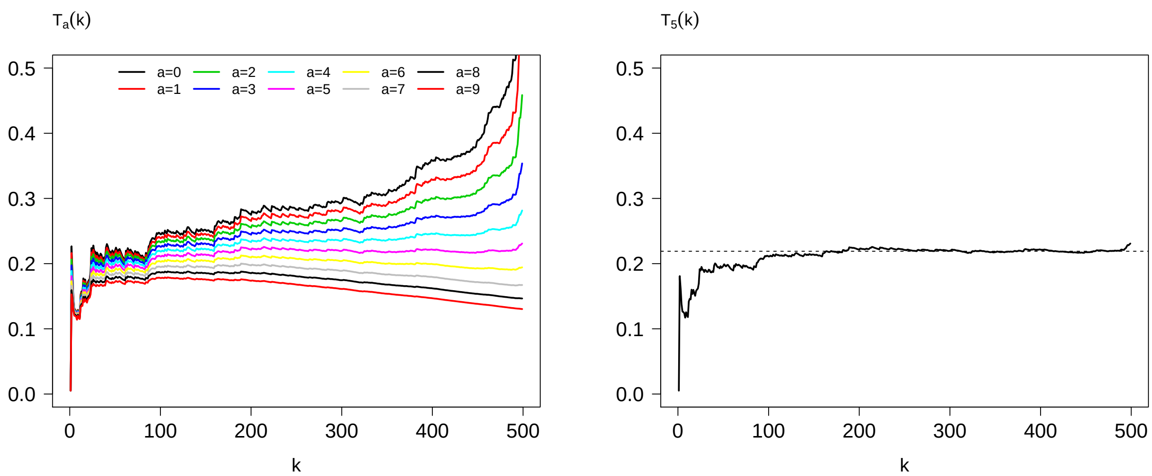

Figure 5: Sample paths of $\mathrm{RB}_p(k)$, $p = a/H_{00}$, $a = 0(1)9$ (*left*), and sample path associated with the selected value of a, together with the final adaptive EVI-estimate (*right*), for the $\mathrm{Burr}_{0.25,-0.75}$ sample

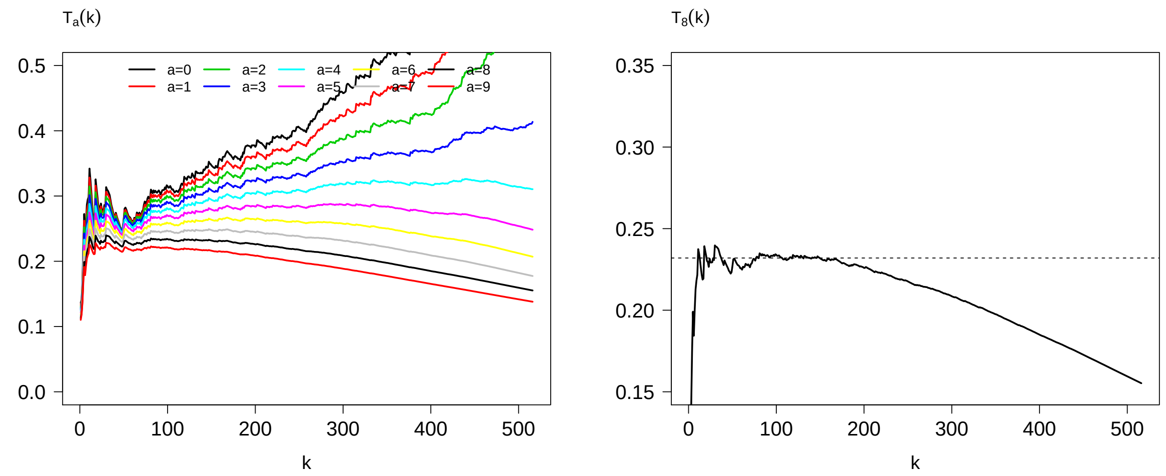

Figure 6: Sample paths of $\text{RB}_p(k)$, $p = a/H_{00}$, $a = 0(1)9$ (*left*), and sample path associated with the selected value of a, together with the final adaptive EVI-estimate (*right*), for the Student t_4 sample

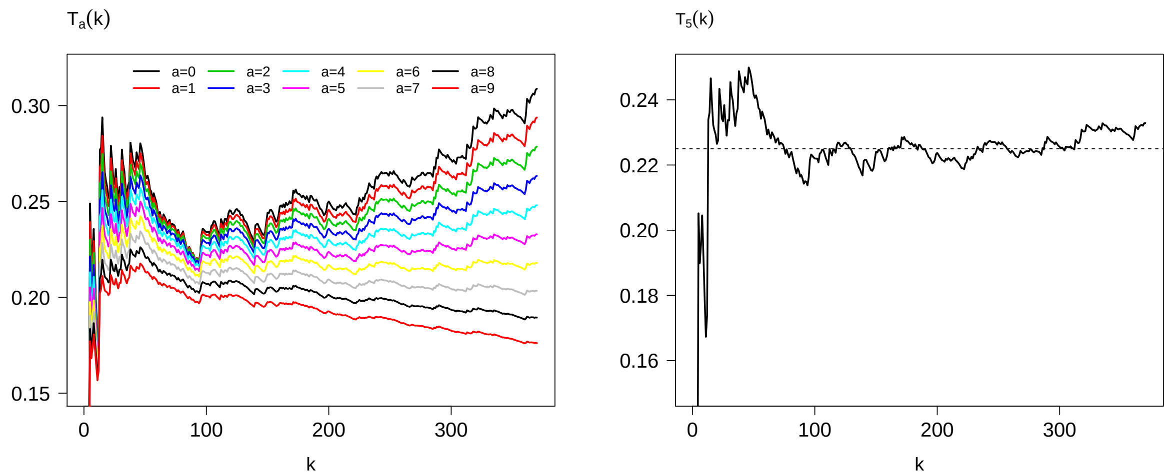

Figure 7: Sample paths of $\text{RB}_p(k)$, $p = a/H_{00}$, $a = 0(1)9$ (*left*), and sample path associated with the selected value of a, together with the final adaptive EVI-estimate (*right*), for SECURA data

The application of **Algorithm 2** to aforementioned samples, but with the inclusion of the value $a = 10$, for curiosity, led us to the results presented in Table 12. For the $\text{Burr}_{0.25,-0.75}$ sample, we have been led to $k^{**} = 499$, $a^{**} = 9$ and $\hat{\xi}_2 = 0.179$. For the Student-t_4 sample ($\xi = 0.25$), we have been led to $k^{**} = 504$, $a^{**} = 9$ and $\hat{\xi}_2 = 0.252$. For the SECURA data, we have been led to $k^{**} = 370$, $a^{**} = 9$ and $\hat{\xi}_2 = 0.218$.

Burr$_{0.25,-0.75}$ sample										
a	0	1	2	3	4	5	6	7	8	9
k_{min}	160	160	182	15	15	15	20	20	21	23
k_{max}	389	449	484	212	460	499	499	499	473	377
s_{T_a}	230	290	303	198	446	**485**	480	480	453	355

Table 9: Values of k_{min}, k_{max} and s_{T_a}, in **Step 8** of **Algorithm 1**, for the Burr$_{0.25,-0.75}$ sample

Student t_4 sample										
a	0	1	2	3	4	5	6	7	8	9
k_{min}	158	186	242	50	50	71	77	31	4	4
k_{max}	285	331	420	291	516	511	351	516	516	465
s_{T_a}	128	146	179	242	467	441	275	**486**	513	462

Table 10: Values of k_{min}, k_{max} and s_{T_a}, in **Step 8** of **Algorithm 1**, for the Student t_4 sample

SECURA data										
a	0	1	2	3	4	5	6	7	8	9
k_{min}	228	232	55	54	50	5	5	5	5	5
k_{max}	370	370	240	316	370	370	370	370	370	370
s_{T_a}	143	139	186	263	321	**366**	366	366	366	366

Table 11: Values of k_{min}, k_{max} and s_{T_a}, in **Step 8** of **Algorithm 1**, for the SECURA data

a	0	1	2	3	4	5	6	7	8	9	10
Burr$_{0.25,-0.75}$ sample											
m_a	70	56	70	71	157	281	310	310	**391**	**391**	23
Student t_4 sample											
m_a	104	104	103	103	104	155	262	289	**431**	**431**	48
SECURA data											
m_a	297	272	276	**324**	**324**	**324**	**324**	**324**	**324**	**324**	104

Table 12: Values of m_a, in **Step 9'** of **Algorithm 2**, for the the three data sets

6 Concluding remarks

- For the simulated samples, we know the true value of ξ, and we can easily assess the reliability of the estimates provided by the algorithms in Section 4, immediately coming to the conclusion that, as expected, the PRBMOP methodology and both algorithms provide a quite reliable EVI-estimation for the Student sample, but under estimate the EVI for the Burr sample.
- It is well-known that the adaptive Hill EVI-estimation usually leads to an over-estimation of the EVI. The adaptive PRBMOP seem to be closer to the target value. However, and again for the Burr sample, even the adaptive Hill EVI-estimate is below the target value.
- Regarding the application to the SECURA real data set, the sample path associated with $a = 6$ is quite stable up to $k = n - 1$, but this had already happened with the application of reduced-bias EVI-estimation to the data (see Gomes *et al.*, 2007a, and Caeiro and Gomes, 2014b, among others).
- These case studies claim obviously for a simulation comparative study of the algorithms. This is however a topic beyond the scope of this paper.

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