Multifractals Tied to Extensions of Panjer's Iterative Procedures

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Abstract. Binomial, Poisson and Negative Binomial are the basic count models whose probability mass function satisfies a simple recursive relation. This has been used by Panjer [8] to iteratively compute the density of randomly stopped sums, namely in the context of making provision for claims in insurance. Pestana and Velosa [9] used probability generating functions of randomly stopped sums whose subordinator is a member of Panjer's family to discuss more involved recursive relations, leading to refinements of infinite divisibility and self-decomposability in count models. After discussing multifractal measures generated by the geometric and by the Poisson laws, as guidelines to define multifractals generated by general count measures with denumerably infinite support, the complex recursivity of Pestana and Velosa [9] classes of randomly stopped sums is exhibited, hinting that randomness can bring in deeper meaning to multifractality, that, as Mandelbrot argues, is a vague concept that remains without an agreed mathematical definition. A simple random extension of binomial and multinomial multifractals, considering that each multiplier of a cascade is the outcome of some stochastic count model, is also discussed in depth.
Keywords: Count models, probability generating functions, multifractal measures, random multipliers.

1 Introduction

Simple introductory texts on multifractals, v.g. Ervertsz and Mandelbrot [2], use binary splitting and multiplicative cascades generating binomial measures as a straightforward and intuitive example. Mandelbrot [6] (p. 83–84 and 89–91) also uses the binomial measure to exhibit the complications that arise when self-similarity and self-affinity are applied to measures rather than to sets, restricting the probability p to take values in the interval $[0, \frac{1}{2}]$.[1]

Ervertsz and Mandelbrot [2], under the heading "Beyond Multinomial Measure" (p. 937–938), briefly mention multifractal measures generated by a

[1] In fact, for $p = 1/2$ the procedure leads to the uniform measure in (0,1), a straightforward consequence of the binary representation of real numbers in the

countably infinite support probability mass function. In Section 2 we detail the construction of such measures starting either from a geometric distribution or from a Poisson distribution.

On the other hand, Mandelbrot [6] (p. 14) states that "*the terms* fractal *and* multifractal *remain without an agreed mathematical definition*", although the fact that self-similarity, self-affinity and the ensuing mild or wild variability play an essential role in their theory. Binomial, negative binomial and Poisson count measures probability mass functions satisfy some sort of self-similarity, in the sense that $p_{n+1} = (a + \frac{b}{n+1})\, p_n,\ \ n = 0, 1, \ldots,$ a recursive expression that has been successfully used by Panjer [8] to iteratively compute densities of randomly stopped sums whose subordinator is one of the above mentioned count models, and our first choice has been to exploit implications and extensions of this extended kind of self-similarity. Observe that the simplest cases are $N \frown Poisson(b)$ for $a = 0$ and $N \frown Geometric(1-a)$ for $b = 0$, leading to simple forms of extended self-similarity, and that for this reason are the topic of Section 2.

In Section 3 we briefly mention the basic count models whose probability mass function satisfies some sort of mitigated self-similarity, extending Panjer's [8] class, and we use probability generating functions investigated in [9] to discuss multiple self-similarity, extending results in [1].

In Section 4 we discuss other pathways to multifractality, extending the construction of binomial/multinomial measures to accommodate the case of countably infinite support discrete generators, using randomness as a device to operate this alternative extension of multifractality.

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interval (0,1)

$$\sum_{k=1}^{\infty} \frac{X_k}{2^k}\,, \quad X_k \frown Bernoulli\left(\frac{1}{2}\right),\ \text{independent}$$

and of Borel's pioneering construction of continuous probability. As Mandelbrot [6] (p. 45) states, "*The definition of multifractality used in this book and almost everywhere else in the literature* [...] *is limited to singular non-negative measures constructed using continuous non-decreasing generators.*"

Feller [3] (p. 141–142), on the same issue, denoting F_p the distribution of

$$Y_p = \sum_{k=1}^{\infty} \frac{X_k}{2^k}, \quad \text{where} \quad X_k \frown Bernoulli(p),\ \text{independent},$$

observes that $Y_{\frac{1}{2}}$ is the standard uniform random variable, and that Y_p is a singular random variable for each $p \neq 1/2$. He further comments that "*A little reflection* [...] *reveals that a decision* [on the fairness of a coin] *after finitely many trials is due to the fact that* F_p *is singular with respect to* $F_{\frac{1}{2}}$ *(provided* $p \neq 1/2$*). The existence of singular distributions is therefore essential to statistical practice.*"

2 Geometric and Poisson generated measures

Let $X \frown Exponential\,(1/\delta)$, and define the countably discrete random variable

$$N = \begin{cases} k = 0, 1, \ldots \\ p_k = \mathbb{P}[N = k] = \mathbb{P}[k \le X < k+1] = (1 - \mathrm{e}^{-\delta})(\mathrm{e}^{-\delta})^k \end{cases}$$

i.e., $N = \lfloor X \rfloor \frown Geometric(1 - \mathrm{e}^{-\delta})$ ($\lfloor x \rfloor$ denotes the integer part of x).

On the other hand, from the probability integral transform,

$$1 - \mathrm{e}^{-\delta X} \stackrel{\mathrm{d}}{=} \mathrm{e}^{-\delta X} \stackrel{\mathrm{d}}{=} U \frown Uniform[0, 1].$$

Thus, starting from the interval [0,1], in the first step [0,1] is splitted in countably many subintervals,

$$[0, 1] = \bigcup_{k=0}^{\infty} \left(\mathrm{e}^{-(k+1)}, \mathrm{e}^{-k} \right] = \bigcup_{k=0}^{\infty} \mathcal{I}_{k(1)}$$

to which we attach probabilities $m_k = (1 - \mathrm{e}^{-\delta})(\mathrm{e}^{-\delta})^k,\ k = 0, 1, \ldots$.

In step 2, each $\mathcal{I}_k$ is treated as a reduction of the original [0,1] interval, i.e., using self-explaining standard notations for the translation and scaling of sets,

$$\mathcal{I}_{k(1)} = \bigcup_{j=0}^{\infty} \left\{ \mathrm{e}^{-(k+1)} + \left(\mathrm{e}^{-k} - \mathrm{e}^{-(k+1)} \right) \left(\mathrm{e}^{-(j+1)}, \mathrm{e}^{-j} \right] \right\} = \bigcup_{j=0}^{\infty} \mathcal{I}_{j_k(2)},$$

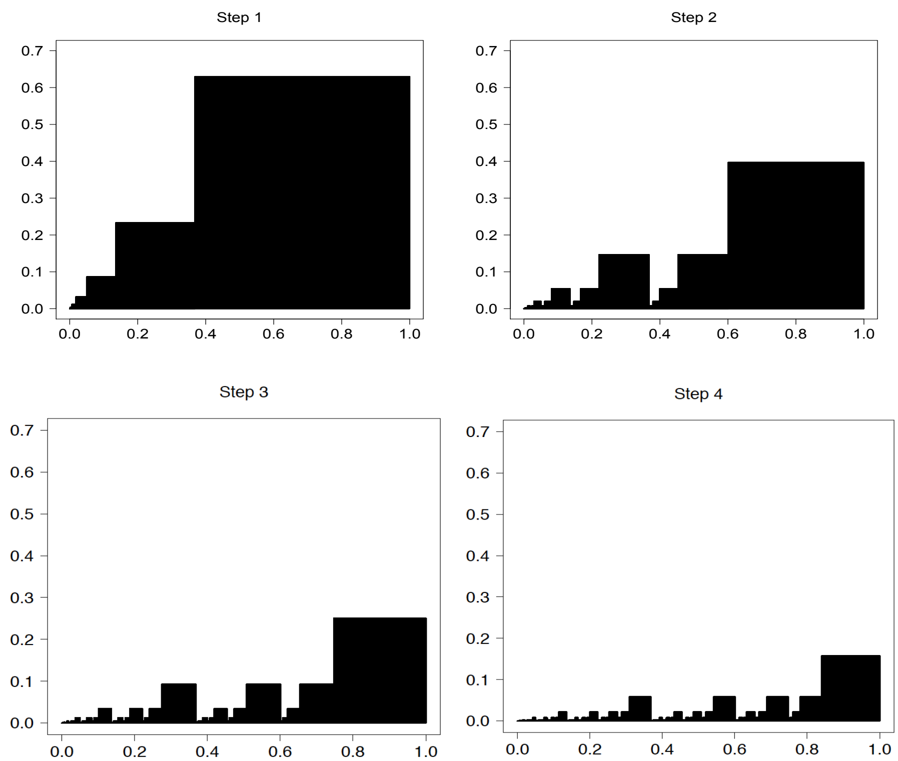
so that $[0, 1] = \bigcup_{k=0}^{\infty} \Big(\bigcup_{j=0}^{\infty} \mathcal{I}_{j_k(2)} \Big)$, and to each interval $\mathcal{I}_{j_k(2)}$ we attach the probability $m_k m_j$.

In step 3, the subintervals $\mathcal{I}_{j_k(2)}$ are treated as the $\mathcal{I}_k$ intervals in step 2, and similarly in the countably infinite steps that follow to build up a multifractal generated by a Geometric initial measure. Notations soon become cumbersome, but the principles used in the build up of the multiplicative cascade $m_{k_1} m_{k_2} \cdots$ are simple. In Figure 1 we show the initial four steps of the construction of the geometric measure with the parameter $1 - e^{-1}$.

The procedure described above is intuitive in view of the geometric discretization of the exponential measure, but it can in fact be used with an initial generator whose support is $\mathbb{N}$, namely $N \frown Poisson(\lambda)$.

$N_G \frown Geometric(p)$ may be looked at as the "unit" of the class of $NegativeBinomial(r, p)$ random variables, in the same sense that $N_B \frown Bernoulli(p)$ is the unit of $Binomial(n, p)$ random variables. On the other hand the sum of independent Poisson random variables is Poisson, and hence we may consider that $N_P \frown Poisson(1)$ is the unit of

Fig. 1. Construction of the geometric measure with parameter $1-a=0.63$ (i.e. $\delta=1$) — the initial four steps



the class of $Poisson(\lambda)$ random variables. Observe also that the Poisson is a yardstick in the perspective of dispersion, since its dispersion index $\mathrm{Var}[N_P]/\mathbb{E}[N_P]=1$, while $Binomial(n,p)$ random variables are underdispersed and $NegativeBinomal(r,p)$ random variables are overdispersed.

Observe also that Binomials, Poissons and NegativeBinomials are the only discrete classes of natural exponential families whose variance is at most a quadratic function of the mean value (Morris [7]), who writes "*Much theory is unified for these* [...] *natural exponential families by appeal to their quadratic variance property, including* [...] *large deviations*", one of the tools routinely used to investigate dimensionality issues in multifractals. Without pursuing the matter further herein, we remark that a differential simile of Panjer's difference iteration is $f'/f=a+b/x$, where f denotes the density function of a positive absolutely continuous random variable, and hence f must be the density of a $Gamma(b+1,-\frac{1}{a})$ random variable, for $b>-1$ and $a<0$. The gamma random variables are the sole Morris continuous random variables with positive support.

3 Extended self-similarity of basic count models

Let

$$N = \begin{cases} k = 0, 1, 2, \ldots \\ \\ p_k = \mathbb{P}[N = k] \end{cases}$$

be a count random variable. Panjer [8] made an important breakthrough in insurance theory by showing that the only non-degenerate random variables whose probability mass function satisfies the recurrence relation

$$p_{n+1} = p_n \left(\alpha + \frac{\beta}{n+1} \right), \ n = 0, 1, \ldots$$

are the Poissons, the Binomals and the Negative Binomials, and that the above recurrence relation can be used to deduce an iterative procedure to compute the density of randomly stopped sums

$$\sum_{k=0}^{N} X_k, \ X_k \text{ independent random variables, independent of } N,$$

often used as models for aggregate claims, cf. [5] or [10]. Further generalizations may be constructed relaxing the iterative expression to hold for $n \geq k_0$, see Hess *et al.* [4] construction of what they call basic count models.

A further generalization can be developed as follows:

Consider discrete random variables $N_{\alpha,\,\beta,\,\gamma}$ whose probability mass functions (p.m.f.) $\left\{p_n = f_{N_{\alpha,\,\beta,\,\gamma}}(n)\right\}_{n\in\mathbb{N}}$ satisfy the relation

$$\frac{f_{N_{\alpha,\,\beta,\,\gamma}}(n+1)}{f_{N_{\alpha,\,\beta,\,\gamma}}(n)} = \alpha + \beta\, \frac{\mathbb{E}(U_0^n)}{\mathbb{E}(U_\gamma^n)} = \alpha + \frac{\beta}{\sum_{k=0}^{n} \gamma^k}\,, \ \alpha,\, \beta \in \ \mathbb{R}, \ n = 0, 1, \ldots$$

where $U_\gamma \frown Uniform(\gamma,\, 1), \ \ \gamma \in (-1,\, 1)$. As

$$\mathbb{E}(U_\gamma^n) = \frac{1}{n+1}\, \frac{1-\gamma^{n+1}}{1-\gamma} \underset{\gamma \to 1}{\longrightarrow} 1,$$

Panjer's class corresponds to the degenerate limit case, letting $\gamma \longrightarrow 1$ so that $U_\gamma \underset{\gamma\to 1}{\longrightarrow} U_1$, the degenerate random variable with unit mass at 1.

The probability generating function $\mathcal{G}_{\alpha,\,\beta,\,\gamma}(s) = \sum_{n=0}^{\infty} f_{N_{\alpha,\,\beta,\,\gamma}}(n)\, s^n$ must then satisfy

$$\mathcal{G}_{\alpha,\,\beta,\,\gamma}(s) = \mathcal{G}_{\alpha,\,\beta,\,\gamma}(\gamma^{n+1} s) \prod_{k=0}^{n} \frac{1 - \alpha\gamma^{k+1} s}{1 - [\alpha + \beta(1-\gamma)]\gamma^k s}\,.$$

Observing that

$$\frac{\mathcal{G}_{\alpha,\,\beta,\,\gamma}(s)}{\mathcal{G}_{\alpha,\,\beta,\,\gamma}(1)} = \frac{\mathcal{G}_{\alpha,\,\beta,\,\gamma}(\gamma^{n+1}s)}{\mathcal{G}_{\alpha,\,\beta,\,\gamma}(\gamma^{n+1})} \prod_{k=0}^{n} \frac{\frac{1-\alpha\gamma^{k+1}s}{1-[\alpha+\beta(1-\gamma)]\gamma^k s}}{\frac{1-\alpha\gamma^{k+1}}{1-[\alpha+\beta(1-\gamma)]\gamma^k}},$$

and letting $n\to\infty$,

$$\mathcal{G}_{\alpha,\,\beta,\,\gamma}(s) = \prod_{k=0}^{\infty} \frac{1-\alpha\gamma^{k+1}s}{1-\alpha\gamma^{k+1}}\ \frac{1-[\alpha+\beta(1-\gamma)]\gamma^k}{1-[\alpha+\beta(1-\gamma)]\gamma^k s}\,. \tag{1}$$

If $\gamma\in[0,1)$, $\alpha<0$ and $\beta\in\left\{-\frac{\alpha}{1-\gamma},\, \frac{1-\alpha}{1-\gamma}\right\}$, we recognize in (1) the probability generating function of an infinite sum of independent random variables, the k-th summand being the result of randomly adding 1, with probability $\alpha\gamma^{k+1}/\big(\alpha\gamma^{k+1}-1\big)$, to an independent *Geometric*$(1-[\alpha+\beta(1-\gamma)]\gamma^k)$ random variable. Each summand exhibits its own scale of extended self-similarity, a characteristic feature observed, in what concerns self-similarity and self-affinity, in strict sense (in Madelbrot's perspective) multifractals.

The limiting case $\gamma=1$ may be approached as follows: observing that

$$\frac{\mathcal{G}_{\alpha,\,\beta,\,\gamma}(s)-\mathcal{G}_{\alpha,\,\beta,\,\gamma}(\gamma s)}{\alpha s[\mathcal{G}_{\alpha,\,\beta,\,\gamma}(s)-\mathcal{G}_{\alpha,\,\beta,\,\gamma}(\gamma s)]+(1-\gamma)s[\beta\mathcal{G}_{\alpha,\,\beta,\,\gamma}(s)+\alpha\mathcal{G}_{\alpha,\,\beta,\,\gamma}(\gamma s)]}=1,$$

dividing the numerator and the denominator by $(1-\gamma)s$ and letting $\gamma\to 1$, we get

$$\frac{\mathcal{G}'_{\alpha,\,\beta,\,1}(s)}{\alpha s\mathcal{G}'_{\alpha,\,\beta,\,1}(s)+\beta\mathcal{G}_{\alpha,\,\beta,\,1}(s)+\alpha\mathcal{G}_{\alpha,\,\beta,\,1}(s)}=1 \Longleftrightarrow \frac{\mathcal{G}'_{\alpha,\,\beta,\,1}(s)}{\mathcal{G}_{\alpha,\,\beta,\,1}(s)}=\frac{\alpha+\beta}{1-\alpha s},$$

the expression we obtain working out the probability generating function in Panjer's iterative expression

$$p_{\alpha,\,\beta}(n+1)=\left(\alpha+\frac{\beta}{n+1}\right)p_{\alpha,\,\beta}(n),\quad \alpha,\,\beta\in\mathbb{R},\ n=0,1,\ldots\,.$$

So, while Panjer's recurrence relation and Hess *et al.* extension for the basic count models exhibit a single scaling, (1) exhibits multi-scaling as typical of multifractals.

4 A simple generalization of the binomial/multinomial measure

There are many pathways to expand the notion of a multiplicative cascade. One is to consider that each multiplier is the outcome of some stochastic rule. These kind of multiplicative iterative schemes are usually called random multiplicative cascades.

In Section 2 we introduced the geometric and Poisson generated measures. In this section we shall expand differently the notion of random multiplicative cascades by allowing the number of subdivisions that each interval undergoes, at each step of the measure construction, to be determined by the outcome of a discrete random variable N, where $\mathbb{P}[N \geq 2] = 1$. This procedure has some similar aspects with the binomial and multinomial measures. However, at step k, $k = 1, 2, \dots$, the outcome of N will dictate the number of subdivisions that each interval suffers. In this new scenario the multipliers used at each step will also depend on the outcome of N, i.e., $m_i = m_i^{(N)}$.

Starting with the interval [0,1], having uniformly distributed unit mass, the new measure is formally constructed as follows:

Step 1: Generate an observation n_1 from the random variable N. Split the interval [0,1] into the n_1 equally length subintervals

$$[in_1^{-1}, (i+1)n_1^{-1}], \quad i = 0, 1, \dots, n_1 - 1, \tag{2}$$

with uniformly distributed masses $m_i^{(n_1)}$, $i = 0, 1, \dots, n_1 - 1$, respectively;

Step 2: Generate a second observation n_2 from N, independent from n_1. Split each interval in (2) into n_2 equally length subintervals and use the multipliers $m_i^{(n_2)}$, $i = 0, 1, \dots, n_2 - 1$, to uniformly distribute the parent interval's mass by these subintervals. After this step is completed the subintervals formed are $[i(n_1 n_2)^{-1}, (i+1)(n_1 n_2)^{-1}]$, $i = 0, 1, \dots, n_1 n_2 - 1$;

Step *k*: Generate an observation n_k from N, independent from the previous $k-1$ observations of N. Split each interval from the previous step into n_k subintervals of equal length and use the multipliers $m_i^{(n_k)}$, $i = 0, 1, \dots, n_k - 1$, to uniformly distribute the parent interval's mass by these subintervals. The subintervals formed after this step are $[i(n_1 n_2 \dots n_k)^{-1}, (i+1)(n_1 n_2 \dots n_k)^{-1}]$, $i = 0, 1, \dots, n_1 n_2 \dots n_k - 1$.

The new measure μ results from applying the previous procedure infinitely.

An example of a family of multipliers that can be used in this type of measure construction is

$$m_i^{(n)} = \frac{2(i+1)}{n(n+1)} \quad i = 0, 1, \dots, n-1, \tag{3}$$

when $N = n$. (Note that with the multipliers defined in (3) we do not get $m_0 = m_1 = 1/2$ if $N = 2$ is observed.)

In order to illustrate how the measure is obtained we give a simple example. Suppose that the random variable N has support on $\{2, 3\}$ with p.m.f. $\mathbb{P}[N = 2] = 1/4$ and $\mathbb{P}[N = 3] = 3/4$. Let us further assume that we observe for the first two steps of the measure's construction the sequence of divisors

$(N_1, N_2) = (3, 2)$, where N_1 and N_2 are independent replicas of N. Using the multipliers defined in (3), we get

$$m_0^{(2)} = \frac{1}{3} \quad \text{and} \quad m_1^{(2)} = \frac{2}{3}\,,$$

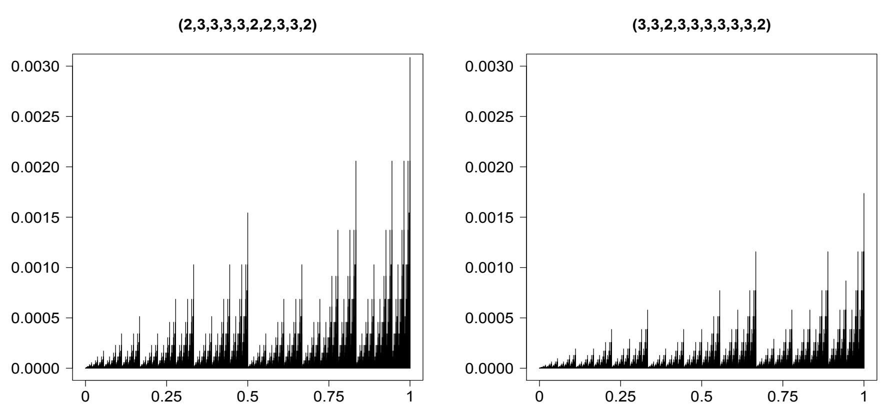
and

$$m_0^{(3)} = \frac{1}{6}, \quad m_1^{(3)} = \frac{1}{3} \quad \text{and} \quad m_2^{(3)} = \frac{1}{2}.$$

At step one we obtain the subintervals $[0, \frac{1}{3}]$, $[\frac{1}{3}, \frac{2}{3}]$ and $[\frac{2}{3}, 1]$, with masses 1/6, 1/3 and 1/2, respectively, and after step two the subintervals $[0, \frac{1}{6}]$, $[\frac{1}{6}, \frac{1}{3}]$, $[\frac{1}{3}, \frac{1}{2}]$, $[\frac{1}{2}, \frac{2}{3}]$, $[\frac{2}{3}, \frac{5}{6}]$ and $[\frac{5}{6}, 1]$, with masses 1/18, 1/9, 1/9, 2/9, 1/6 and 1/3, respectively. We should point out that when a measure of this type is being formed, one actually does not know which generator sequence of divisors $(N_1, N_2, \dots)$ is being used in the construction, and consequently which multipliers are being used at each step.

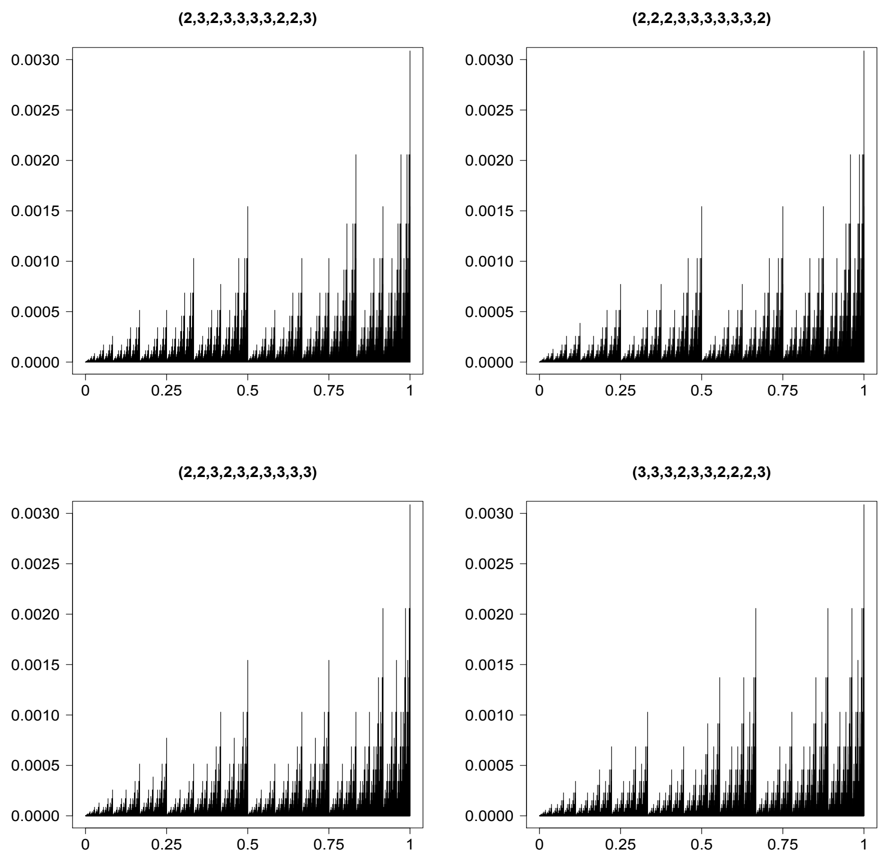
In Figure 2 we show the measure obtained after 10 steps for two different generator sequences, when working with the above random variable N. The patterns clearly reveal that the first divisor was 2 in the left plot and 3 in the right plot. In Figure 3 we show the effect of some permutations of a sequence of divisors of length 10 on the measure's construction (note that in this case there are a total of $2^{10} = 1024$ possible permutations for the sequence of digits). As we can see all four plots have different patterns.

Fig. 2. The measure obtained after 10 steps for two different generator sequences



In the binomial and multinomial measures the multipliers used throughout all steps are fixed in value and in number. In this new scenario each multiplier

Fig. 3. The measure obtained after 10 steps for four different permutations of a generator sequence



should be regarded as a random variable, since the magnitude and number of the multipliers used are directly determined by the distribution of N.

Let us go back to the example to see how this is the case. For the multipliers defined in (3) we can have $m_0 = 1/3$ or $m_0 = 1/6$, with probability $1/4$ and $3/4$, respectively, and for this example there are 3 random multipliers that need to be defined. If M_i denotes the random variable that represents the value of the i-th random multiplier,

$$M_0 = \begin{cases} \frac{1}{3} & \frac{1}{6} \\ \frac{1}{4} & \frac{3}{4} \end{cases}, \quad M_1 = \begin{cases} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{4} & \frac{3}{4} \end{cases} \quad \text{and} \quad M_2 = \begin{cases} 0 & \frac{1}{2} \\ \frac{1}{4} & \frac{3}{4} \end{cases}. \tag{4}$$

The expected values for the multipliers given in (4) are $\mathbb{E}[M_0] = 5/24$, $\mathbb{E}[M_1] = 5/12$ and $\mathbb{E}[M_2] = 3/8$. For an arbitrarily random variable N, the number of random multipliers M_i will depend on the number of points where N has non null mass.

In each step of this new multiplicative cascade we can also attach an address (location) to each interval generated, as is done in the binomial and multinomial measures (for more details on this subject see e.g. Ervertsz and Mandelbrot [2]). However, given the way the measure is constructed, we can have different intervals for the same address. In order to illustrate this situation we indicate in Table 1 the intervals and corresponding addresses and masses for the first two steps of all possible cases for (N_1, N_2) (in brackets we indicate the probability of observing each sequence of length 2).

Table 1. Intervals, addresses and masses for all possible sequences (N_1, N_2)

	$(N_1, N_2) = (2, 2)$ $(\frac{1}{16})$								
Interval	$[0, \frac{1}{4}]$	$[\frac{1}{4}, \frac{1}{2}]$	$[\frac{1}{2}, \frac{3}{4}]$	$[\frac{3}{4}, 1]$					
Address	0.00	0.01	0.10	0.11					
$\boldsymbol{\mu}$	1/9	2/9	2/9	4/9					
	$(N_1, N_2) = (2, 3)$ $(\frac{3}{16})$								
Interval	$[0, \frac{1}{6}]$	$[\frac{1}{6}, \frac{1}{3}]$	$[\frac{1}{3}, \frac{1}{2}]$	$[\frac{1}{2}, \frac{2}{3}]$	$[\frac{2}{3}, \frac{5}{6}]$	$[\frac{5}{6}, 1]$			
Address	0.00	0.01	0.02	0.10	0.11	0.12			
$\boldsymbol{\mu}$	1/18	1/9	1/6	1/9	2/9	1/3			
	$(N_1, N_2) = (3, 2)$ $(\frac{3}{16})$								
Interval	$[0, \frac{1}{6}]$	$[\frac{1}{6}, \frac{1}{3}]$	$[\frac{1}{3}, \frac{1}{2}]$	$[\frac{1}{2}, \frac{2}{3}]$	$[\frac{2}{3}, \frac{5}{6}]$	$[\frac{5}{6}, 1]$			
Address	0.00	0.01	0.10	0.11	0.20	0.21			
$\boldsymbol{\mu}$	1/18	1/9	1/9	2/9	1/6	1/3			
	$(N_1, N_2) = (3, 3)$ $(\frac{9}{16})$								
Interval	$[0, \frac{1}{9}]$	$[\frac{1}{9}, \frac{2}{9}]$	$[\frac{2}{9}, \frac{1}{3}]$	$[\frac{1}{3}, \frac{4}{9}]$	$[\frac{4}{9}, \frac{5}{9}]$	$[\frac{5}{9}, \frac{2}{3}]$	$[\frac{2}{3}, \frac{7}{9}]$	$[\frac{7}{9}, \frac{8}{9}]$	$[\frac{8}{9}, 1]$
Address	0.00	0.01	0.02	0.10	0.11	0.12	0.20	0.21	0.22
$\boldsymbol{\mu}$	1/36	1/18	1/12	1/18	1/9	1/6	1/12	1/6	1/4

From Table 1 we observe that there is no one-to-one correspondence between address and interval, contrarily to what happens with the binomial and multinomial measures. We also observe that intervals with the same address do not have necessarily the same mass. Thus the definitions of coarse and local Hölder exponents given in the literature can not be applied directly to this type of measure.

We recall that the coarse Hölder exponent is defined as

$$\alpha_k(x) = \frac{\log(\mu(\mathrm{I}_{0.\beta_1\beta_2\ldots\beta_k}))}{\log \epsilon}, \quad k = 1, 2, \ldots, \tag{5}$$

where $\mu(\mathrm{I}_{0.\beta_1\beta_2\ldots\beta_k})$ indicates the measure of the interval $x = \mathrm{I}_{0.\beta_1\beta_2\ldots\beta_k}$ having address $0.\beta_1\beta_2\ldots\beta_k$ and size (length) ϵ, with $\beta_i = 0, 1, \ldots, b-1$ and $b \geq 2$. On the other hand, the local Hölder exponent is defined as

$$\alpha(x) = \lim_{k\to\infty} \alpha_k(x) \tag{6}$$

(i.e. for $\epsilon \to 0$), if the limit exists.

However, expressions (5) and (6) can be generalized to accommodate this new measure. All we have to do is to consider that the measure associated with an address is the mean value of the masses of the intervals which can have the address. This becomes clearer by examining Table 2 for the working example. It is also clear from Table 2 that addresses that are permutations of one another have the same mean mass (this remains true at any step).

Table 2. Masses for all possible addresses obtained after 2 steps

Address	(2,2)	(2,3)	(3,2)	(3,3)	Mean mass
0.00	1/9	1/18	1/18	1/36	25/576
0.01	2/9	1/9	1/9	1/18	25/288
0.10	2/9	1/9	1/9	1/18	25/288
0.11	4/9	2/9	2/9	1/9	25/144
0.02	0	1/6	0	1/12	5/64
0.20	0	0	1/6	1/12	5/64
0.12	0	1/3	0	1/6	5/32
0.21	0	0	1/3	1/6	5/32
0.22	0	0	0	1/4	9/64

The question now is how to determine the measure of a particular address $0.\beta_1\beta_2\ldots\beta_k$, $\beta_i = 0, 1, \ldots$, $i = 1, 2, \ldots, k$, which can have a multitude of intervals attached to it, if one does not know which generator sequence $(N_1, N_2, \ldots, N_k)$ was used? As Table 2 suggests, we use the random multipliers expectations. We can prove that the address $0.\beta_1\beta_2\ldots\beta_k$ has expected measure

$$\mu_E(0.\beta_1\beta_2\ldots\beta_k) = \mathbb{E}(M_{\beta_1})\mathbb{E}(M_{\beta_2})\ldots\mathbb{E}(M_{\beta_k}),$$

which does not depend on the generator sequence. We remark that the only kind of dependence that exists between the generator sequence and the expected measure is through the influence of N on the random multipliers M_i. For example, both addresses 0.01 and 0.10 have expected measure $\mathbb{E}(M_0)\mathbb{E}(M_1) = 25/288$.

The generalization of the definitions (5) and (6) to this new measure is now straightforward. For the generalized coarse Hölder exponent we have

$$\alpha_k(0.\beta_1\beta_2\ldots\beta_k) = \frac{\log(\mu_E(0.\beta_1\beta_2\ldots\beta_k))}{\log\left(\mathbb{E}\left[\left(\prod_{i=1}^{k} N_i\right)^{-1}\right]\right)} \approx -\frac{\log(\mu_E(0.\beta_1\beta_2\ldots\beta_k))}{k\log(\mathbb{E}[N])}$$

and for the generalized local Hölder exponent,

$$\alpha = \lim_{k\to\infty}\alpha_k(0.\beta_1\beta_2\ldots\beta_k) \approx -\lim_{k\to\infty}\frac{\log(\mu_E(0.\beta_1\beta_2\ldots\beta_k))}{k\log(\mathbb{E}[N])},$$

if the limit exists. Note that $\left(\prod_{i=1}^{k} N_i\right)^{-1}$ represents the (random) length of the intervals at step k.

On the other hand, if at step k we randomly select an address $0.\beta_1\beta_2\ldots\beta_k$,

$$\mathbb{P}[\beta_i = j|N = n] = \frac{1}{n}, \quad j = 0, 1, \ldots, n-1,$$

and from applying the law of total probability, it follows that

$$\mathbb{P}[\beta_i = j] = \sum_{n=2}^{\infty}\mathbb{P}[\beta_i = j|N = n]\mathbb{P}[N = n], \quad j = 0, 1, \ldots . \tag{7}$$

Therefore, randomly selecting an address in this case corresponds to generating a sequence $\beta_1\beta_2\ldots\beta_k$, where the β_i's satisfy (7). Considering again the example, we get $\mathbb{P}[\beta_i = 0] = \mathbb{P}[\beta_i = 1] = 3/8$ and $\mathbb{P}[\beta_i = 2] = 1/4$.

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